Math 126
Summer 2018
Dr. Lily Yen

Test 3

Show all your work

Name: _____
Number: _____
Signature: _____
Score: ___/40

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Solve for y in the following initial value problem. Express your solution explicitly as $y = f(x)$. Include the domain of f.

$$x^2y' + e^{-y} = 0, \quad y(2) = 4.$$

Here $x^2\frac{dy}{dx} = -e^{-y}$, so $\int e^y\,dy = \int \frac{-1}{x^2}\,dx$, so $e^y = \frac{1}{x} + C$, so $y = \ln(\frac{1}{x} + C)$.
Since $4 = y(2) = \ln(\frac{1}{2} + C)$, it follows that $e^4 = \frac{1}{2} + C$, so $C = e^4 - \frac{1}{2}$. Thus

$$y = \ln\left(\frac{1}{x} + e^4 - \frac{1}{2}\right)$$

Score: /5

Problem 2: Find the orthogonal trajectories of the family of curves $kx^5 = y^4$, where k is an arbitrary constant.

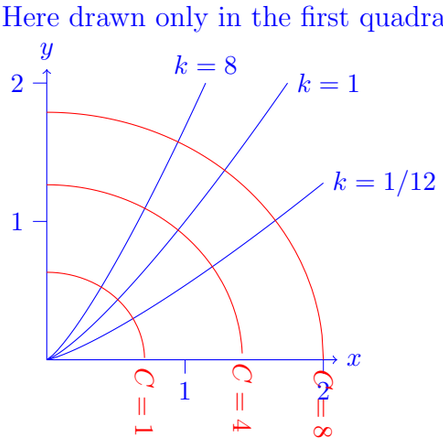
If $kx^5 = y^4$, then $k = \frac{y^4}{x^5}$, so

$$0 = \frac{4y^3\frac{dy}{dx}x^5 - 5y^4x^4}{x^{10}},$$

so $4y^3\frac{dy}{dx}x^5 - 5y^4x^4 = 0$, so $\frac{dy}{dx} = \frac{5y^4x^4}{4y^3x^5} = \frac{5y}{4x}$. The orthogonal trajectories therefore satisfy $\frac{dy}{dx} = -\frac{4x}{5y}$, so $\int 5y\,dy = -\int 4x\,dx$, so $\frac{5}{2}y^2 = -2x^2 + C$, so

$$y = \pm\sqrt{\tfrac{2}{5}C - \tfrac{4}{5}x^2}$$

Here drawn only in the first quadrant; the others by symmetry.



Score: /4

Problem 3: Determine whether each sequence below converges as $n \to \infty$. If so, find the limit.

a. $\left\{\dfrac{3n-(-1)^n}{3n+(-1)^n}\right\}$

$\frac{3n-(-1)^n}{3n+(-1)^n} = \frac{n-\frac{(-1)^n}{3}}{n+\frac{(-1)^n}{3}} \to 1$ as $n \to \infty$.

b. $a_n = \dfrac{n\sin(n\pi/2)}{2n+1}$

If $n = 2k$, then $\frac{n\sin(n\pi/2)}{2n+1} = \frac{2k\sin(k\pi)}{4k+1} = 0$. On the other hand, if $n = 4k+1$, then $\frac{n\sin(n\pi/2)}{2n+1} = \frac{4k+1}{8k+2} \geq \frac{5}{11}$. Since these are incompatible, the series DIVERGES.

c. $s_n = \displaystyle\sum_{k=0}^{n}\left(\frac{3^k}{2^{2k}} - \frac{1}{k!}\right)$

$s_n = \sum_{k=0}^{n}\left(\frac{3^k}{(2^2)^k} - \frac{1}{k!}\right) = \sum_{k=0}^{n}\left(\frac{3}{4}\right)^k - \sum_{k=0}^{n}\frac{1}{k!} \to \frac{1}{1-\frac{3}{4}} - e^1 = 4 - e$

d. The sequence a_n satisfying the recurrence: $a_n = 3a_{n-1} - a_{n-2}$ for integers $n \geq 2$ and $a_0 = 1$, and $a_1 = 1$.

Claim: $a_n \geq 2a_{n-1}$ for $n \geq 2$.
Proof: Since $a_2 = 2$, the claim is true for $n = 2$. Now assume the claim for $n-1$, that is $a_{n-1} \geq 2a_{n-2}$, so $a_{n-2} \leq \frac{1}{2}a_{n-1}$. Then
$a_n = 3a_{n-1} - a_{n-2} \geq 3a_{n-1} - \frac{1}{2}a_{n-1} = \frac{5}{2}a_{n-1} > 2a_{n-1}$. QED.
The claim shows that $a_n \geq 2^{n-1}$, but $\{2^{n-1}\}$ is divergent, so $\{a_n\}$ is DIVERGENT.

Score: /12

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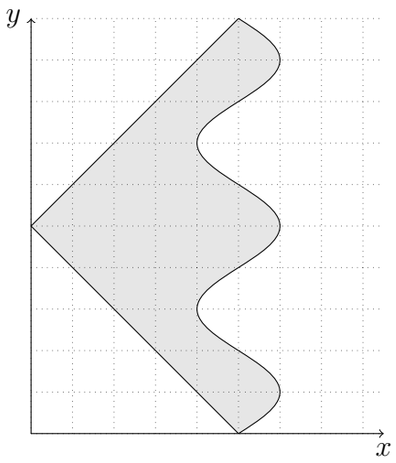
Calculators permitted in this part.

Problem 4: The shaded region shown is bounded between the curves:

$$x = |y-5|, \qquad \text{and} \qquad x = 5 + \cos(\tfrac{\pi}{2}(y-5))$$



Use integrals to express the perimeter. Give your answer to 4-decimal place accuracy.

$$\int_0^{10} \sqrt{1 + \left(\frac{d}{dy}|y-5|\right)^2}\, dy + \int_0^{10} \sqrt{1 + \left(\frac{d}{dy}\left(5 + \cos(\tfrac{\pi}{2}(y-5))\right)\right)^2}\, dy$$

$$= \int_0^{10} \sqrt{1 + (\pm 1)^2}\, dy + \int_0^{10} \sqrt{1 + \left(-\tfrac{\pi}{2}\sin(\tfrac{\pi}{2}(y-5))\right)^2}\, dy$$

$$= 10\sqrt{2} + \int_0^{10} \sqrt{1 + \tfrac{\pi^2}{4}\sin^2(\tfrac{\pi}{2}(y-5))}\, dy$$

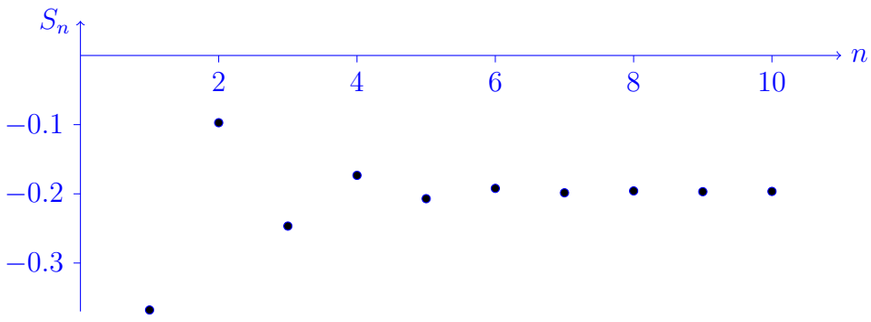
$$\approx 28.7791$$

Score: /5

Problem 5: For the series

$$S = \sum_{k=1}^{\infty} \frac{(-1)^k k}{e^k}$$

a. Graph the n-th partial sum of the series for the first 10 values of n.



b. Suppose you use S_{49} to estimate S. Bound the error $|S - S_{49}|$.

Since the series is alternating, the error is at most the 50th term, $\frac{50}{e^{50}} \approx 9.644 \times 10^{-21}$.

c. Determine whether the series S converges absolutely.

Since $k < 2^k$, it follows that $\left|\frac{(-1)^k k}{e^k}\right| = \frac{k}{e^k} < \frac{2^k}{e^k} = \left(\frac{2}{e}\right)^k$. Now $\sum_{k=1}^{\infty}\left(\frac{2}{e}\right)^k$ is convergent (since it is a geometric series), so S CONVERGES ABSOLUTELY.

Score: /6

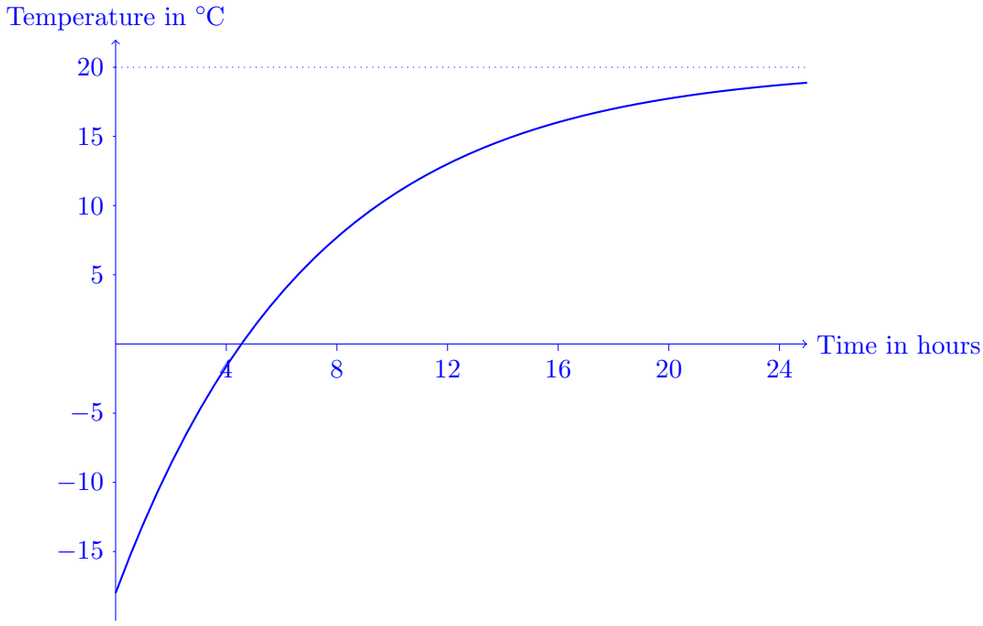
Problem 6: Recall Newton's law of cooling: the rate of change in temperature of an object varies proportionally to the difference in temperature between the object and the surrounding medium. Write a differential equation whose solution is the temperature as a function of time for a bottle of frozen elder flower juice taken out of a $-18\,^\circ$C deep freezer and left in a $20\,^\circ$C kitchen, given that after one hour, the temperature of the bottle rose $5\,^\circ$C. Include all conditions. Solve the equation for the temperature of the object at time t and graph the solution.

Let $y(t)$ be the temperature (in °C) of the bottle at time t (in hours since we took the bottle out of the freezer). By Newton's Law of Cooling,

$$\frac{dy}{dt} = k(y-20), \qquad y(0) = -18 \text{ and } y(1) = -13.$$

Thus $\int \frac{1}{y-20}\,dy = \int k\,dt$, so $\ln|y-20| = kt + C$, so $|y-20| = e^{kt+C} = e^C e^{kt}$, so $y - 20 = Ae^{kt}$ where $A = \pm e^C$, so $y = 20 + Ae^{kt}$.
Moreover, $-18 = y(0) = 20 + Ae^0 = 20 + A$, so $A = -38$. Hence, $y(t) = 20 - 38e^{kt}$. We also know that $y(1) = -13$, so $-13 = 20 - 38e^k$, so $e^k = \frac{33}{38}$, so $k = \ln(\frac{33}{38}) \approx -0.1411$. Thus

$$y(t) = 20 - 38e^{-0.1411t}$$



Score: /8