Math 126
Summer 2018
Dr. Lily Yen

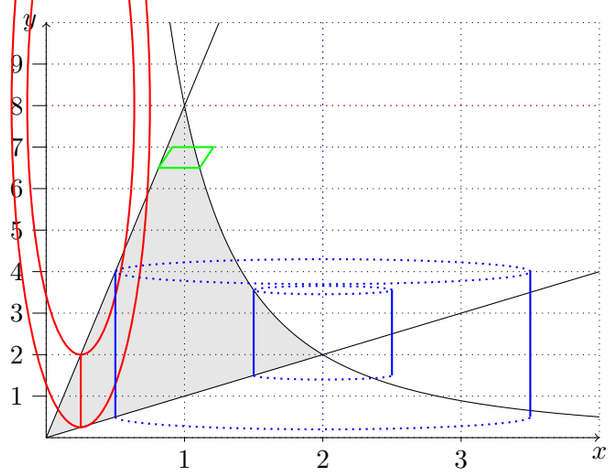
Test 2

Show all your work

Name: _____
Number: _____
Signature: _____
Score: ___/48

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: First draw $y = 8/(x^2)$, $y = 8x$, and $y = x$. Shade the region enclosed by these three graphs. Clearly label the intersections.



Score: /3

Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

If $x^2y = 8$ and $y = x$, then $x^3 = 8$, so $x = 2$. If $x^2y = 8$ and $y = 8x$, then $8x^3 = 8$, so $x^3 = 1$, so $x = 1$. If $y = x$ and $y = 8x$, then $x = 8x$, so $x = 0$. Thus the intersection points are $(2, 2)$, $(1, 8)$, and $(0, 0)$. The area is then
$\int_0^1 8x - x\,dx + \int_1^2 \frac{8}{x^2} - x\,dx = \int_0^1 7x\,dx + \int_1^2 \frac{8}{x^2} - x\,dx$ which easily evaluates to 6.

Score: /3

b. Use the washer method to find the volume obtained if the region is rotated about $y = 8$.

$$\int_0^1 \pi\left((8-x)^2 - (8-8x)^2\right)\,dx + \int_1^2 \pi\left((8-x)^2 - \left(8 - \frac{8}{x^2}\right)^2\right)\,dx$$

which is easy to evaluate to $\frac{176}{3}\pi$.

Score: /3

c. Use the method of cylindrical shells to find the volume obtained if the region is rotated about $x = 2$.

$$\int_0^1 2\pi(2-x)(8x-x)\,dx + \int_1^2 2\pi(2-x)\left(\frac{8}{x^2} - x\right)\,dx$$

which is easy to evaluate to $8\pi(3 - 2\ln(2))$.

Score: /3

d. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the y-axis are squares.

$$\int_0^2 \left(y - \frac{y}{8}\right)^2\,dy + \int_2^8 \left(\sqrt{\frac{8}{y}} - \frac{y}{8}\right)^2\,dy$$

Score: /3

Problem 2: Evaluate the following integrals exactly.

a. $\displaystyle\int e^x \sin(2x)\,dx$

Using integration by parts twice,

$$\begin{aligned}\int e^x \sin(2x)\,dx &= e^x \sin(2x) - 2\int e^x \cos(2x)\,dx \\ &= e^x \sin(2x) - 2\left(e^x \cos(2x) + 2\int e^x \sin(2x)\,dx\right) \\ &= e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x)\,dx,\end{aligned}$$

so $5\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x)$, so

$$\int e^x \sin(2x)\,dx = \tfrac{1}{5}e^x \sin(2x) - \tfrac{2}{5}e^x \cos(2x) + C.$$

Score: /4

b. $\displaystyle\int \sqrt{x}e^{\sqrt{x}}\,dx$. Hint: Substitute $u = \sqrt{x}$.

If $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, so $dx = 2\sqrt{x}\,du$. Therefore

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = \int \sqrt{x}e^u 2\sqrt{x}\,du = \int 2u^2 e^u\,du.$$

Using integration by parts twice, this equals

$$\begin{aligned}2u^2e^u - \int 4ue^u\,du = 2u^2e^u - 4ue^u + \int 4e^u\,du &= 2u^2e^u - 4ue^u + 4e^u + C \\ &= 2(u^2 - 4u + 4)e^u + C = 2(x - 2\sqrt{x} + 2)e^{\sqrt{x}} + C\end{aligned}$$

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Calculators permitted from here on.

Problem 3: The acceleration of a particle is $a(t) = 60t - 4t^3$ in meters per second squared. Compute the average acceleration and the average speed over the time interval $[2, 6]$, assuming that the particle's initial velocity is zero.

The average acceleration is

$$\bar{a} = \frac{1}{6-2}\int_2^6 60t - 4t^3\,dt = \frac{1}{4}(30t^2 - t^4)\Big|_2^6 = -80\,\text{m/s}^2$$

The velocity is $v(t) = \int a(t)\,dt = 30t^2 - t^4 + C$, but since the initial velocity is zero, $C = 0$. If $v(t) = 0$, then $t = 0$ or $t = \pm\sqrt{30}$. Thus the average *speed* is

$$\begin{aligned}\frac{1}{6-2}\int_2^6 |v(t)|\,dt &= \frac{1}{4}\int_2^{\sqrt{30}} 30t^2 - t^4\,dt + \frac{1}{4}\int_{\sqrt{30}}^6 t^4 - 30t^2\,dt \\ &= \frac{1}{4}(10t^3 - \tfrac{1}{5}t^5)\Big|_2^{\sqrt{30}} + \frac{1}{4}(\tfrac{1}{5}t^5 - 10t^3)\Big|_{\sqrt{30}}^6 = 636\,\text{m/s}\end{aligned}$$

Score: /4

Problem 4: Circle improper integrals below. State a brief reason next to each item.

a. $\displaystyle\int_0^1 \frac{dx}{\sqrt{3-x^2}}$

PROPER since $3 - x^2 > 0$ for all x in $[0, 1]$.

b. $\displaystyle\int_0^1 \frac{dx}{xe^x + x^2}$

IMPROPER since the integrand is undefined at $x = 0$.

c. $\displaystyle\int_0^1 \frac{dx}{(2x-1)^2}$

IMPROPER since the integrand is undefined at $x = \frac{1}{2}$.

d. $\displaystyle\int_0^1 \sec(\pi x)\,dx$

IMPROPER since the integrand is undefined at $x = \frac{1}{2}$.

Score: /6

Problem 5: Find the volume of the solid obtained by rotating the region bounded by $f(x) = x^{-1/2}$, y-axis, and x-axis restricted to the domain $[0, 9]$ around the y-axis.

Cylindrical shells:

$$\int_0^9 2\pi x \frac{1}{\sqrt{x}}\,dx = \tfrac{4}{3}\pi x^{3/2}\Big|_0^9 = 36\pi \approx 113$$

Score: /4

Problem 6: Evaluate using the method of partial fractions. $\displaystyle\int \frac{18}{(x+3)(x^2+9)}\,dx$

Using partial fractions, if $\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}$, then

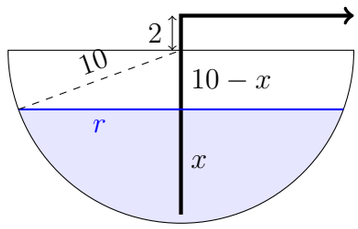
$$18 = A(x^2+9) + (Bx+C)(x+3).$$

If $x=-3$, then $18 = 18A + 0$, so $A = 1$. If $x = 0$, then $18 = 9A + 3C = 9 + 3C$, so $C = 3$. If $x = 1$, then $18 = 10A + 4(B+C) = 10 + 4(B+3) = 10 + 4B + 12 = 22 + 4B$, so $B = -1$. Hence (if $u = \frac{1}{3}x$),

$$\begin{aligned}
\int \frac{18}{(x+3)(x^2+9)}\,dx &= \int \frac{1}{x+3} + \frac{-x+3}{x^2+9}\,dx \\
&= \ln|x+3| + \int \frac{-x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx \\
&= \ln|x+3| - \tfrac{1}{2}\int \frac{2x}{x^2+9}\,dx + \int \frac{9}{(3u)^2+9}\,du \\
&= \ln|x+3| - \tfrac{1}{2}\ln|x^2+9| + \int \frac{1}{u^2+1}\,du \\
&= \ln|x+3| - \tfrac{1}{2}\ln(x^2+9) + \arctan(u) + C \\
&= \ln|x+3| - \tfrac{1}{2}\ln(x^2+9) + \arctan(x/3) + C
\end{aligned}$$

Score: /6

Problem 7: Compute the work (in joules) required to pump all of the water out of the full hemispherical tank shown. Distances are in meters, $g = 9.81\,\text{m/s}^2$, and the density of water is $1000\,\text{kg/m}^3$.



The slice at height x is a disk with radius $r = \sqrt{10^2 - (10-x)^2} = \sqrt{20x - x^2}$. Therefore the volume of the disk is $\pi r^2\,dx = \pi(20x - x^2)\,dx$. Since the density is $1000\,\text{kg/m}^3$, the mass of the disk is $1000\pi(20x - x^2)\,dx$, so the force of gravity on the disk is $9.81 \times 1000\pi(20x - x^2)\,dx = 9810\pi(20x - x^2)\,dx$. The disk has to be lifted $12 - x$ metres, which then requires a work of $9810\pi(20x - x^2)(12 - x)\,dx = 9810\pi(240x - 32x^2 + x^3)\,dx$. The total amount of work required is thus

$$\int_0^{10} 9810\pi(240x - 32x^2 + x^3)\,dx = \cdots = 37605000\pi \approx 118\,\text{MJ}$$

Score: /5