Math 126
Summer 2016
Dr. Lily Yen

Test 3

Show all your work

Name: _____

Score: ___/43

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Use separation of variables to find the general solution of the following differential equation.

$$y' = y^2(1 - x^2)$$

If $\frac{dy}{dx} = y^2(1-x^2)$, then $\int \frac{1}{y^2}\, dy = \int (1-x^2)\, dx$, so $-\frac{1}{y} = x - \frac{1}{3}x^3 + C$, so

$$y = \frac{-1}{x - \frac{1}{3}x^3 + C} = \frac{3}{x^3 - 3x - 3C}$$

Score: /4

Problem 2: Find all values of a such that $y = x^a$ is a solution of

$$y'' - 12x^{-2}y = 0$$

If $y = x^a$, then $y'' = a(a-1)x^{a-2}$. Thus filing into the differential equation yields that $a(a-1)x^{a-2} - 12x^{-2}x^a = 0$, so $a(a-1)x^{a-2} - 12x^{a-2} = 0$ for all x, so $a(a-1) - 12 = 0$, so $a = 4$ or $a = -3$.

Score: /5

Problem 3: Find the first 5 terms of $a_n = a_{n-1} + 2a_{n-2}$ for $n > 2$ and $a_1 = 1$, $a_2 = 5$.

$a_1 = 1$
$a_2 = 5$
$a_3 = a_2 + 2a_1 = 5 + 2 \times 1 = 7$
$a_4 = 7 + 2 \times 5 = 17$
$a_5 = 17 + 2 \times 7 = 31$
$\vdots$
$a_n = 2^n + (-1)^n$

Score: /3

Problem 4: For each sequence or series below, determine if it converges; find the limit if possible, or prove that it diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}$ If this series converges, find its limit. Is it absolutely convergent?

This is a geometric series with common ratio $-\frac{1}{e}$ which therefore converges absolutely to $\frac{1}{1+e}$.

Score: /3

b. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$.

Since $\frac{1}{\ln(n)} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, it follows from Comparison Test that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is divergent.

Score: /3

c. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n!}$.

Note that $\cos(n\pi) = \pm 1$ and that $n! > n^2$ (if $n \geq 4$). Therefore $\left|\frac{\cos(n\pi)}{n!}\right| = \frac{1}{n!} < \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-Test, it follows from Comparison Test that $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n!}$ converges absolutely.

Actually $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n!} = \frac{1}{e} - 1$.

Score: /3

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Calculators permitted in this part.

Problem 5: For the function $f(x) = \ln(\sin(x))$, set up the integral expression for its arc length for $x \in [\pi/4, \pi/2]$. Evaluate the integral accurate to 4 decimal places.

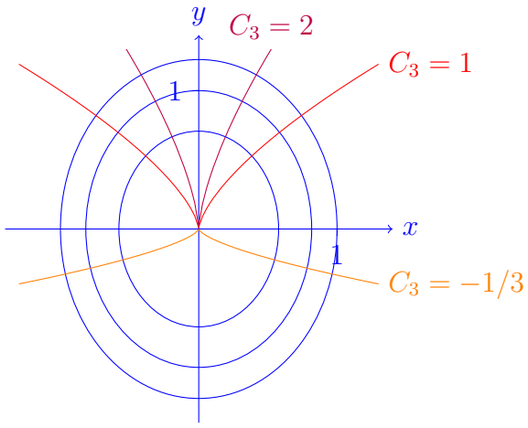
Since $f'(x) = \frac{1}{\sin(x)}\cos(x) = \cot(x)$, the arc length is

$$\int_{\pi/4}^{\pi/2} \sqrt{1+\cot^2(x)}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{\csc^2(x)}\,dx = \int_{\pi/4}^{\pi/2} \csc(x)\,dx$$
$$= -\ln(\csc(x)+\cot(x))\Big|_{\pi/4}^{\pi/2} = -\ln(1+0) + \ln(1+\sqrt{2})$$
$$= \ln(1+\sqrt{2}) \approx 0.8814$$

Score: /4

Problem 6: Find the orthogonal trajectories of $3x^2 + 2y^2 = c$, for arbitrary non-negative constants c. Sketch at least three members of each family (the function and its orthogonal trajectory). Label each curve with its parameter value.

To draw the original family, solve for $y = \pm\sqrt{\frac{1}{2}x - \frac{3}{2}x^2}$ **or** parametrize as $(x, y) = \left(\sqrt{\frac{c}{3}}\cos(\theta), \sqrt{\frac{c}{2}}\sin(\theta)\right)$ using that $\cos^2(\theta) + \sin^2(\theta) = 1$ **or** simply know that $\frac{x^2}{c/3} + \frac{y^2}{c/2} = 1$ is the equation of an ellipse with horizontal half-axis ("radius") $\sqrt{c/3}$ and vertical half-axis $\sqrt{c/2}$.
Now, if $3x^2 + 2y^2 = c$, then $6x + 4yy' = 0$, so $y' = -\frac{6x}{4y} = -\frac{3x}{2y}$. Hence the orthogonal trajectories satisfy $\frac{dy}{dx} = \frac{2y}{3x}$, so $\int \frac{1}{2y}\,dy = \int \frac{1}{3x}\,dx$, so $\frac{1}{2}\ln|2y| = \frac{1}{3}\ln|3x| + C_1$, so $\ln\left(\sqrt{|2y|}\right) - \ln\left(\sqrt[3]{|3x|}\right) = C_1$, so $\ln\frac{\sqrt{|2y|}}{\sqrt[3]{|3x|}} = C_1$, so $\frac{\sqrt{|2y|}}{\sqrt[3]{|3x|}} = e^{C_1}$, so $2y = C_2(3x)^{2/3}$, so $y = C_3x^{2/3}$.



The ellipses with $c = 1, 2, 3$ from the inside out

Score: /5

Problem 7: A patient is given the drug theophylline intravenously at a rate of 43.2 mg/hour to relieve acute asthma. The rate at which the quantity of drug leaves the body (in mg/hour) is proportional to the quantity there, with proportionality constant 0.082 /hour. Assume the patient's body contains none of the drug initially.

a. Describe in words how you would expect the quantity of theophylline in the patient to vary with time and sketch an approximate graph.

See Part c.

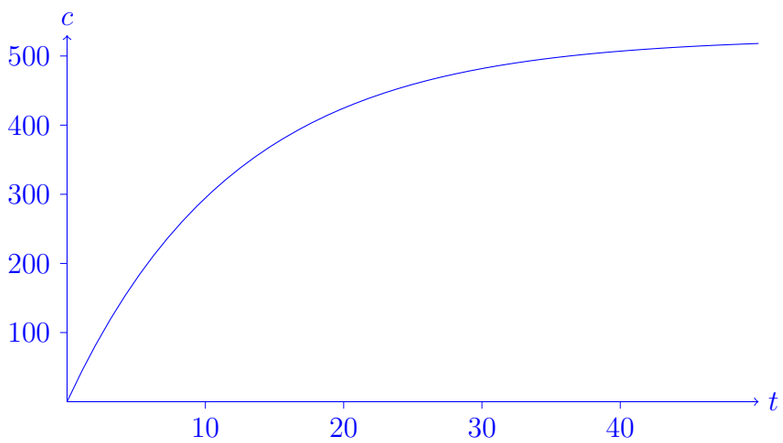
b. Write a differential equation satisfied by the quantity of the drug, $A(t)$ in mg.

$$\frac{dc}{dt} = 43.2 - 0.082c, \qquad c(0) = 0$$

c. Solve the differential equation and graph the solution. What happens to the quantity in the long run?

If $\frac{dc}{dt} = 43.2 - 0.082c$, then $\int \frac{1}{43.2-0.082c}\,dc = \int 1\,dt$, so $-\frac{1}{0.082}\ln|43.2 - 0.082c| = t + C$, so $|43.2 - 0.082c| = e^{-0.082(t+C)}$, so $43.2 - 0.082c = \pm e^{-0.082(t+C)} = Ae^{-0.082t}$. Since $c(0) = 0$, $43.2 = A$. Thus

$$c = \frac{43.2(1 - e^{-0.082t})}{0.082} = 526.8(1 - e^{-0.082t})$$



The concentration levels off at $c = 526.8$ mg.

Score: /8

Problem 8: Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

a. Evaluate the Nth partial sum for the first four N's.

$$\sum_{n=1}^{1} \frac{1}{n^2} = \frac{1}{1^2} = 1$$
$$\sum_{n=1}^{2} \frac{1}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$$
$$\sum_{n=1}^{3} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36} \approx 0.8611$$
$$\sum_{n=1}^{4} \frac{1}{n^2} = \frac{31}{36} - \frac{1}{16} = \frac{115}{144} \approx 0.798611$$

Score: /2

b. With your calculator, evaluate the 20-th partial sum. Give 4-decimal place accuracy.

Set up $n_{min} = 1$, $u(n) = \frac{(-1)^{n-1}}{n^2} + u(n-1)$, $u_{min} = 1$.

$$\sum_{n=1}^{20} \frac{(-1)^{n-1}}{n^2} \approx 0.8213$$

Score: /2

c. The sum is equal to $\pi^2/12$. Find the smallest N-th partial sum that is accurate to 4 decimal places of the answer.

$N = 99$ because the error of S_{99} is bounded by the absolute value of the next term $1/100^2 = 1/10000$, or 4-decimal place accuracy.

Score: /1