Math 126
Summer 2016
Dr. Lily Yen

Test 2

Show all your work

Name: _____

Score: ___/45

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Evaluate the following integrals analytically.

a. $\int x^5 \ln(x)\, dx$

Using integration by parts,
$\int x^5 \ln(x)\, dx = \frac{1}{6}x^6 \ln(x) - \int \frac{1}{6}x^6 \cdot \frac{1}{x}\, dx = \frac{1}{6}x^6 \ln(x) - \int \frac{1}{6}x^5\, dx = \frac{1}{6}x^6 \ln(x) - \frac{1}{36}x^6 + C.$

Score: /3

b. $\int \dfrac{1}{\tan(6x)}\, dx$

If $u = \sin(6x)$, then $\frac{du}{dx} = 6\cos(6x)$, so $dx = \frac{1}{6\cos(6x)}\, du$. Therefore, $\int \frac{1}{\tan(6x)}\, dx =$
$\int \frac{\cos(6x)}{\sin(6x)}\, dx = \int \frac{\cos(6x)}{u} \cdot \frac{1}{6\cos(6x)}\, du = \int \frac{1}{6} \cdot \frac{1}{u}\, du = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|\sin(6x)| + C.$

Score: /3

c. $\int e^{2x}\sin(3x)\, dx$

Using integration by parts twice,
$\int e^{2x}\sin(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \int \frac{1}{2}e^{2x}3\cos(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\int e^{2x}\cos(3x)\, dx =$
$\frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos(3x) - \int \frac{1}{2}e^{2x}(-3)\sin(3x)\, dx\right) =$
$\frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) - \frac{9}{4}\int e^{2x}\sin(3x)\, dx + C.$ Therefore
$\frac{13}{4}\int e^{2x}\sin(3x)\, dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x) + C$, so

$$\int e^{2x}\sin(3x)\, dx = \frac{2}{13}e^{2x}\sin(3x) - \frac{3}{13}e^{2x}\cos(3x) + C_2$$

Score: /4

d. $\displaystyle\int \frac{4+x}{x^2+x}\,dx$

By the method of partial fractions, constants A and B exist such that $\frac{4+x}{x^2+x} = \frac{4+x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$, so $4 + x = A(x+1) + Bx$. Substituting $x = 0$ yields that $4 = A$, and substituting $x = -1$ yields that $3 = -B$, so $B = -3$. Hence

$$\int \frac{4+x}{x^2+x}\,dx = \int \frac{4}{x} - \frac{3}{x+1}\,dx = 4\ln|x| - 3\ln|x+1| + C$$

Score: /4

Problem 2: Evaluate if possible, otherwise state why the integral does not exist.

$$\int_0^7 \frac{1}{(x-5)^{2/3}}\,dx$$

The integrand is undefined at $x = 5$, so split the interval of integration: $\int_0^5 \frac{1}{(x-5)^{2/3}}\,dx = \lim_{z\to 5^-} \int_0^z \frac{1}{(x-5)^{2/3}}\,dx = \lim_{z\to 5^-} 3(x-5)^{1/3}\Big|_0^z = \lim_{z\to 5} 3\sqrt[3]{z-5} - 3\sqrt[3]{-5} = 0 + 3\sqrt[3]{5}$. Similarly, $\int_5^7 \frac{1}{(x-5)^{2/3}}\,dx = 3\sqrt[3]{(2)}$. Hence

$$\int_0^7 \frac{1}{(x-5)^{2/3}}\,dx = 3\sqrt[3]{5} + 3\sqrt[3]{2}$$

Score: /4

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Calculators permitted from here on.

Problem 3: Determine analytically the convergence of the following integral.

$$\int_1^\infty \frac{\sin(x)+2}{\sqrt{x}}\,dx$$

Note that $\frac{\sin(x)+2}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for all x. Therefore
$\int_1^\infty \frac{\sin(x)+2}{\sqrt{x}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{z\to\infty} \int_1^z \frac{1}{\sqrt{x}}\,dx = \lim_{z\to\infty} 2\sqrt{x}\Big|_1^z = \lim_{z\to\infty} 2\sqrt{z} - 2 = \infty$, so

$$\int_1^\infty \frac{\sin(x)+2}{\sqrt{x}}\,dx = \infty$$

Score: /3

Problem 4: Suppose $\int_{-1}^1 h(z)\,dz = 7$, and that $h(z)$ is an even function. Calculate the following:

a. $\int_0^1 h(z)\,dz =$ $\boxed{7/2}$

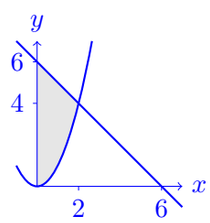
Since h is even, $h(-x) = h(x)$, so $\int_{-1}^0 h(z)\,dz = \int_0^1 h(z)\,dz$.

b. $\int_{-4}^{-2} 5h(z+3)\,dz =$ $\boxed{35}$

Substituting $z = u - 3$, $\int_{-4}^{-2} 5h(z+3)\,dz = \int_{-1}^1 5h(u)\,du = 5\int_{-1}^1 h(u)\,du = 5 \times 7 = 35$.

Score: /3

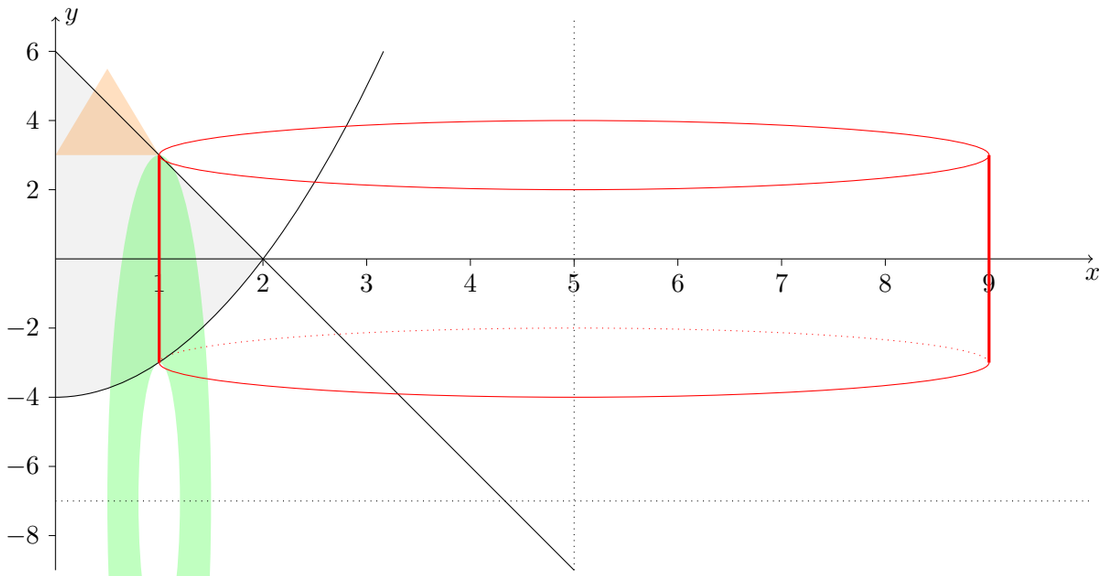
Problem 5: Consider the area bounded by the y-axis, $x + y = 6$, and $y = x^2$ in the first quadrant. Find the average vertical height of this area.



The height at x is $(6 - x) - x^2$, so the average height is
$\frac{1}{2-0}\int_0^2 6 - x - x^2\,dx = \frac{1}{2}(6x - \frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_0^2 = \frac{1}{2}(12 - 2 - \frac{8}{3}) = \frac{11}{3}$.

Score: /4

Problem 6: Draw $y = 6 - 3x$ and $y = x^2 - 4$ on the grid. Shade the region bounded by these two graphs and the y-axis for non-negative values of x.



Use integrals to express the following. DO NOT EVALUATE YOUR INTEGRALS. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

$$\int_0^2 (6 - 3x) - (x^2 - 4)\,dx = \int_0^2 10 - 3x - x^2\,dx = 10x - \frac{3}{2}x^2 - \frac{1}{3}x^3\Big|_0^2 = \frac{34}{3}$$

Score: /2

b. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the y-axis are equilateral triangles.

For $x \le 0$, the base of each triangle is $x = \sqrt{y+4}$, so the area is $\frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4}(y+4)$. For $x \ge 0$, the side of each triangle is $x = 2 - \frac{1}{3}y$, so the area is $\frac{\sqrt{3}}{4}(2 - \frac{1}{3}y)^2$. Hence the volume is

$$\int_{-4}^0 \tfrac{\sqrt{3}}{4}(y+4)\,dy + \int_0^6 \tfrac{\sqrt{3}}{4}(2 - \tfrac{1}{3}y)^2\,dy = \cdots = 4\sqrt{3} \approx 6.928$$

Score: /2

c. The volume of the solid obtained by rotating the region around $x = 5$.

Using cylindrical shells, the height of each shell is $(6 - 3x) - (x^2 - 4) = 10 - 3x - x^2$, so the volume is

$$\int_0^2 2\pi(5 - x)(10 - 3x - x^2)\,dx = \cdots = \frac{292}{3}\pi \approx 305.8$$

Score: /2

d. The volume of the solid obtained by rotating the region around $y = -7$.

Using washers, the outer radius is $y + 7 = 13 - 3x$ and the inner radius is $y + 7 = 3 + x^2$, so the volume is

$$\int_0^2 \pi(13 - 3x)^2 - \pi(3 + x^2)^2\,dx = \cdots = \frac{828}{5}\pi \approx 520.2$$

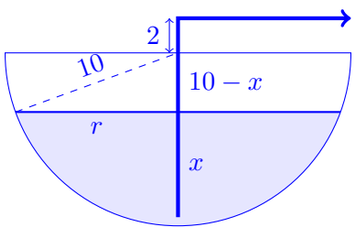
Score: /2

Problem 7: For your summer job, you work on a scaffolding 75 ft above the ground. Suppose you need to lift a 500 lb bucket of cement from the ground to a point 30 ft above the ground by pulling on a rope weighing 0.5 lb/ft. How much work is required?

Lifting just the bucket of cement requires $30\,\text{ft} \times 500\,\text{lb} = 15\,000\,\text{ft}\cdot\text{lb}$ of work.
Similarly, lifting the bottom 45 ft of rope requires $30\,\text{ft} \times 0.5\,\text{lb/ft} \times 45\,\text{ft} = 675\,\text{ft}\cdot\text{lb}$.
Lifting the top 30 ft of the rope—the part that ends up on the scaffolding with you—requires $\int_0^{30}(30-x)0.5\,dx = \int_0^{30} 15 - 0.5x\,dx = 15x - 0.25x^2\Big|_0^{30} = 225\,\text{ft}\cdot\text{lb}$.
Total $15\,000\,\text{ft}\cdot\text{lb} + 675\,\text{ft}\cdot\text{lb} + 225\,\text{ft}\cdot\text{lb} = 15\,900\,\text{ft}\cdot\text{lb}$.

Score: /4

Problem 8: Calculate the work done in pumping oil from a full (lower) hemispherical tank of radius 10 m out of a spout 2 m above the top of the hemisphere. Assume that the oil has density $800\,\text{kg/m}^3$.



The slice at height x is a disk with radius $r = \sqrt{10^2 - (10-x)^2} = \sqrt{20x - x^2}$. Therefore the volume of the disk is $\pi r^2\,dx = \pi(20x - x^2)\,dx$. Since the density is $800\,\text{kg/m}^3$, the mass of the disk is $800\pi(20x - x^2)\,dx$, so the force of gravity on the disk is $9.8 \times 800\pi(20x - x^2)\,dx = 7840\pi(20x - x^2)\,dx$. The disk has to be lifted $12 - x$ metres, which then requires a work of $7840\pi(20x - x^2)(12 - x)\,dx = 7840\pi(240x - 32x^2 + x^3)\,dx$. The total amount of work required is thus

$$\int_0^{10} 7840\pi(240x - 32x^2 + x^3)\,dx = \cdots = \frac{90160000}{3}\pi \approx 94.4\,\text{MJ}$$

Score: /5