Math 126
Summer 2015
Dr. Lily Yen

Test 3

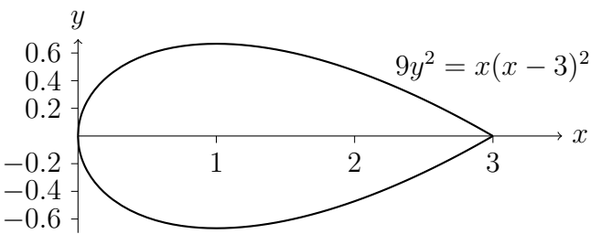
Show all your work

Name: _____

Score: ___/44

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Find the exact arc length of the curve shown.



If $y \geq 0$, then $y = \frac{1}{3}\sqrt{x}(3-x)$. Taking derivative yields that
$18y\frac{dy}{dx} = (3-x)^2 - 2x(3-x) = 9 - 6x + x^2 - 6x + 2x^2 = 9 - 12x + 3x^2 = 3(x-1)(x-3)$,
so $\frac{dy}{dx} = \frac{3(x-1)(x-3)}{18y} = \frac{(x-1)(x-3)}{6y} = \frac{(x-1)(x-3)}{2\sqrt{x}(3-x)} = \frac{1-x}{2\sqrt{x}}$. Therefore the length os the upper arch is $\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1 + \frac{(1-x)^2}{4x}}\,dx = \int_0^3 \sqrt{\frac{4x}{4x} + \frac{1-2x+x^2}{4x}}\,dx = \int_0^3 \sqrt{\frac{1+2x+x^2}{4x}}\,dx =$
$\int_0^3 \sqrt{\frac{(1+x)^2}{4x}}\,dx = \int_0^3 \frac{1+x}{2\sqrt{x}}\,dx = \frac{1}{2}\int_0^3 x^{-1/2} + x^{1/2}\,dx = x^{1/2} + \frac{1}{3}x^{3/2}\Big|_0^3 = 3^{1/2} + \frac{1}{3}\cdot 3^{3/2} = 2\sqrt{3}$.
The entire arc length is then $4\sqrt{3}$.

Score: /5

Problem 2: Solve the following initial value problem.

$$(1-t)\frac{dy}{dt} - y = 0, \qquad y(2) = -4.$$

The differential equation is separable: if $(1-t)\frac{dy}{dt} - y = 0$, then $\frac{dy}{dt} = \frac{y}{1-t}$, so
$\int \frac{1}{y}\,dy = \frac{1}{1-t}\,dt$, so $\ln|y| = -\ln|1-t| + C$, so $|y| = -e^C\frac{1}{1-t}$, so $y = \frac{A}{1-t}$.
Now $-4 = y(2) = \frac{A}{1-2} = -A$, so $A = 4$. Thus

$$y = \frac{4}{1-t}$$

Score: /4

Problem 3: State the logistic differential equation and discuss how it differs from the exponential differential equation with respect to modelling population growth.

Logistic DE: $y' = ky(1 - \frac{y}{A})$ vs. exponential DE: $y' = ky$. When used to model population growth, the exponential DE grows exponentially regardless of space and resource constraints, an unlikely situation in the long run. However, the logistic DE models exponential growths when the population is small in the beginning, but slows down as it approaches the carrying capacity A. This is a more realistic model which considers natural constraints.

Score: /4

Problem 4: For each sequence or series below, determine if it converges; find the limit if possible, or prove that it diverges.

a. $a_n = \dfrac{(-1)^n n^3 + 2^{-n}}{4^{-n} + 5n^3}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n^3 + 2^{-n}}{4^{-n}+5n^3} = \lim_{n\to\infty} \frac{(-1)^n n^3}{5n^3} = \lim_{n\to\infty} \frac{(-1)^n}{5}$ does not exist.

Score: /3

b. $a_0 = 0, a_1 = 1, a_n = a_{n-1} + a_{n-2}$ for all integers $n \geq 2$.

The terms clearly increase, so $\lim_{n\to\infty} a_n$ diverges to ∞.

$a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad a_5 = 5, \quad a_6 = 8, \quad a_7 = 13, \quad a_8 = 21, \quad a_9 = 34, \ldots$

The sequence is well known as the Fibonacci Sequence.

Score: /3

c. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 8}{3^n}$

Let $a_n = \frac{2^n-8}{3^n}$. Then $\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}-8}{3^{n+1}}}{\frac{2^n-8}{3^n}} = \frac{(2^{n+1}-8)3^n}{(2^n-8)3^{n+1}} = \frac{2^{n+1}-8}{(2^n-8)3}$, so
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}-8}{(2^n-8)3}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n \cdot 3}\right| = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3} < 1$. Thus $\sum_{n=0}^{\infty} a_n$ converges (absolutely) by Ratio Test.
If you are to find the value of an infinite sum, it is very likely to be a geometric series. Indeed, $\sum_{n=0}^{\infty} \frac{2^n-8}{3^n} = \sum_{n=0}^{\infty} \frac{2^n}{3^n} - \sum_{n=0}^{\infty} \frac{8}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 8\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{2}{3}} - \frac{8}{1-\frac{1}{3}} = \frac{1}{1/3} - \frac{8}{2/3} = 3 - 12 = -9$

Score: /4

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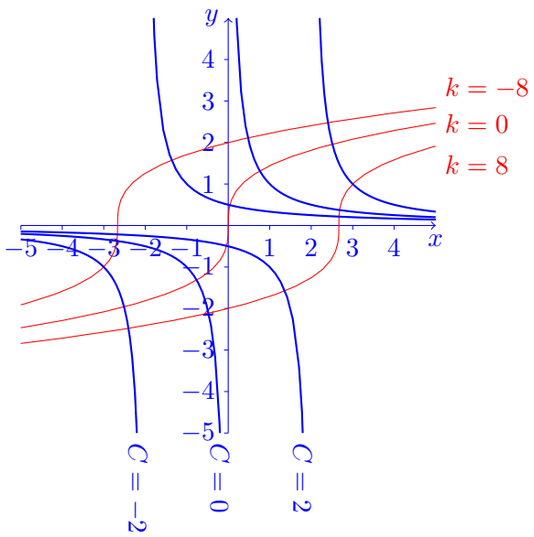
Calculators permitted in this part.

Problem 5: Find the orthogonal trajectories of the families of curves where k is a parameter. Sketch at least three members of each family on the same coordinate system. Label each curve with its parameter value.

$$3x - y^3 = k$$

Taking derivative, $3 - 3y^2 \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{1}{y^2}$. The orthogonal trajectories are then given by $\frac{dy}{dx} = -y^2$, so $\int -\frac{1}{y^2}\, dy = \int 1\, dx$, so $\frac{1}{y} = x + C$, so $y = \frac{1}{x+C}$.



Score: /6

Problem 6: Determine the sum of the series: $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$.

Rewrite as $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ and break ln into a difference to use the technique of telescoping sum. Thus

$$\sum_{n=1}^{m} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{m} (\ln(n+1) - \ln(n)) = \ln(m+1)$$

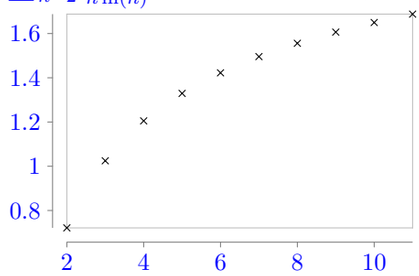
Clearly, as $m \to \infty$, $\ln(m+1) = \infty$.

Score: /3

Problem 7: Determine whether the series converges absolutely, conditionally, or not at all. Draw the first 5 partial sums on the coordinate plane for each problem.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Substituting $u = \ln(x)$ in $\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ yields $\int_{\ln(2)}^{\infty} \frac{1}{u}\,du = \ln(u)\big|_{\ln(2)}^{\infty} = \infty$, so the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges by Integral Test.

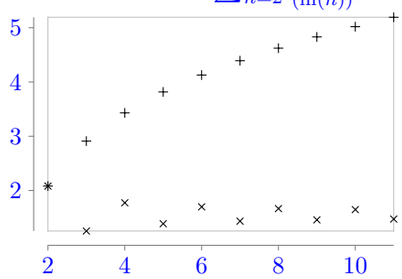


Score: /3

b. $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{(\ln(n))^2}$

Since $\cos(n\pi)$ alternates between 1 and -1, and since $\frac{\cos(n\pi)}{(\ln(n))^2} \to 0$ as $n \to \infty$, the series converges.

However, since $\ln(n) < n$ for all n, $\frac{1}{(\ln(n))^2} > \frac{1}{n\ln(n)}$, it follows from Comparison Test and Part a that $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$ diverges. Thus $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{(\ln(n))^2}$ converges conditionally.



Score: /3

Problem 8: A 100 L tank contains 50 L of a salt-water solution, with salt concentration of 0.03 kg/L of water. At time zero, pure water is poured into the tank at a rate of 6 L/min and simultaneously, the thoroughly mixed solution is drained from the tank at a rate of 10 L/min. How long does it take to have half the initial amount of salt left in the tank? Provide your answer to the nearest minute.

Let $y(t)$ be the amount of salt (in kg) in the tank at time t (minutes). Then $y(0) = 0.03 \times 50 = 1.5$. Moreover, the amount of salt water in the tank is $50 - 4t$ (litres). Therefore the concentration of salt is $\frac{y}{50-4t}$, so $\frac{dy}{dt} = -\frac{10y}{50-4t}$, so $\int \frac{1}{y}\,dy = \int -\frac{10}{50-4t}\,dt$, so $\ln|y| = \frac{5}{2}\ln|50-4t| + C$. Since both y (the amount of salt) and $50-4t$ (the amount of salt water) are positive, $\ln(y) = \frac{5}{2}\ln(50-4t) + C = \ln\big((50-4t)^{5/2}\big) + C$, so $y(t) = e^C(50-4t)^{5/2}$.
Now $1.5 = y(0) = e^C(50)^{5/2}$, so $e^C = \frac{1.5}{50^{5/2}} \approx 0.8485 \times 10^{-4}$. Hence

$$y(t) = 0.8485 \times 10^{-4}(50-4t)^{5/2}$$

You now need to solve $y(t) = \frac{1}{2}y(0)$, so $e^C(50-4t)^{5/2} = \frac{1}{2}e^C(50)^{5/2}$, so $(50-4t)^{5/2} = \frac{1}{2}(50)^{5/2}$, so $50 - 4t = \left(\frac{1}{2}\right)^{2/5} \times 50$, so $4t = 50 - \left(\frac{1}{2}\right)^{2/5} \times 50$, so $t = \frac{25}{2} - \left(\frac{1}{2}\right)^{2/5} \times \frac{25}{2} \approx 3.027$ minutes.

Score: /6