Math 126
Summer 2015
Dr. Lily Yen

# Test 1

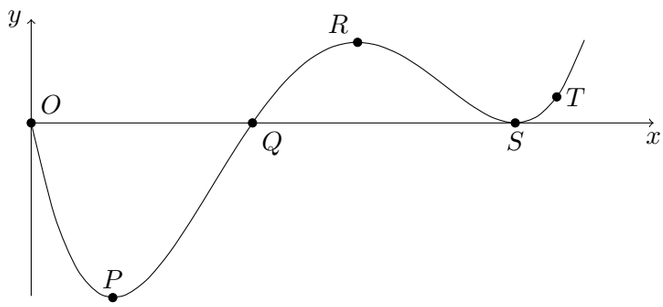
Show all your work

Name: _____

Score: ___/50

**No Calculator permitted** in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

**Problem 1**: Let $A(x) = \int_0^x f(t)\,dt$, where $f$ is the function graphed below (and $x$ is in the interval shown).




a. $A(x)$ has a local maximum at the $x$-coordinates of (circle all that apply):

$O$ $P$ $Q$ $R$ $S$ $T$

b. $A(x)$ has a local minimum at the $x$-coordinates of (circle all that apply):

$O$ $P$ $Q$ $R$ $S$ $T$

c. $A(x)$ has points of inflection at the $x$-coordinates of (circle all that apply):

$O$ $P$ $Q$ $R$ $S$ $T$

d. Which inequality is true for all $x$ in the interval over which the graph of $f$ is shown? (Circle all that apply.)

$A(x) \le 0$ $A(x) \ge 0$ neither both

e. $A(x)$ has an absolute minimum at the $x$-coordinates of (circle all that apply):

$O$ $P$ $Q$ $R$ $S$ $T$

Score: /5

**Problem 2**: Describe the area represented by the following limit as an integral. Do not evaluate.

$$\lim_{N\to\infty} \frac{\pi}{2N} \sum_{j=1}^{N} \cos\left(\frac{\pi}{3} - \frac{\pi}{4N} + \frac{j\pi}{2N}\right)$$

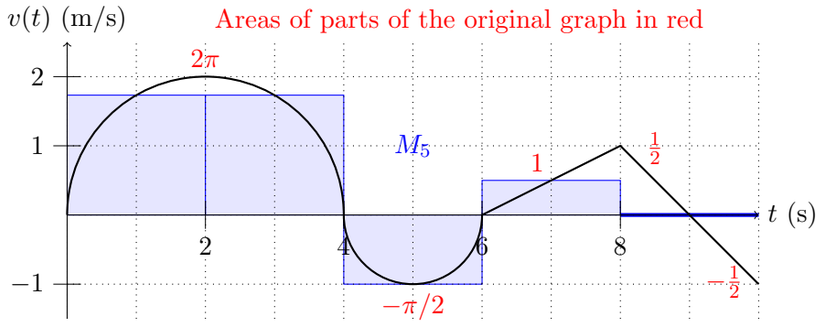
This is the middle-point Riemann sum for

$$\int_0^{\pi/2} \cos(\tfrac{\pi}{3} + x)\,dx = \frac{1-\sqrt{3}}{2}$$

That is, the net area between the $x$-axis and $y = \cos(\frac{\pi}{3} + x)$ for $0 \le x \le \frac{\pi}{2}$.

Score: /3

**Problem 3**: A particle begins at the origin at time $t = 0$ and moves with velocity $v(t)$ as shown below. (The graph consists of semicircles and line segments.)




a. Find the distance travelled by the particle after 6 seconds. $2.5\pi$ metres

b. Find the displacement of the particle after 8 seconds. $1.5\pi + 1$ metres

c. Find the time the particle is farthest from the origin. 4 seconds

d. Directly on the graph, draw in and shade the approximating rectangles representing $M_5$.

Score: /5

**Problem 4**: Evaluate the following integrals exactly using the Fundamental Theorem of Calculus.

a. $\dfrac{d}{dt}\displaystyle\int_{100}^{t} \sec(5x-9)\,dx$

$\sec(5t-9)$

b. $\displaystyle\int \frac{d}{dx}(23x^2 - 8e^x + \cos(x^2))\,dx$

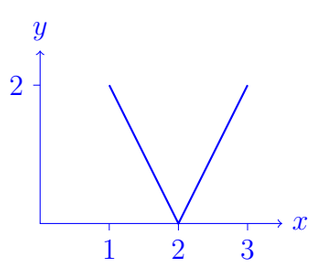
$23x^2 - 8e^x + \cos(x^2) + C$

c. $\dfrac{d}{du}\displaystyle\int_{-u}^{\sqrt{u}} \tan\left(\frac{1}{1+x^2}\right)dx$

Let $f$ be the function defined by $f(t) = \int_0^t \tan(\frac{1}{1+x^2})\,dx$. Then $f'(t) = \tan(\frac{1}{1+t^2})$ and the integral you have to find is
$\frac{d}{du}(f(\sqrt{u}) - f(-u)) = f'(\sqrt{u}) \cdot \frac{1}{2\sqrt{u}} - f'(-u) \cdot (-1) = \tan(\frac{1}{1+u}) \cdot \frac{1}{2\sqrt{u}} + \tan(\frac{1}{1+u^2})$.

Score: /4

**Problem 5**: Evaluate exactly $\int_1^3 |2x-4|\,dx$. Sketch the graph.




The area under the graph is clearly 2.

Score: /2

**Problem 6**: Find an expression of a function $f$ and a corresponding interval $[a, b]$ such that

$$\left| \int_a^b f(x)\,dx \right| = \frac{1}{2} \quad \text{and} \quad \int_a^b |f(x)|\,dx = \frac{3}{2}.$$

For example $f(x) = x$ on $[-1, \sqrt{2}]$.

Another: $f(x) = \begin{cases} x - 1, & x \le 2 \\ -x + 3, & x > 2 \end{cases}$ on $[0, 3]$.

Score: /3

**Problem 7**: Integrate the following analytically.

a. $\displaystyle\int \frac{2x^2 e^x - 3\sqrt{x} + 6}{x^2}\,dx$

$\int \frac{2x^2e^x - 3\sqrt{x}+6}{x^2}\,dx = \int 2e^x - 3x^{-3/2} + 6x^{-2}\,dx = 2e^x + 6x^{-1/2} - 6x^{-1} + C$

Score: /3

b. $\displaystyle\int (x+1)(x^2+2x)^{3/4}\,dx$

If $u = x^2 + 2x$, then $\frac{du}{dx} = 2x + 2 = 2(x+1)$, so $dx = \frac{1}{2(x+1)}\,du$. Thus
$\int (x+1)(x^2+2x)^{3/4}\,dx = \int (x+1)u^{3/4}\frac{1}{2(x+1)}\,du = \frac{1}{2}\int u^{3/4}\,du = \frac{1}{2}\cdot\frac{4}{7}u^{7/4} + C =$
$\frac{2}{7}(x^2+2x)^{7/4} + C$

Score: /2

c. $\displaystyle\int \frac{\tan(\ln(x))}{x}\,dx$

If $u = \ln(x)$, then $\frac{du}{dx} = \frac{1}{x}$, so $dx = x\,du$. Thus
$\int \frac{\tan(\ln(x))}{x}\,dx = \int \frac{\tan(u)}{x}x\,du = \int \tan(u)\,du = -\ln|\cos(u)| + C = -\ln|\cos(\ln(x))| + C$

Score: /3

Math 126
Summer 2015
Dr. Lily Yen

# Test 1

Show all your work

Name: _____

**Calculators permitted** from here on.

**Problem 8**: Express the following integral as a limit of sums.

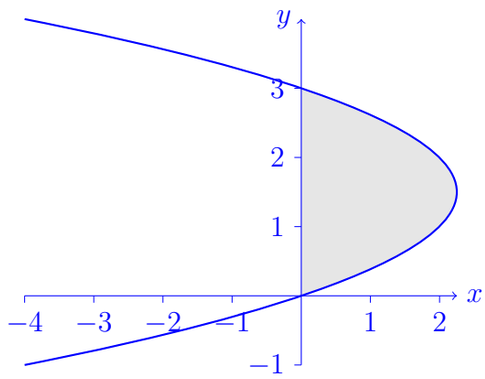
$$\int_1^4 e^x \sin(x)\,dx$$

Using the left Riemann sum,

$$\lim_{n\to\infty} \frac{3}{n} \sum_{j=0}^{n-1} e^{1+3j/n} \sin(1+\tfrac{3j}{n})$$

Score: /3

**Problem 9**: Graph $x = 3y - y^2$ and find the exact area bounded by the curve and $y$-axis.




The area is $\int_0^3 3y - y^2\,dy = \frac{3}{2}y^2 - \frac{1}{3}y^3\Big|_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$.

Score: /4

**Problem 10**: Evaluate the following integral exactly.

$$\int_{-3}^{10} \frac{1}{4+25x^2}\,dx$$

If $u = \frac{5}{2}x$, then $\frac{du}{dx} = \frac{5}{2}$, so $dx = \frac{2}{5}\,du$. Thus
$\int_{-3}^{10} \frac{1}{4+25x^2}\,dx = \int_{x=-3}^{x=10} \frac{1}{4+4u^2} \cdot \frac{2}{5}\,du = \int_{-15/2}^{25} \frac{1}{4(1+u^2)} \cdot 25\,du = \frac{1}{10}\int_{-15/2}^{25} \frac{1}{1+u^2},du =$
$\frac{1}{10}\arctan(u)\Big|_{-15/2}^{25} = \frac{1}{10}\arctan(25) - \frac{1}{10}\arctan(-\frac{15}{2}) = \frac{1}{10}\arctan(25) + \frac{1}{10}\arctan(\frac{15}{2})$

Score: /4

**Problem 11**: The rate (in litres per minute) at which water drains from a tank is recorded at half-minute intervals. Compute the average left- and right-endpoint approximations to estimate the total amount of water drained during the first 3 minutes.

| $t$ (min) | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $r$ (L/min) | 50 | 48 | 46 | 44 | 42 | 40 | 38 |

The left-sum is $\frac{1}{2}(50+48+46+44+42+40) = 135$ and the right-sum is $\frac{1}{2}(48+46+44+42+40+38) = 129$, so the average is 132 litres.

Score: /5

**Problem 12**: Wind engineers have found that wind speed $v$ (in metres per second) at a given location follows a Rayleigh distribution of the type

$$W(v) = \frac{1}{32} v e^{-v^2/64}.$$

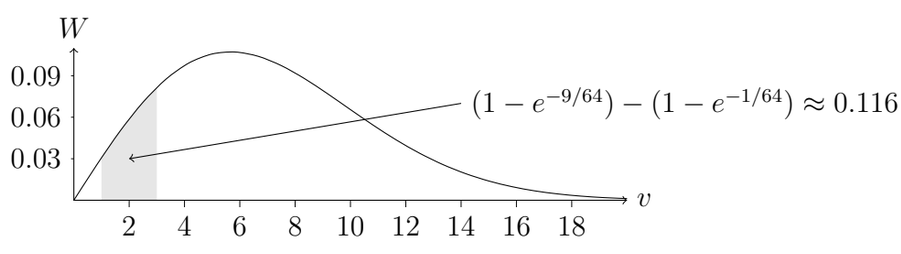
This means that at a given moment in time, the probability that $v$ lies between $a$ and $b$ is equal to the shaded area under $W$ in the interval $[a, b]$.

a. Find analytically the probability that $v \in [0, b]$.

You have to find $\int_0^b W(v)\,dv$. Now, if $u = \frac{-v^2}{64}$, then $\frac{du}{dv} = -\frac{v}{32}$, so $dv = -\frac{32}{v}, du$. Thus $\int_0^b W(v)\,dv = \int_0^b \frac{1}{32} v e^{-v^2/64}\,dv = \int_{v=0}^{v=b} \frac{1}{32} v e^u (-\frac{32}{v})\,du = -\int_{v=0}^{v=b} e^u\,du = -e^u\Big|_{v=0}^{v=b} = -e^{-v^2/64}\Big|_0^b = -e^{-b^2/64} - (-1) = 1 - e^{-b^2/64}$.

Score: /3

b. Compute the probability that $v \in [1, 3]$ to 6 decimal place accuracy. Show the shaded region on the graph of $W$.




Score: /1