Math 126
Summer 2014
Dr. Lily Yen

Test 3

Show all your work

Name: _____

Score: ___/34

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Integrate the following.

a. $\int (x-3)e^{-12x}\,dx$

Integrating by parts,

$$\begin{aligned}\int (x-3)e^{-12x}\,dx &= (x-3)\tfrac{-1}{12}e^{-12x} - \int \tfrac{-1}{12}e^{-12x}\,dx \\ &= -\tfrac{1}{12}(x-3)e^{-12x} - \tfrac{1}{144}e^{-12x} + C \\ &= \tfrac{35}{144}e^{-12x} - \tfrac{1}{12}xe^{-12x} + C\end{aligned}$$

Score: /3

b. $\int \dfrac{6x+4}{x^2-1}\,dx$

Using partial fractions, if $\frac{6x+4}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$, then $6x+4 = (x+1)A + (x-1)B$. When $x=1$ this yields that $10 = 2A$, so $A = 5$. When $x = -1$, you get that $-2 = -2B$, so $B = 1$. Therefore

$$\begin{aligned}\int \frac{6x+4}{x^2-1}\,dx &= \int \frac{5}{x-1} + \frac{1}{x+1}\,dx \\ &= 5\ln|x-1| + \ln|x+1| + C\end{aligned}$$

Score: /3

c. $\int e^x \cos(x)\,dx$

Using integration by parts twice,
$\int e^x\cos(x)\,dx = e^x\cos(x) + \int e^x\sin(x)\,dx = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,dx$.
Therefore $2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$, so

$$\int e^x\cos(x)\,dx = \tfrac{1}{2}e^x(\cos(x)+\sin(x)) + C.$$

Score: /4

Problem 2: Does the following integral converge? If so, evaluate it. If not, show where convergence fails.

$$\int_0^{\pi/2} \cot(\theta)\, d\theta$$

First note that the integrand, $\cot(\theta)$, is defined for all θ in the half-open interval $(0, \frac{\pi}{2}]$. Then substitute $u = \sin(\theta)$. Then $\frac{du}{d\theta} = \cos(\theta)$, so $d\theta = \frac{1}{\cos(\theta)}$, and

$$\begin{aligned}\int_a^b \cot(\theta)\, d\theta = \int_a^b \frac{\cos(\theta)}{\sin(\theta)}\, d\theta &= \int_{\sin(a)}^{\sin(b)} \frac{1}{u}\, du \\ &= \ln|u|\Big|_{\sin(a)}^{\sin(b)} = \ln|\sin(b)| - \ln|\sin(b)|\end{aligned}$$

Now $\lim_{b\to\pi/2} \ln|\sin(b)| = \ln(1) = 0$, but $\lim_{a\to 0+} \ln|\sin(b)| = \infty$.
Therefore $\int_0^{\pi/2} \cot(\theta)\, d\theta$ is divergent, $\int_0^{\pi/2} \cot(\theta)\, d\theta = +\infty$.

Score: /4

Problem 3: Evaluate analytically the arc length of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}, \quad a > 0.$$

If $x = a\cos^3(t)$ and $y = a\sin^3(t)$, then
$x^{2/3} + y^{2/3} = a^{2/3}\cos^2(t) + a^{2/3}\sin^2(t) = a^{2/3}(\cos^2(t) + \sin^2(t)) = a^{2/3}$, so the curve can be parameterized as $x = a\cos^3(t)$, $y = a\sin^3(t)$, $0 \le t \le 2\pi$. Therefore the arc length is

$$\begin{aligned}\int_0^{2\pi} \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt &= \int_0^{2\pi} \sqrt{\left(-3a\cos^2(t)\sin(t)\right)^2 + \left(3a\sin^2(t)\cos(t)\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{9a^2\cos^4(t)\sin^2(t) + 9a^2\sin^4(t)\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 3a\sqrt{\cos^2(t)\sin^2(t)(\cos^2(t) + \sin^2(t))}\, dt \\ = \int_0^{2\pi} 3a\sqrt{\cos^2(t)\sin^2(t)}\, dt &= 12a\int_0^{\pi/2} \sqrt{\cos^2(t)\sin^2(t)}\, dt = 12a\int_0^{\pi/2} \cos(t)\sin(t)\, dt \\ &= 6a\int_0^{\pi/2} 2\cos(t)\sin(t)\, dt = 6a\int_0^{\pi/2} \sin(2t)\, dt = -3a\cos(2t)\Big|_0^{\pi/2} = 6a.\end{aligned}$$

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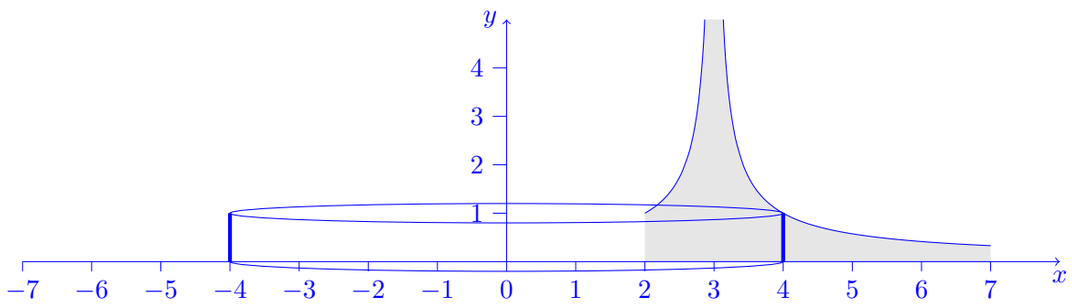
Calculators permitted in this part.

Problem 4: Find the volume of the solid S obtained by rotating the region R bounded by the function $f(x) = \dfrac{1}{(x-3)^{4/5}}$ and the x-axis on the interval $[2, 7]$ about the y-axis. Sketch the region R and a cross-sectional volume element as part of your solution.



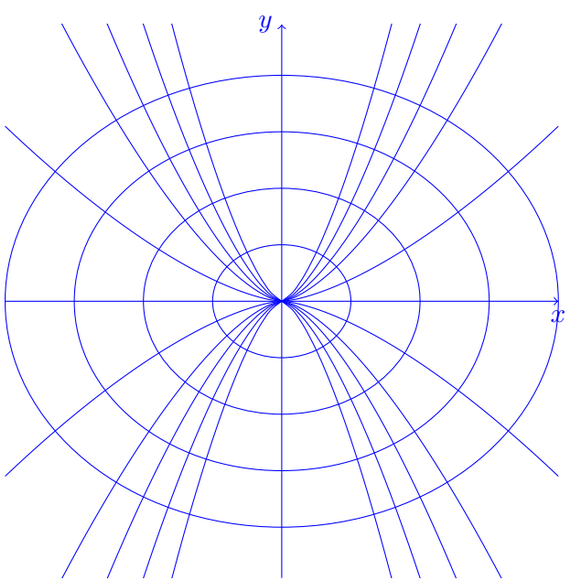
By the method of cylindrical shells, the volume is
$\int_2^7 2\pi x(x-3)^{-4/5}\,dx = \int_{-1}^4 2\pi(u+3)u^{-4/5}\,du = 2\pi\int_{-1}^4 u^{1/5}+3u^{-4/5}\,du =$
$2\pi \lim_{a\to 0-}\int_{-1}^a u^{1/5}+3u^{-4/5}\,du + 2\pi\lim_{b\to 0+}\int_b^4 u^{1/5}+3u^{-4/5}\,du =$
$2\pi \lim_{a\to 0-} \frac{5}{6}u^{6/5}+15u^{1/5}\Big|_{-1}^{a} + 2\pi\lim_{b\to 0+}\frac{5}{6}u^{6/5}+15u^{1/5}\Big|_{b}^{4} =$
$2\pi(0-\frac{5}{6}+15) + 2\pi(\frac{5}{6}\cdot 4^{6/5}+15\cdot 4^{1/5}-0) = \frac{85}{3}\pi + \frac{110}{3}\pi\cdot 4^{1/5} \approx 241.0$

Score: /6

Problem 5: Find the orthogonal trajectories of the families of curves where k is a parameter and sketch at least three members of each family. Label each curve with its parameter value.

$$kx^3 - y^2 = 0$$

Note that $k = \frac{y^2}{x^3}$. Now, differentiating the given equation with respect to x yields that $3kx^2 - 2yy' = 0$, so $y' = \frac{3kx^2}{2y} = \frac{3\frac{y^2}{x^3}x^2}{2y} = \frac{3y}{2x}$.
Therefore the orthogonal trajectories satisfy the separable differential equation $\frac{dy}{dx} = -\frac{2x}{3y}$, so $\int 3y\,dy = -\int 2x\,dx$, so $\frac{3}{2}y^2 = -x^2 + C$, so $\frac{x^2}{C} + \frac{y^2}{2C/3} = 1$, which is a family of ellipses.



$k = \frac{1}{10}, \frac{1}{2}, 1, 2, 4$

$C = 1, \sqrt{2}, \sqrt{3}, 2$

Score: /4

Problem 6: Capilano Lake is stocked with 2000 rainbow trout, and after 1 year, the population has grown to 4500. Assuming logistic growth with a carrying capacity of 20 000, find the growth constant k (including units) and determine when the population will increase to 10 000.

If a population, P, follows logistic growth with capacity M, then $\frac{dP}{dt} = kP(M-P)$, so $\int \frac{1}{P(M-P)}\,dP = \int k\,dt$. Using partial fractions, if $\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$, then $1 = A(M-P) + BP$, so if $P = 0$, $1 = AM$, so $A = \frac{1}{M}$; and if $P = M$, $1 = BM$, so $B = \frac{1}{M}$. Hence $\int \frac{1}{MP} + \frac{1}{M(M-P)}\,dP = \int k\,dt$, so $\frac{1}{M}\ln|MP| - \frac{1}{M}\ln|M(M-P)| = kt + C$, so $\ln\left|\frac{P}{M-P}\right| = M(kt+C)$, so $\frac{P}{M-P} = \pm e^{M(kt+C)} = \frac{1}{a}e^{bt}$, where $a = \pm e^{-MC}$ and $b = Mk$ are suitable constants. Thus $\frac{1}{\frac{M}{P}-1} = \frac{1}{a}e^{bt}$, so $\frac{M}{P} - 1 = ae^{-bt}$, so $\frac{M}{P} = 1 + ae^{-bt}$, so

$$P = \frac{M}{1 + ae^{-bt}}.$$

In the case of the trout, $M = 20\,000$, $P(0) = 2000$, and $P(1) = 4500$. Thus $2000 = \frac{20\,000}{1+a}$, so $1 + a = 10$, so $a = 9$; and $4500 = \frac{20\,000}{1+ae^{-b}}$, so $1 + 9e^{-b} = \frac{20\,000}{4500} = \frac{40}{9}$, so $b = \ln(\frac{81}{31})$. You need to find when $P(t) = 10\,000$, so $10\,000 = \frac{20\,000}{1+9e^{-bt}}$, so $\frac{1}{2} = \frac{1}{1+9e^{-bt}}$, so $2 = 1 + 9e^{-bt}$, so $\frac{1}{9} = e^{-bt}$, so $bt = \ln(9)$, so $t = \frac{\ln(9)}{b} = \frac{\ln(9)}{\ln(81/31)} \approx 2.29$ years.

Score: /6