Math 126
Summer 2014
Dr. Lily Yen

Test 2

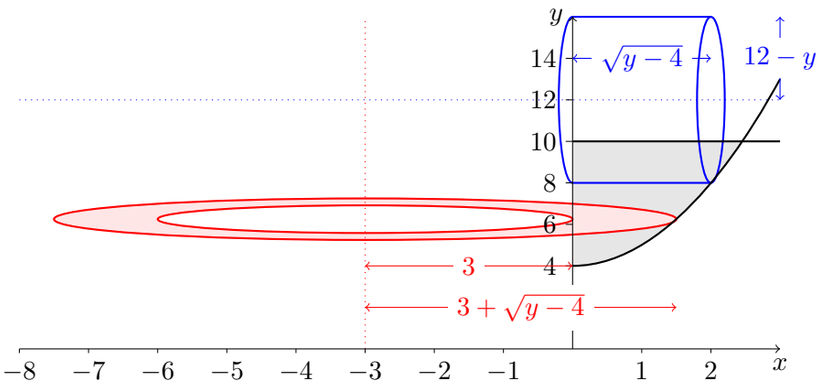
Show all your work

Name: _____

Score: ___/33

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: First draw $y = x^2 + 4$ and $y = 10$. Shade the region in the first quadrant bounded by these two graphs and the y-axis.



Use integrals to express the following. DO NOT EVALUATE YOUR INTEGRALS. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

If $x^2 + 4 = 10$ and $x \geq 0$, then $x = \sqrt{6}$. Therefore the area is

$$\int_0^{\sqrt{6}} 10 - (x^2 + 4)\, dx = \int_0^{\sqrt{6}} 6 - x^2\, dx \qquad = 4\sqrt{6}$$

Score: /2

b. The volume obtained if the region is rotated about $x = -3$, using the method of slicing (washers/disks).

$$\pi \int_4^{10} (\sqrt{y-4} + 3)^2 - 3^2\, dy$$

$= \pi(\frac{1}{2}y^2 - 4y + 4(y-4)^{3/2})\big|_4^{10} = \pi(18 + 24\sqrt{6})$

Score: /2

c. The volume obtained if the region is rotated about $y = 12$, using the method of cylindrical shells.

$$2\pi \int_4^{10} (12 - y)\sqrt{y-4}\, dy$$

$= \pi(-\frac{2}{5}(y-4)^{5/2} + \frac{16}{3}(y-4)^{3/2})\big|_4^{10} = \frac{176}{5}\pi\sqrt{6}$

Score: /2

d. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the x-axis are squares.

$$\int_0^{\sqrt{6}} (10 - (x^2 + 4))^2\, dx = \int_0^{\sqrt{6}} (6 - x^2)^2\, dx$$

$= \frac{1}{5}x^5 - 4x^3 + 36x\big|_0^{\sqrt{6}} = \frac{96}{5}\sqrt{6}$

Score: /2

e. The volume of a solid with the shaded region as its base, and cross-sections perpendicular to the y-axis are equilateral triangles.

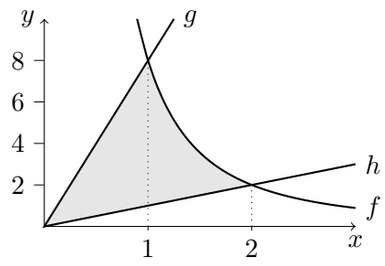
$$\int_4^{10} \frac{\sqrt{3}}{4}(\sqrt{y-4})^2\, dy = \int_4^{10} \frac{\sqrt{3}}{4}(y-4)\, dy$$

Score: /2

Problem 2: First, draw the following graphs on the coordinate system, then shade the region bounded by them:

$$f(x) = \frac{8}{x^2}, \quad g(x) = 8x, \quad \text{and} \quad h(x) = x.$$

Set up the integral for the area of the shaded region, then evaluate the integral.

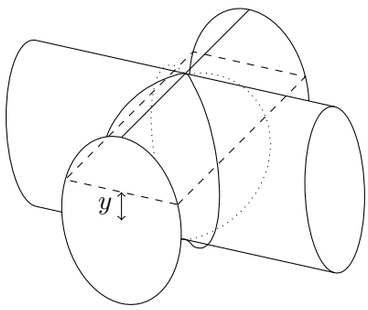


If $f(x) = g(x)$, then $\frac{8}{x^2} = 8x$, so $1 = x^3$, so $x = 1$. If $f(x) = h(x)$, then $\frac{8}{x^2} = x$, so $8 = x^3$, so $x = 2$. If $g(x) = h(x)$, then $8x = x$, so $x = 0$. Thus the intersection points are located as indicated in the graph.
The area is then

$$\begin{aligned}\int_0^1 g(x) - h(x)\,dx + \int_1^2 f(x) - h(x)\,dx &= \int_0^1 8x - x\,dx + \int_1^2 \frac{8}{x^2} - x\,dx \\ &= \int_0^1 7x\,dx + \int_1^2 8x^{-2} - x\,dx \\ &= \tfrac{7}{2}x^2\Big|_0^1 + (-8)x^{-1} - \tfrac{1}{2}x^2\Big|_1^2 \\ &= \tfrac{7}{2} - \tfrac{8}{2} - 2 + 8 + \tfrac{1}{2} \\ &= 6\end{aligned}$$

Score: /4

Problem 3: The solid S is the intersection of two cylinders of radius r whose axes are perpendicular. Find the volume of S as a function of r.



The dashed rectangle in the figure would have width $2\sqrt{r^2 - y^2}$. The shown dashed rectangle intersects with a similar rectangle inside the other cylinder creating a square with side length $2\sqrt{r^2 - y^2}$. The solid S is formed by stacking all these squares, so the volume of S is

$$\int_{-r}^{r} 4(r^2 - y^2)\,dy = 4r^2y - \tfrac{4}{3}y^3\Big|_{-r}^{r} = \tfrac{16}{3}r^3$$

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Calculators permitted from here on.

Problem 4: The acceleration of a Nerf bullet is $a(t) = 60t - 4t^3$ meters per second squared after it is fired. Compute the average acceleration and the average velocity over the time interval $[2, 6]$, assuming that the particle's initial velocity is zero (that is, $v(0) = 0$).

The average acceleration is $\frac{1}{6-2}\int_2^6 a(t)\,dt = \frac{1}{4}\int_2^6 60t - 4t^3\,dt = \frac{1}{4}(30t^2 - t^4)\big|_2^6 = -80\,\mathrm{m/s^2}$.
The velocity of the bullet is $v(t) = \int_0^t a(u)\,du = \int_0^t 60u - 4u^3\,du = 30u^2 - u^4\big|_0^t = 30t^2 - t^4$.
Therefore the average velocity is
$\frac{1}{6-2}\int_2^6 v(t)\,dt = \frac{1}{4}\int_2^6 30t^2 - t^4\,dt = \frac{1}{4}(10t^3 - \frac{1}{5}t^5)\big|_2^6 = 132.8\,\mathrm{m/s}$

Score: /4

Problem 5: If 5 J of work are needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond equilibrium?

Work is force times distance, $E = F \times d$, and the force of a spring is $F = k \times d$, where k is the spring constant. Thus the work required to stretch a spring a distance of d beyond its equilibrium is $E = \int_0^d F\,dx = \int_0^d kx\,dx = \frac{1}{2}kx^2\big|_0^d = \frac{1}{2}kd^2$.
In standard units, $10\,\mathrm{cm} = 0.10\,\mathrm{m}$, so $5\,\mathrm{J} = \frac{1}{2}k(0.10\,\mathrm{m})^2$, so
$k = \frac{2\times 5}{0.10^2} = 1000\,\mathrm{J/m^2} = 1000\,\mathrm{N/m}$.
Therefore the work required to stretch the spring $15\,\mathrm{cm} = 0.15\,\mathrm{m}$ is
$E = \frac{1}{2} \times 1000\,\mathrm{N/m} \times (0.15\,\mathrm{m})^2 = 11.25\,\mathrm{J}$.

Score: /4

Problem 6: Calculate the work required to lift a 10 m chain over the side of a building. Assume that the chain has a density of 8 kg/m.

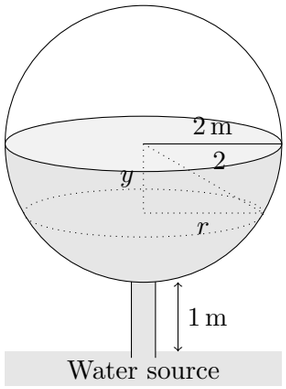
The work required to list a mass m the distance x is mgx, where g is the acceleration due to gravity.
The short piece of chain with length dy at height y has mass $8\,dy$ and needs to be lifted a distance of $10 - y$. Thus the work required to lift the chain is

$$\int_0^{10} 8g(10-y)\,dy = 8g(10y - \tfrac{1}{2}y^2)\big|_0^1 0 = 400g = 3920\,\text{J}$$

using $9.8\,\text{m/s}^2$ for g.

Score: /3

Problem 7: Water is pumped into a spherical tank of radius 2 m from a source located 1 m below a hole at the bottom. The density of water is $1000\,\text{kg/m}^3$. Calculate the work $F(h)$ required to fill the tank to level h metres in the sphere.



The cross section of the water at a distance y from the centre of the tank has radius $r = \sqrt{2^2 - y^2}$. Hence the slab of water of thickness dy has volume $\pi r^2\,dy = \pi(4-y^2)\,dy$ and mass $1000\pi(4-y^2)\,dy$. Since the slab has to be lifted a distance of $1 + 2 + y$, the work required for the slab is $1000\pi(4-y^2)(3+y)g\,dy = 1000\pi(12 + 4y - 3y^2 - y^3)g\,dy$, where $g = 9.8\,\text{m/s}^2$ is the acceleration due to gravity. Thus

$$\begin{aligned} F(h) &= \int_{-2}^{h-2} 1000\pi(12 + 4y - 3y^2 - y^3)g\,dy \\ &= 1000\pi(12y + 2y^2 - y^3 - \tfrac{1}{4}y^4)g\big|_{-2}^{h-2} \\ &= 1000\pi(-12 + 2h^2 + h^3 - \tfrac{1}{4}h^4 - (-12))g \\ &= 1000\pi(2h^2 + h^3 - \tfrac{1}{4}h^4)g \end{aligned}$$

(in joule, J).

Score: /4