Math 126
Spring 2017
Dr. Lily Yen

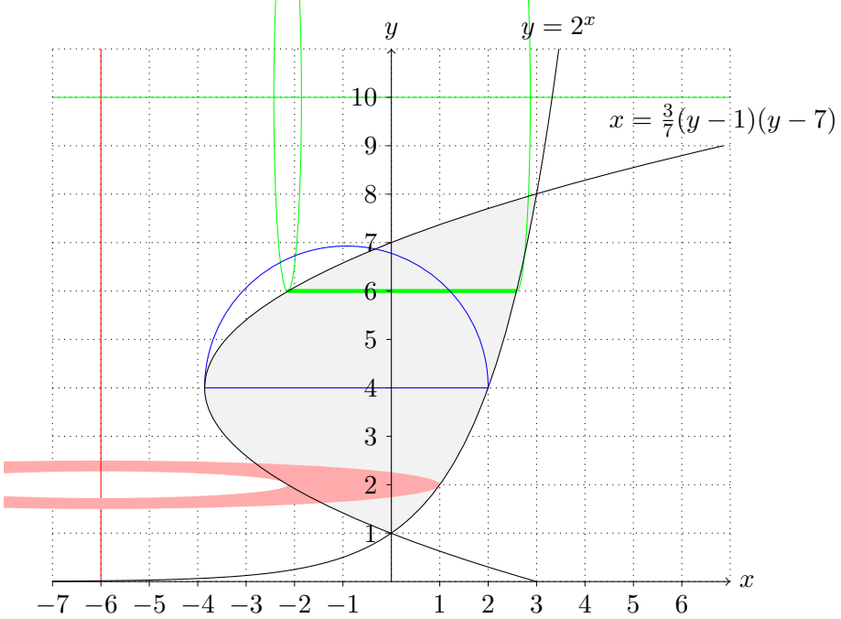
Test 2

Show all your work

Name: _____
Number: _____
Signature: _____
Score: ___/40

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Draw $y = 2^x$ and $x = \frac{3}{7}(y-1)(y-7)$ on the grid. One of the intersections of these two curves is $(3, 8)$. Shade the region bounded by these two curves.



Use integrals to express the following. DO NOT EVALUATE YOUR INTEGRALS. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

$$\int_1^8 \log_2(y) - \tfrac{3}{7}(y-1)(y-7)\,dy = 38 - \frac{7}{\ln(2)} \approx 27.90$$

Score: /2

b. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the y-axis are semi-circles.

$$\frac{\pi}{8}\int_1^8 \left(\log_2(y) - \tfrac{3}{7}(y-1)(y-7)\right)^2\,dy \approx 132.7$$

Score: /2

c. The volume of the solid obtained by rotating the region around $x = -6$.

$$\pi\int_1^8 (6+\log_2(y))^2 - (6+\tfrac{3}{7}(y-1)(y-7))^2\,dy \approx 993.7$$

Score: /2

d. The volume of the solid obtained by rotating the region around $y = 10$.

$$2\pi\int_1^8 (10-y)(\log_2(y) - \tfrac{3}{7}(y-1)(y-7))\,dy \approx 973.8$$

Score: /2

Problem 2: Evaluate the following integrals analytically.

a. $\displaystyle\int x^3 e^{x^2}\,dx$

If $u = x^2$, then $\frac{du}{dx} = 2x$, so $dx = \frac{1}{2x}\,du$. Therefore

$$\int x^3 e^{x^2}\,dx = \int x^3 e^u \frac{1}{2x}\,du = \tfrac{1}{2}\int x^2 e^u\,du = \tfrac{1}{2}\int u e^u\,du.$$

Using integration by parts, this equals

$$\tfrac{1}{2}\left(ue^u - \int e^u\,du\right) = \tfrac{1}{2}\left(ue^u - e^u\right) + C = \tfrac{1}{2}(x^2-1)e^{x^2} + C$$

Score: /3

b. $\displaystyle\int e^{-x}\cos(2x)\,dx$

Using integration by parts twice,

$$\begin{aligned}\int e^{-x}\cos(2x)\,dx &= -e^{-x}\cos(2x) - 2\int e^{-x}\sin(2x)\,dx \\ &= -e^{-x}\cos(2x) - 2\left(-e^{-x}\sin(2x) + 2\int e^{-x}\cos(2x)\,dx\right) \\ &= -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) - 4\int e^{-x}\cos(2x)\,dx,\end{aligned}$$

so $5\int e^{-x}\cos(2x)\,dx = -e^{-x}\cos(2x) + 2e^{-x}\sin(2x)$, so

$$\int e^{-x}\cos(2x)\,dx = \tfrac{1}{5}\left(2\sin(2x) - \cos(2x)\right)e^{-x}$$

Score: /4

c. $\displaystyle\int \frac{2x^2 - x - 1}{(x-2)(x^2+1)}\,dx$

By the method of partial fractions, $\frac{2x^2-x-1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$, so $2x^2 - x - 1 = A(x^2+1) + (Bx+C)(x-2)$. If $x = 2$, this yields $5 = 5A$, so $A = 1$. If $x = 0$ (and $A = 1$), the equation yields that $-1 = 1 - 2C$, so $C = 1$. Using this and, say, $x = 1$ you get that $0 = 1 - B$, so $B = 1$.
Thus

$$\begin{aligned}\int \frac{2x^2 - x - 1}{(x-2)(x^2+1)}\,dx &= \int \frac{1}{x-2} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= \ln|x-2| + \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C\end{aligned}$$

Score: /5

Math 126
Spring 2017
Dr. Lily Yen
Calculators permitted from here on.

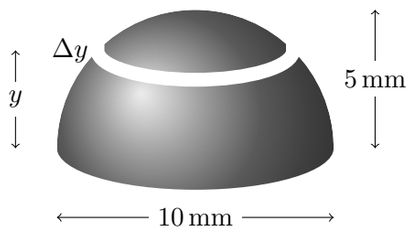
Test 2

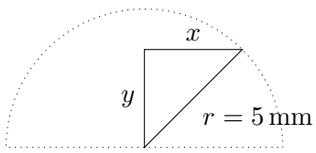
Show all your work

Name: _____

Number: _____

Problem 3: Write a Riemann sum and then a definite integral representing the volume of the hemisphere, using the slice shown. Evaluate the integral exactly.





If each slice is Δy thick, then we have $n = \frac{1}{\Delta y}$ of them. The slice at height y is a circle with radius x (see the figure above), and $x^2 + y^2 = r^2 = 25$, so each slice has volume $\pi x^2\,\Delta y = \pi(25 - y^2)\,\Delta y$. Therefore the (left) Riemann sum is

$$\lim_{\Delta y \to 0} \sum_{j=0}^{\frac{1}{\Delta y}-1} \pi(25-(j\,\Delta y)^2)\,\Delta y = \pi \int_0^5 (25-y^2)\,dy = \pi(25y-\tfrac{1}{3}y^3)\Big|_0^5 = \pi(125-\tfrac{1}{3}\cdot 125) = \frac{2}{3}\pi 125$$

Score: /4

Problem 4: Evaluate if possible; otherwise, state why the integral does not exist.

$$\int_1^3 \frac{1}{(x-2)^2}\,dx$$

The integrand is not defined at $x = 2$, so you must evaluate on both sides separately. However,

$$\int_1^2 \frac{1}{(x-2)^2}\,dx = \lim_{z\to 2^-} \int_1^z \frac{1}{(x-2)^2}\,dx = \lim_{z\to 2^-} \frac{-1}{x-2}\,dx\Big|_1^z = \lim_{z\to 2^-} \frac{-1}{z-2} - 1 = \infty,$$

so $\int_1^3 \frac{1}{(x-2)^2}\,dx$ does not exist.

Score: /4

Problem 5: Suppose that f is continuous for all real numbers and that $\int_0^\infty f(x)\,dx$ converges to a real number, c. Let a be any positive number. Decide which of the statements below are true and which are false. Give an explanation for your answer.

a. $\int_0^\infty af(x)\,dx$ converges.
Equals $a\int_0^\infty f(x)\,dx$, so true.

b. $\int_0^\infty f(x+a)\,dx$ converges.
$\int_0^\infty f(x+a)\,dx = \int_a^\infty f(x)\,dx$ so the difference between the convergent integral $\int_0^\infty f(x)\,dx$ and the desired integral $\int_a^\infty f(x)\,dx$ is $\int_0^a f(x)\,dx$ which must be convergent since a is positive. Therefore the statement is true.

c. $\int_0^\infty f(ax)\,dx$ converges.
Equals (by substitution method) $\frac{1}{a}\int_0^\infty f(x)\,dx$, so true.

d. $\int_0^\infty (a+f(x))\,dx$ converges.
$\int_0^\infty (a+f(x))\,dx = \int_0^\infty a\,dx + \int_0^\infty f(x)\,dx$ and the first integral clearly diverges, so the statement is false.

Score: /4

Problem 6: A 20 m uniform chain with a mass density of 1.5 kg/m is dangling from the roof of a building. How much work is required to pull half the chain up onto the top of the building? Use $g = 9.81\,\text{m/s}^2$. Provide your answer accurate to 2 decimal places.

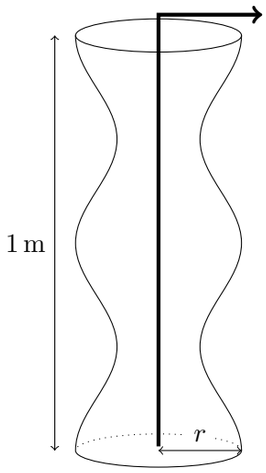
The bottom half of the chain simply need to be lifted 10 m. The mass of the bottom half of the chain is $1.5 \times 10 = 15$ kg, so lifting it requires $15g \times 10 = 1471.5000$ J.
For the top half of the chain, the short piece of length dy at distance y from the roof has mass $1.5\,dy$ and (obviously) needs to be lifted the distance y which required $1.5gy\,dy$ amount of work. In total (for the top half of the chain)

$$\int_0^{10} 1.5gy\,dy = \frac{1.5g}{2}y^2\Big|_0^{10} = 75g = 735.7500\,\text{J}$$

Grand total 2207.2500 J

Score: /4

Problem 7: A display fish tank has a circular cross section with radius r metres at a height of y metres above the ground given by $r = (3 + \cos(4\pi y))/100$. The height of the tank is 1 m. A spout is placed 0.05 m above the tank. Find the work required to pump all the water from a full tank out through the spout. Provide your answer accurate to the nearest integer. Hint: $1\,\text{m}^3$ of water has a mass of 1000 kg. (Drawing is not to scale.)



The slice at distance y metres from the bottom is a cylinder with radius r and height dy, so volume $\pi r^2\,dy$ (in m^3), so the mass of the slice is $1000\pi r^2\,dy$ (in kg). The work required to lift the slice $1.05 - y$ metres is therefore $1000\pi r^2(1.05 - y)g\,dy$ Joules.
The total work is then

$$\begin{aligned}\int_0^1 1000\pi r^2(1.05 - y)g\,dy &= \frac{1000}{100^2}\pi g\int_0^1 (3 + \cos(4\pi y))^2(1.05 - y)\,dy \\ &= \frac{\pi g}{10}\int_0^1 (3 + \cos(4\pi y))^2(1.05 - y)\,dy \\ &= \frac{209}{400}\pi g \approx 16.10\,\text{J}\end{aligned}$$

(using $g = 9.81\,\text{m/s}^2$).

Score: /4