Math 126 Fall 2025 Dr. Lily Yen	**Test 2** Show all your work	Full Name: ____________ Student Number: ____________ Signature: ____________ Score: ___/41

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Evaluate the following integrals analytically.

a. $\displaystyle\int t^3 \ln(t)\, dt$

Using integration by parts,

$$\begin{aligned}\int t^3 \ln(t)\, dt &= \frac{1}{4}t^4 \ln(t) - \int \frac{1}{4}t^4 \cdot \frac{1}{t}\, dt\\ &= \frac{1}{4}t^4 \ln(t) - \int \frac{1}{4}t^3\, dt\\ &= \frac{1}{4}t^4 \ln(t) - \frac{1}{16}t^4 + C\end{aligned}$$

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b. $\displaystyle\int x \tan^{-1}(x)\, dx$

Using integration by parts,

$$\begin{aligned}\int x\tan^{-1}(x)\, dx &= \frac{1}{2}x^2 \tan^{-1}(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x^2+1}\, dx\\ &= \frac{1}{2}x^2 \tan^{-1}(x) - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}\, dx\\ &= \frac{1}{2}x^2 \tan^{-1}(x) - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\, dx\\ &= \frac{1}{2}x^2 \tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C\end{aligned}$$

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Problem 2: Determine analytically the convergence of the following integral.

$$\int_1^\infty \frac{\cos(x)+2}{\sqrt{x}}\, dx$$

Note that $\frac{\cos(x)+2}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for all x. Therefore
$\int_1^\infty \frac{\cos(x)+2}{\sqrt{x}}\, dx \geq \int_1^\infty \frac{1}{\sqrt{x}}\, dx = \lim_{z\to\infty} \int_1^z \frac{1}{\sqrt{x}}\, dx = \lim_{z\to\infty} 2\sqrt{x}\Big|_1^z = \lim_{z\to\infty} 2\sqrt{z} - 2 = \infty$, so

$$\int_1^\infty \frac{\cos(x)+2}{\sqrt{x}}\, dx = \infty$$

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Problem 3: Evaluate if possible, otherwise state why the integral does not exist.

$$\int_0^5 \frac{1}{\sqrt[3]{x-3}}\,dx$$

The integrand is undefined at $x = 3$, so split the interval of integration:

$$\begin{aligned}\int_0^3 \frac{1}{(x-3)^{1/3}}\,dx &= \lim_{z\to 3^-} \int_0^z \frac{1}{(x-3)^{1/3}}\,dx \\ &= \lim_{z\to 3^-} \frac{3}{2}(x-3)^{2/3}\Big|_0^z \\ &= \lim_{z\to 3^-} \frac{3}{2}(z-3)^{2/3} - \frac{3}{2}(-3)^{2/3} = 0 - \frac{3}{2}3^{2/3}\end{aligned}$$

Similarly, $\int_3^5 \frac{1}{(x-3)^{1/3}}\,dx = \frac{3}{2}2^{2/3}$. Hence

$$\int_0^5 \frac{1}{(x-3)^{1/3}}\,dx = \frac{3}{2}2^{2/3} - \frac{3}{2}3^{2/3}$$

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Problem 4: Find the indefinite integral: $\displaystyle\int e^{-x}\cos(2x)\,dx$.

Using integration by parts twice,

$$\begin{aligned}\int e^{-x}\cos(2x)\,dx &= -e^{-x}\cos(2x) - \int -e^{-x}(-2)\sin(2x)\,dx \\ &= -e^{-x}\cos(2x) - 2\int e^{-x}\sin(2x)\,dx \\ &= -e^{-x}\cos(2x) - 2\left(-e^{-x}\sin(2x) - \int -e^{-x}2\cos(2x)\,dx\right) \\ &= -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) - 4\int e^{-x}\cos(2x)\,dx + C\end{aligned}$$

Therefore $5\int e^{-x}\cos(2x)\,dx = -e^{-x}\cos(2x) + 2e^{-x}\sin(2x) + C$, so

$$\int e^{-x}\cos(2x)\,dx = -\frac{1}{5}e^{-x}\cos(2x) + \frac{2}{5}e^{-x}\sin(2x) + C_2$$

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Calculators permitted from here on.

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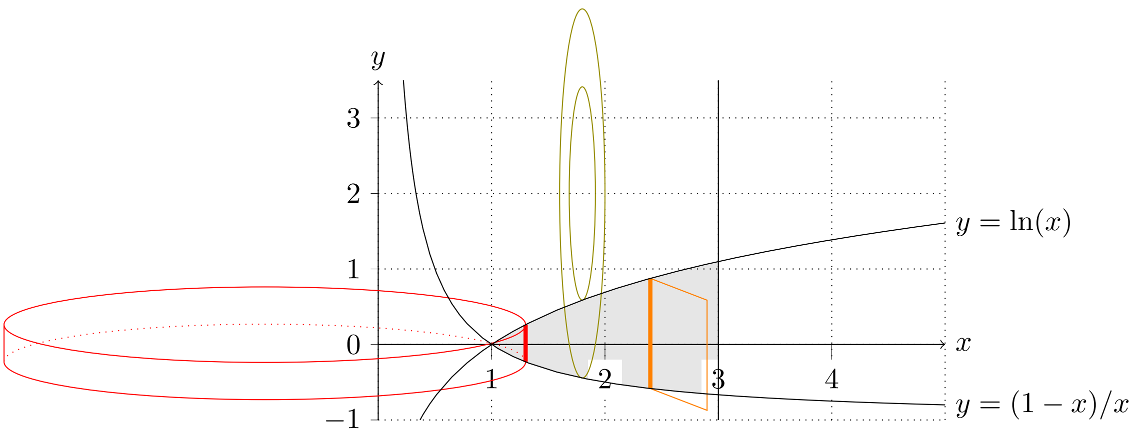
Problem 5: Draw $y_1 = \dfrac{1-x}{x}$ and $y_2 = \ln(x)$ on the grid. Shade the region bounded by these two graphs and the line $x = 3$.



Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid.

a. The area of the region enclosed by y_1, y_2, and $x = 3$.

$$\int_1^3 \ln(x) - \frac{1-x}{x}\,dx = x\ln(x) - (-x + \ln(x))\Big|_1^3 = x\ln(x) - \ln(x)\Big|_1^3 = 2\ln(3)$$

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b. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the x-axis are squares .

$$\int_1^3 \left(\ln(x) - \frac{1-x}{x}\right)^2 dx = x(\ln x)^2 - (\ln x)^2 + x - 2\ln(x) - \frac{1}{x}\Big|_1^3 = 2(\ln 3)^2 - 2\ln(x) + \frac{8}{3}$$

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c. The volume of the solid obtained by rotating the region around $x = -1$.

Using cylindrical shells, the volume is

$$\int_1^3 2\pi(x+1)\left(\ln(x) - \frac{1-x}{x}\right) dx = \cdots = 13\pi\ln(3) \approx 44.87$$

Score: /2

d. The volume of the solid obtained by rotating the region around $y = 2$.

Using washers, the volume is

$$\int_1^3 \pi\left(2 - \frac{1-x}{x}\right)^2 - \pi(2-\ln(x))^2\,dx = \cdots = 12\pi\ln(3) - 3\pi(\ln 3)^2 - \frac{4}{3}\pi \approx 25.85$$

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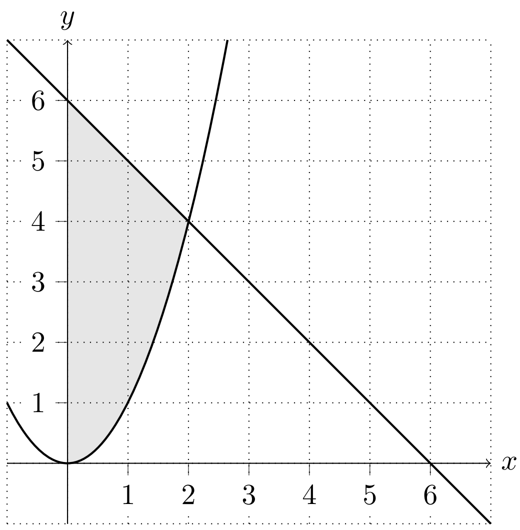
Problem 6: Consider the region bounded by the y-axis, $x+y=6$, and $y=x^2$ in the first quadrant. Draw the region and set up the expression for the perimeter of the region. All straight line segments in the perimeter must be computed exactly as a number or square root; only the curved border may be expressed as a definite integral. DO NO EVALUATE.



$$6+2\sqrt{2}+\int_0^2 \sqrt{1+(2x)^2}\,dx = 6+2\sqrt{2}+\left(\frac{1}{2}x\sqrt{4x^2+1}+\frac{1}{4}\sinh^{-1}(2x)\right)\Bigg|_0^2$$
$$= 6+2\sqrt{2}+\sqrt{17}-\frac{1}{4}\ln(\sqrt{17}-4)$$
$$\approx 13.48$$

Score: /4

Problem 7: Use the method of partial fractions to integrate the following.

$$\int \frac{3x-2}{(2x-5)(x+1)}\,dx$$

By partial fractions,

$$\frac{3x-2}{(2x-5)(x+1)} = \frac{A}{2x-5}+\frac{B}{x+1},$$

so $3x-2 = A(x+1)+B(2x-5)$. If $x=\frac{5}{2}$, that yields that $\frac{11}{2}=\frac{7}{2}A+0$, so $A=\frac{11}{7}$.
Similarly, is $x=-1$, then $-5=0-7B$, so $B=\frac{5}{7}$.
Hence,

$$\int \frac{3x-2}{(2x-5)(x+1)}\,dx = \int \frac{11}{7(2x-5)}+\frac{5}{7(x+1}\,dx = \frac{11}{14}\ln|2x-5|+\frac{5}{7}\ln|x+1|+C$$

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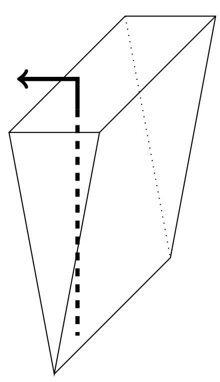
Problem 8: A spring has a natural length of 0.8 m. A force of 24 N stretches the spring to a total length of 2 m. Find the work required to stretch the spring 1.5 m beyond its natural length.

The force needed to stretch a spring a distance of x beyond the natural length is $F = kx$.
Here $k = \frac{F}{x} = \frac{24\,\mathrm{N}}{1.2\,\mathrm{m}} = 20\,\mathrm{N/m}$.
The work is then

$$\begin{aligned}\int_{0\,\mathrm{m}}^{1.5\,\mathrm{m}} F\,dx &= \int_{0\,\mathrm{m}}^{1.5\,\mathrm{m}} kx\,dx = \frac{1}{2}kx^2\Big|_{0\,\mathrm{m}}^{1.5\,\mathrm{m}} \\ &= \frac{1}{2} \times 20\,\mathrm{N/m} \times 2.25\,\mathrm{m}^2 - 0 = 22.5\,\mathrm{Nm} = 22.5\,\mathrm{J}\end{aligned}$$

Score: /3

Problem 9: A tank in the shape of a triangular prism of length 10 m whose vertical cross sections are isosceles triangles of height 8 m and base 3 m is buried 1 m below ground and filled with a fluid of density 200 kg/m^3. Find the work required to pump all the fluid out of the tank from a valve at ground level.



The liquid at distance x from the bottom has width $\frac{3}{8}x$, so it's volume element from a rectangular horizontal slice is $\frac{3}{8}x \times 10\,dx = \frac{15}{4}x\,dx$, so it has mass $200 \times \frac{15}{4}x\,dx = 750x\,dx$. That mass has to be lifted a distance of $9 - x$ against gravity, so the required work is

$$\int_0^8 750xg(9-x)\,dx = 750g\left(\frac{9}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^8 \approx 863\,\mathrm{kJ}$$

if $g = 9.81\,\mathrm{m/s^2}$.

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