Math 126
Fall 2022
Dr. Lily Yen

Test 3

Show all your work

Full Name: _____
Student Number: _____
Signature: _____
Score: ___/35

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Determine whether each sequence below converges as $n \to \infty$. If so, find the limit, else state why you cannot find the sum.

a. Write out the first 3 terms of the sequence, $\left\{\dfrac{2n-(-1)^{n-1}}{5n+(-1)^n}\right\}_{n=0}^{\infty}$, before determining convergence as $n \to \infty$.

The sequence begins $\frac{1}{4}, \frac{5}{11}, \frac{5}{14}$.

$$\frac{2n-(-1)^{n-1}}{5n+(-1)^n} = \frac{n\left(2-\frac{(-1)^{n-1}}{n}\right)}{n\left(5+\frac{(-1)^n}{n}\right)} = \frac{2-\frac{(-1)^{n-1}}{n}}{5+\frac{(-1)^n}{n}} \to \frac{2}{5} \quad \text{as } n \to \infty.$$

b. Write out the first 3 terms of $s_n = \displaystyle\sum_{k=1}^{n} \frac{\sec(k\pi)}{k+1}$, for natural numbers n before determining convergence as $n \to \infty$.

The sequence begins $s_1 = -\frac{1}{2}$, $s_2 = \frac{1}{3} - \frac{1}{2}$ and ,$s_3 = -\frac{1}{4} + \frac{1}{3} - \frac{1}{2}$. Now $\sec(n\pi) = \pm 1$, so $|a_n| = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} a_n = 0.$$

By Alternating series test, the nth partial sum, s_n converges, but the limit is not possible to find with the techniques taught.

c. Write out the first 3 terms of $s_n = \displaystyle\sum_{k=0}^{n} \left(\frac{5^k}{(-4)^{2k}}\right)$ before determining convergence as $n \to \infty$.

The first few terms of $\left\{\frac{5^k}{(-4)^{2k}}\right\}_{k=0}^{\infty}$ are $1, \frac{5}{16}, \frac{25}{256}$, so

$$s_0 = 1, \quad s_1 = 1 + \frac{5}{16} = \frac{21}{16}, \quad s_2 = 1 + \frac{5}{16} + \frac{25}{256} = \frac{361}{256}, \quad s_3 = \frac{5901}{4096}$$

Since $\frac{5^k}{(-4)^{2k}} = \frac{5^k}{((-4)^2)^k} = \left(\frac{5}{16}\right)^k$, $\sum_{k=0}^{n}\left(\frac{5^k}{(-4)^{2k}}\right)$ is a geometric sum with common ratio $\frac{5}{16} < 1$, so

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty}\left(\frac{5}{16}\right)^k = \frac{1}{1-\frac{5}{16}} = \frac{16}{11}$$

Score: /8

Problem 2: Solve for y in the following initial value problem. Express your solution explicitly as $y = f(x)$. Include the domain of f.

$$x\frac{dy}{dx} - e^y = 0, \quad y(1) = 2.$$

If $x\frac{dy}{dx} - e^y = 0$, then $x\frac{dy}{dx} = e^y$, so $\int e^{-y}\,dy = \int \frac{1}{x}\,dx$, so $-e^{-y} = \ln|x| + C$. Since $y(1) = 2$, $-e^{-2} = \ln(1) + C = C$, so $-e^{-y} = \ln|x| - e^{-2}$, so $e^{-y} = e^{-2} - \ln|x|$, so $-y = \ln\big(e^{-2} - \ln|x|\big)$, so $y = -\ln\big(e^{-2} - \ln|x|\big)$.
Since the domain on ln is all positive reals, we need that $x \neq 0$ and $e^{-2} > \ln|x|$, so $0 < x < e^{e^{-2}}$ or $-e^{e^{-2}} < x < 0$. Since the initial condition applies to positive x, the final solution is

$$y = -\ln\big(e^{-2} - \ln|x|\big), \quad 0 < x < e^{e^{-2}}$$

Score: /5

Problem 3: Find the orthogonal trajectories of the family of curves $kx^3 = y^2$, where k is an arbitrary constant.

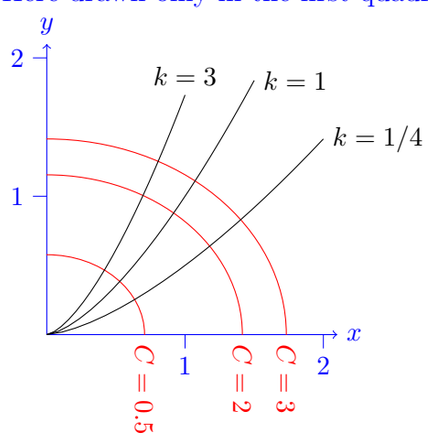
If $kx^3 = y^2$, then $k = \frac{y^2}{x^3}$, so

$$0 = \frac{2y\frac{dy}{dx}x^3 - 3y^2x^2}{x^6},$$

so $2y\frac{dy}{dx}x^3 - 3y^2x^2 = 0$, so $\frac{dy}{dx} = \frac{3y^2x^2}{2yx^3} = \frac{3y}{2x}$. The orthogonal trajectories therefore satisfy $\frac{dy}{dx} = -\frac{2x}{3y}$, so $\int 3y\,dy = -\int 2x\,dx$, so $\frac{3}{2}y^2 = -x^2 + C$, so

$$y = \pm\sqrt{\tfrac{2}{3}C - \tfrac{2}{3}x^2}$$

Here drawn only in the first quadrant; the others by symmetry.



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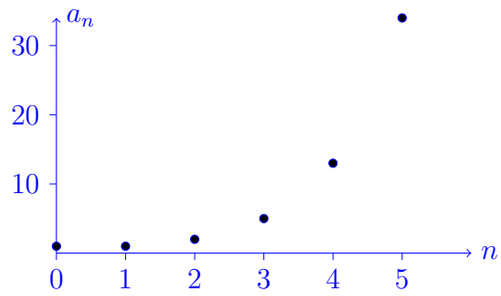
Calculators permitted in this part.

Problem 4: Graph the sequence a_n satisfying the recurrence: $a_n = 3a_{n-1} - a_{n-2}$ for integers $n \geq 2$ and $a_0 = 1$, and $a_1 = 1$. Determine a_5 and whether a_n is convergent.



$a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 13$,
$a_5 = 34$, $a_6 = 89$, $a_7 = 233$, $a_8 = 610$,
$a_9 = 1597$, $a_{10} = 4181$, ...

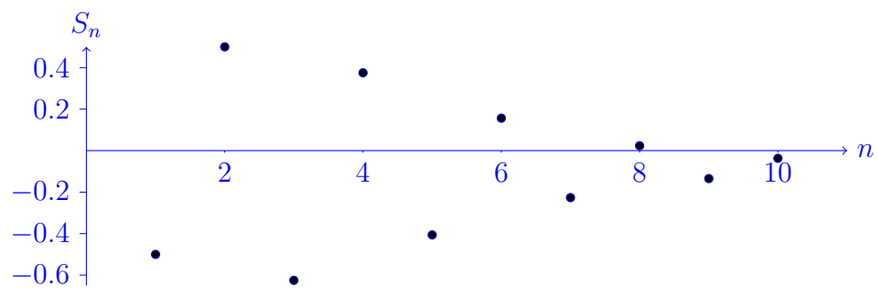
Claim: $a_n \geq 2a_{n-1}$ for $n \geq 2$.
Proof: Since $a_2 = 2$, the claim is true for $n = 2$. Now assume the claim for $n - 1$, that is $a_{n-1} \geq 2a_{n-2}$, so $a_{n-2} \leq \frac{1}{2}a_{n-1}$. Then $a_n = 3a_{n-1} - a_{n-2} \geq 3a_{n-1} - \frac{1}{2}a_{n-1} = \frac{5}{2}a_{n-1} > 2a_{n-1}$.
QED.
The claim shows that $a_n \geq 2^{n-1}$, but $\{2^{n-1}\}$ is divergent, so $\{a_n\}$ is DIVERGENT.

Score: /4

Problem 5: For the series

$$S = \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$$

a. Graph the n-th partial sum of the series for the first 10 values of n. State the first three terms explicitly to support your graph.



$S_1 = -1/2$,
$S_2 = \frac{1}{2}$, $S_3 = -\frac{5}{8}$.

b. Determine whether the series S converges absolutely.

Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2\, 2^n}{2^{n+1}\, n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{2} \quad \text{as } n \to \infty,$$

it follows from Ratio Test that $\sum_{k=1}^{\infty} a_k$ converges, so S converges absolutely.
Actually, $S = -\frac{2}{27}$.

Score: /5

Problem 6: Recall Newton's law of cooling: the rate of change in temperature of an object varies proportionally to the difference in temperature between the object and the surrounding medium. Write a differential equation whose solution is the temperature as a function of time for a bottle of frozen black currant juice taken out of a $-18\,^\circ$C deep freezer and left in a $25\,^\circ$C kitchen, given that after one hour, the temperature of the bottle rose $7\,^\circ$C. Include all conditions. Solve the equation for the temperature of the object at time t.

Let $y(t)$ be the temperature (in °C) of the bottle at time t (in hours since we took the bottle out of the freezer). By Newton's Law of Cooling,

$$\frac{dy}{dt} = k(y - 25), \qquad y(0) = -18 \text{ and } y(1) = -11.$$

Thus $\int \frac{1}{y-25}\,dy = \int k\,dt$, so $\ln|y-25| = kt + C$, so $|y-25| = e^{kt+C} = e^C e^{kt}$, so $y - 25 = Ae^{kt}$ where $A = \pm e^C$, so $y = 25 + Ae^{kt}$.
Moreover, $-18 = y(0) = 25 + Ae^0 = 20 + A$, so $A = -43$. Hence, $y(t) = 25 - 43e^{kt}$. We also know that $y(1) = -11$, so $-11 = 25 - 43e^k$, so $e^k = \frac{36}{43}$, so $k = \ln(\frac{36}{43}) \approx -0.1777$. Thus

$$y(t) = 25 - 43e^{-0.1777t}, \quad t \geq 0.$$

Score: /4

Problem 7: A tank holds 100 litres of pure water. A tap is opened and a solution of 70 % alcohol and 30 % water is fed into the tank at a rate of 5 L/min. Simultaneously, a drain plug at the bottom of the tank is opened and the thoroughly mixed solution leaves the tank at the same rate. How much water is in the tank after 75 minutes? Express your answer correct to 2 decimal places.

Let y be the amount of (pure) water (in litres) in the tank. Then

$$\frac{dy}{dt} = \left(0.30 - \frac{y}{100}\right) 5 = \frac{15}{10} - \frac{1}{20}y, \qquad y(0) = 100.$$

Thus $\int \frac{20}{30-y}\,dy = \int 1\,dt$, so $-20\ln|30-y| = t + C$, so $|30-y| = e^{-(t+C)/20} = e^{-t/20}e^{-C/20}$, so $30 - y = Ae^{-t/20}$, so $y = 30 - Ae^{-t/20}$. Since $100 = y(0) = 30 - A$, it follows that $A = -70$. Hence

$$y = 30 + 70e^{-t/20}$$

and after 75 minutes, the amount of water is 31.65 L.

Score: /5