Math 126
Fall 2022
Dr. Lily Yen

Test 1

Show all your work

Name: _____
Number: _____
Signature: _____
Score: ___/35

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Integrate the following. Use correct notation and show all steps.

a. $\int \sqrt[3]{u}(u+1)^2\,du$

$$\int \sqrt[3]{u}(u+1)^2\,du = \int u^{1/3}(u^2+2u+1)\,du = \int u^{7/3}+2u^{4/3}+u^{1/3}\,du$$
$$= \frac{3}{10}u^{10/3}+\frac{6}{7}u^{7/3}+\frac{3}{4}u^{4/3}+C$$

Score: /2

b. $\int \dfrac{\cos(x)}{\sqrt{4-\sin^2(x)}}\,dx$

If $u=\frac{1}{2}\sin(x)$, then $\frac{du}{dx}=\frac{1}{2}\cos(x)$, so $dx=\frac{2}{\cos(x)}\,du$. Moreover, $2u=\sin(x)$. Therefore

$$\int \frac{\cos(x)}{\sqrt{4-\sin^2(x)}}\,dx = \int \frac{\cos(x)}{\sqrt{4-(2u)^2}}\,\frac{2}{\cos(x)}\,du$$
$$= \int \frac{2}{\sqrt{4(1-u^2)}}\,du = \int \frac{2}{2\sqrt{1-u^2}}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du$$
$$= \sin^{-1}(u)+C = \sin^{-1}(\tfrac{1}{2}\sin(x))+C$$

Score: /2

c. $\int \dfrac{\ln(\sin(\theta))}{\tan(\theta)}\,d\theta$

If $u=\ln(\sin(\theta))$, then $\frac{du}{d\theta}=\frac{1}{\sin(\theta)}\cos(\theta)=\frac{1}{\tan(\theta)}$, so $d\theta=\tan(\theta)\,du$. Therefore

$$\int \frac{\ln(\sin(\theta))}{\tan(\theta)}\,d\theta = \int \frac{u}{\tan(\theta)}\tan(\theta)\,du = \int u\,du = \tfrac{1}{2}u^2+C = \tfrac{1}{2}\big(\ln(\sin(\theta))\big)^2+C$$

Score: /2

d. $\int \dfrac{t-2}{3t^2+1}\,dt$

$$\int \frac{t-2}{3t^2+1}\,dt = \int \frac{t}{3t^2+1}\,dt - \int \frac{2}{3t^2+1}\,dt$$

If $u=3t^2+1$, then $\frac{du}{dt}=6t$, so $dt=\frac{1}{6t}\,du$. Moreover, if $v=\sqrt{3}t$, then $v^2=3t^2$ and $\frac{dv}{dt}=\sqrt{3}$, so $dt=\frac{1}{\sqrt{3}}\,dv$. Therefore the expression above equals

$$\int \frac{t}{u}\,\frac{1}{6t}\,du - \int \frac{2}{v^2+1}\,\frac{1}{\sqrt{3}}\,dv = \frac{1}{6}\int \frac{1}{u}\,du - \frac{2}{\sqrt{3}}\int \frac{1}{1+v^2}\,dv$$
$$= \frac{1}{6}\ln|u| - \frac{2}{\sqrt{3}}\tan^{-1}(v)+C = \frac{1}{6}\ln(3t^2+1) - \frac{2}{\sqrt{3}}\tan^{-1}(\sqrt{3}t)+C$$

Score: /3

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Name: _____

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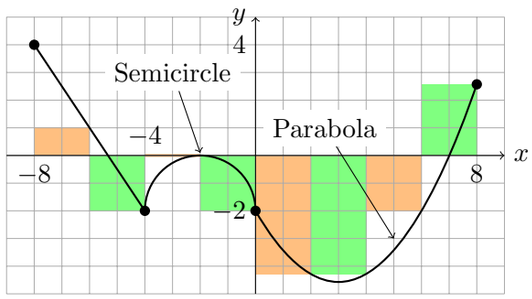
A TI83 or TI84 Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 2: Given the graph of f, evaluate the following integrals exactly.



a. $\displaystyle\int_{-8}^{-4} f(x)\,dx$

The line through $(-8,4)$ and $(-4,-2)$ is given by $y+2=-\frac{3}{2}(x+4)$. If $y=0$, then $2=-\frac{3}{2}(x+4)$, so $x=-\frac{16}{3}$ which, then, is the x-intercept. The value of the integral is then $\frac{1}{2}(-\frac{16}{3}+8)4-\frac{1}{2}(-4+\frac{16}{3})2=4$.

b. $\displaystyle\int_{-4}^{0} |f(x)|\,dx$

This is a rectangle minus a semicircle: $4\times 2-\frac{1}{2}\pi 2^2=8-2\pi$.

c. $\displaystyle\int_{-8}^{0} |f(x)|\,dx$

$\frac{1}{2}(-\frac{16}{3}+8)4+\frac{1}{2}(-4+\frac{16}{3})2+(8-2\pi)=\frac{44}{3}-2\pi$.

d. Draw R_8 over $[-8,8]$ directly on the grid. Lightly shade the rectangles.

Score: /6

Problem 3: Give a function g that has the following expression as a left Riemann sum when $n=10$, $\Delta x=3$, and the left end point of the closed interval is -4:

$$\sum_{k=0}^{9} \frac{1}{6k-2}$$

Recall that $L_{10}=\sum_{k=0}^{9} f(x_k)\cdot 3$. We begin by introducing a factor 3 in the summand and undo the factor by multiplying the sum by $1/3$.

$$L_{10}=\frac{1}{3}\sum_{k=0}^{9}\frac{1}{6k-2}\cdot 3$$

We also know that with the indices of k running from 0 to 9, $x_k=-4+3k$, so solve for k in terms of x_k to get an expression for $6k-2$, the denominator of $f(x_k)$ inside the summation notation.

$$k=\frac{x_k+4}{3},\quad \text{so}\quad 6k=2(x_k+4)$$

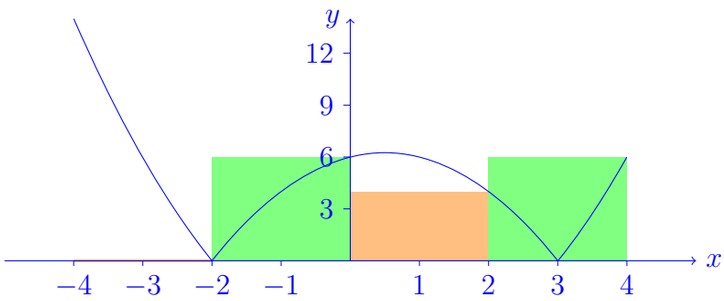
With $6k-2=2x_k+8-2=2x_k+6$, we replace x_k by x to get

$$g(x)=\frac{1}{3}\cdot\frac{1}{2x+6}$$

Score: /3

Problem 4: Given the function $f(u) = |u^2 - u - 6|$, answer the following questions.

a. First draw f on the interval $[-4, 4]$, then draw R_4 for f on the same coordinate system.



b. Compute exactly R_4 on the interval $[-4, 4]$ by listing the area for each rectangle.

$$2 \times 0 + 2 \times 6 + 2 \times 4 + 2 \times 6 = 0 + 12 + 8 + 12 = 32$$

c. Use your graphing calculator to fill the following table for the interval, $[-4, 4]$. Provide at least 2 decimal places.

Approx	$n = 10$	$n = 100$
L_n	40.70	36.66
R_n	34.30	36.02

Score: /5

Problem 5: Suppose the number of hours of daylight on a given day in North Vancouver is modelled by $-4.1\cos(\frac{\pi t}{6}) + 12.5$, where t is given in months and $t = 0$ corresponds to the winter solstice. Answer the following.

a. Find the average number of daylight hours in a year.

$$\frac{1}{12}\int_0^1 2 - 4.1\cos(\frac{\pi t}{6}) + 12.5\,dt = 12.5$$

The average number of daylight hours in a year is 12.5 hours.

b. Find the times t_1 and t_2 in a year where the number of daylight hours equal the average number.

If $-4.1\cos(\frac{\pi t}{6}) + 12.5 = 12.5$, then $\cos(\frac{\pi t}{6}) = 0$, so $\frac{\pi t}{6} = \frac{\pi}{2}$ or $\frac{\pi t}{6} = \frac{3\pi}{2}$, so $t = 3$ or $t = 9$.
You can also graph both $D(x) = -4.1\cos(\frac{\pi t}{6}) + 12.5$ as Y_1 and $Y_2 = 12.5$ in your graphing calculator to find the intersections of these two functions.
Thus, the times in a year where the number of daylight hours equal the average number happen in March and September, or Spring and Fall equinox.

Score: /4

Problem 6: Use the Fundamental Theorem of Calculus to find the following.

a. If $F(x) = \displaystyle\int_{3\sqrt{x}}^{\pi} 2t\,dt$, find $F'(4)$ exactly.

Let $h(x) = \int_{\pi}^{x} 2t\,dt$. Then $h'(x) = 2x$ by the FTC. Moreover, $F(x) = -h(3\sqrt{x})$, so $F'(x) = -h'(3\sqrt{x})\frac{3}{2\sqrt{x}} = -6\sqrt{x}\frac{3}{2\sqrt{x}} = -9$, so $F'(4) = -9$.

Score: /2

b. If $G(x) = \displaystyle\int_{\tan(\pi/4)}^{\ln(x)} \pi\tan^{-1}(t) + 2e^t\,dt$, find $dG(x)/dx$.

Let $h(x) = \int_{0}^{x} \pi\tan^{-1}(t) + 2e^t\,dt$. Then $h'(x) = \pi\tan^{-1}(x) + 2e^x$ by the FTC. Moreover, $G(x) = \int_{\tan(\pi/4)}^{0} \pi\tan^{-1}(t) + 2e^t\,dt + \int_{0}^{\ln(x)} \pi\tan^{-1}(t) + 2e^t\,dt = -\int_{0}^{\tan(\pi/4)} \pi\tan^{-1}(t) + 2e^t\,dt + \int_{0}^{\ln(x)} \pi\tan^{-1}(t) + 2e^t\,dt = C + h(\ln(x))$, where C is a constant because of the definite integral; so

$$\begin{aligned} G'(x) &= 0 + h'(\ln(x))\frac{1}{x} \\ &= \left(\pi\tan^{-1}(\ln(x)) + 2e^{\ln(x)}\right)\frac{1}{x} \\ &= \pi\tan^{-1}(\ln(x))\frac{1}{x} + 2 \end{aligned}$$

Score: /2

Problem 7: The following table shows the change in historic average cost per gigabyte of RAM on a computer over 5-year increments from 1980 to 2010.

Year	1980	1985	1990	1995	2000	2005	2010
5-Year Change ($)	0	−5 468 750	−755 495	−73 005	−29 768	−918	−177

The table shows that the average cost per gigabyte of RAM in 1985 dropped almost 5.5 million dollars compared to that of 1980. Answer the following questions.

a. If the average cost per gigabyte of RAM in 2010 is \$12, find the average cost per gigabyte of RAM in 2000.

$12 + 177 + 918 = 1107$, so \$1107.

b. The average cost per gigabyte of RAM can be approximated by the function $C(t) = 6.5(0.65)^t$ where t is measured in years since 1980 and C is cost in millions of dollars (US). Find the average cost per gigabyte of RAM from 1980 to 2010.

$$\frac{1}{30}\int_{0}^{30} 6.5(0.65)^t\,dt = \frac{1}{30}\left.\frac{6.5(0.65)^t}{\ln(0.65)}\right|_{0}^{30} \approx 0.5030$$

The average cost per gigabyte of RAM from 1980 to 2010 according to the model C is 0.503 million US dollars.

Score: /4