Math 126
Fall 2016
Dr. Lily Yen

Test 3

Show all your work

Name: _____

Score: ___/41

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: For each sequence or series below, determine if it converges; find the limit or sum if possible, or prove that it diverges to ∞, or $-\infty$, or does not exist.

a. $a_n = \dfrac{2-5n\sqrt{n}}{3+n}$. Find the limit of the sequence.

$a_n = \frac{2-5n\sqrt{n}}{3+n} = \frac{2-5n^{3/2}}{3+n} = \frac{n^{3/2}(2/n^{3/2}-5)}{n(3/n+1)} = \sqrt{n}\frac{2/n^{3/2}-5}{3/n+1} \to \sqrt{n}(-5/1) \to -\infty$ as $n \to \infty$.

Score: /2

b. $\left\{\dfrac{\ln n}{n} + \cot\left(\dfrac{n\pi}{n+1}\right)\right\}_{n=1}^{\infty}$. Find the limit of the sequence.

If you consider the graphs of $y = \ln(x)$ and $y = x$, it is clear that $\frac{\ln(n)}{n} \to 0$ as $n \to \infty$. However, $\lim_{n\to\infty} \cot\left(\frac{n\pi}{n+1}\right) = \lim_{x\to\pi^-} \cot(x) = -\infty$, so the given sequence tends to $-\infty$.

Score: /3

c. $\displaystyle\sum_{n=0}^{\infty} \frac{2(5^n)}{3^{2n}}$.

$\sum_{n=0}^{\infty} \frac{2(5^n)}{3^{2n}} = \sum_{n=0}^{\infty} 2(\frac{5}{9})^n$ is a geometric series with first term 2 and common ratio $\frac{5}{9} < 1$, so it converges to $\frac{2}{1-5/9} = \frac{2}{4/9} = 9/2$.

Score: /3

d. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.

$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} = \sum_{n=2}^{\infty} (\frac{1}{n-1} - \frac{1}{n+1}) =$
$(\frac{1}{1} - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + \cdots = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$

Score: /3

e. $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Determine both convergence and absolute convergence of the series.

If $a_n = \frac{(-10)^n}{n!}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-10)^{n+1} n!}{(-10)^n (n+1)!}\right| = \left|\frac{-10}{n+1}\right| = \frac{10}{n+1} \to 0 < 1$ as $n \to \infty$, so the given sequence converges absolutely by Ratio Test.

Score: /4

Math 126
Fall 2016
Dr. Lily Yen

Test 3

Show all your work

Name: _____

Calculators permitted in this part.

Problem 2: Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$$

a. Evaluate the Nth partial sum for the first four N's. Provide exact answers only.

$$\sum_{n=1}^{1} \frac{(-1)^{n-1} n}{n^2+1} = \frac{1}{2}$$

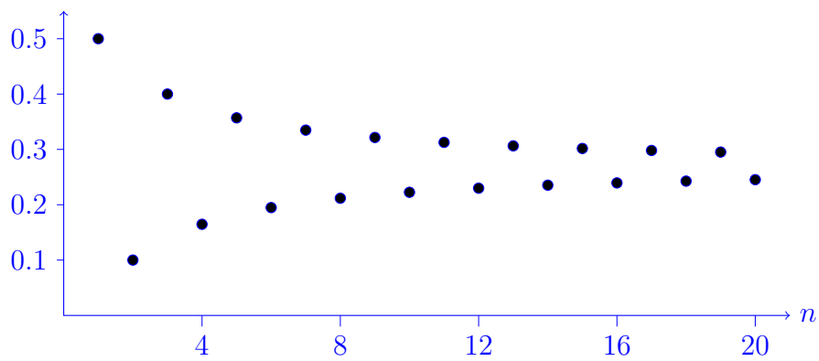
$$\sum_{n=1}^{2} \frac{(-1)^{n-1} n}{n^2+1} = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\sum_{n=1}^{3} \frac{(-1)^{n-1} n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} = \frac{2}{5}$$

$$\sum_{n=1}^{4} \frac{(-1)^{n-1} n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} = \frac{14}{85}$$

Score: /2

b. With your calculator, plot the Nth partial sum for the first 20 N's. You must specify your TI entries.



Score: /2

c. Approximate this infinite series with the 100th partial sum and find the error bound. Provide a 6-decimal place accuracy.

$S_{100} \approx 0.264636$.
Since the series is alternating, the size of the error is bounded by the size of the absolute value of the first omitted term: $\left|\frac{(-1)^{100}101}{101^2+1}\right| = \frac{101}{10202} \approx 0.0099$

Score: /2

Problem 3: For the function $f(x) = e^x + x^2$, set up the integral expression for its arc length for $x \in [1, 4]$. Approximate the integral accurate to 4 decimal places.

$$\int_1^4 \sqrt{1 + (f'(x))^2}\, dx = \int_1^4 \sqrt{1 + (e^2 + 2x)^2}\, dx \approx 66.9916$$

Score: /3

Problem 4: Use separation of variables to find the general solution of the following differential equation.

$$y' = \frac{\sqrt{1-y^2}}{1+x^2}$$

If $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$, then $\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{1+x^2}\,dx$, so $\arcsin(y) = \arctan(x) + C$, so

$$y = \sin(\arctan(x) + C)$$

Using that $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$, that $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$, and that $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$, you can rewrite the solution as

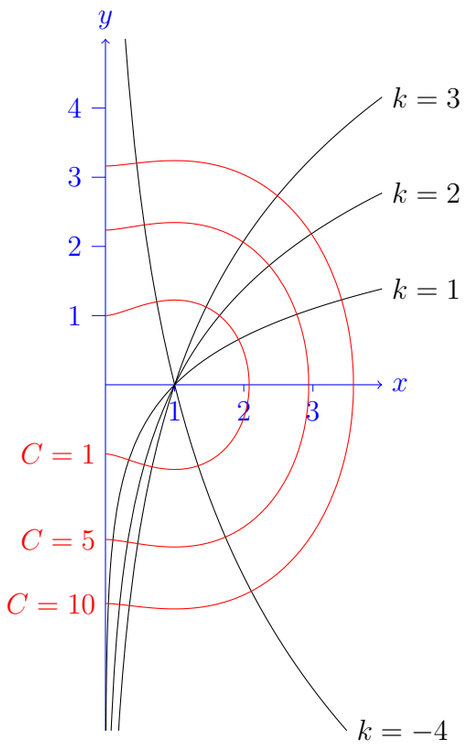
$$y = \frac{x\cos(C) + \sin(C)}{\sqrt{1+x^2}}$$

Score: /3

Problem 5: Find the orthogonal trajectories of $y = k\ln(x)$, for arbitrary constants k and positive x. Sketch at least three members of each family (the function and its orthogonal trajectory). Label each curve with its parameter value.

If $y = k\ln(x)$, then $k = \frac{y}{\ln(x)}$ and $y' = \frac{k}{x}$. Thus the orthogonal trajectories satisfy $\frac{dy}{dx} = -\frac{1}{k}x = -\frac{\ln(x)}{y}x$, so $\int y\,dy = -\int x\ln(x)\,dx$, so, using integration by parts, $\frac{1}{2}y^2 = -\frac{1}{2}x^2\ln(x) + \int \frac{1}{2}x^2\frac{1}{x}\,dx = -\frac{1}{2}x^2\ln(x) + \int \frac{1}{2}x\,dx = -\frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C$. Hence

$$y = \pm\sqrt{-x^2\ln(x) + \tfrac{1}{2}x^2 + 2C}$$



Score: /6

Problem 6: A stomach flu spread through Capilano University. Let $y(t)$ be the fraction of the population at Cap that has gotten the flu at time t days from first observation. Assume that the rate at which the flu spreads is proportional to the product of the fraction y of the population that has the flu and the fraction $1 - y$ that has not yet contracted the flu.

a. Write down the differential equation satisfied by y in terms of a proportionality constant k.

$$\frac{dy}{dt} = ky(1 - y)$$

b. Find k, assuming that 10 % of the population at Cap got the flu at $t = 0$, and 40 % got the flu at $t = 2$ days.

If $\frac{dy}{dt} = ky(1 - y)$, then $\int \frac{1}{y(1-y)}\, dy = \int k\, dt$, so by partial fractions,
$\int \frac{1}{y} + \frac{1}{1-y}\, dy = \int k\, dt$, so $\ln(y) - \ln(1 - y) = kt + C$.
If $t = 0$, then $y = 0.10$, so $C = \ln(0.10) - \ln(0.90) = -2.197$. If $t = 2$, then $y = 0.40$, so $\ln(0.40) - \ln(0.60) = 2k - 2.197$, so $k = 0.8959$.

c. Using the assumption of the previous part, determine when three quarters of the population at Cap will have the flu. Provide a 2-decimal place accuracy.

If $y = 0.75$, then $\ln(0.75) - \ln(0.25) = 0.90t - 2.20$, so $t = 3.68$ days.

Score: /8