Math 126
Fall 2016
Dr. Lily Yen

Test 2

Show all your work

Name: _____

Score: ___/45

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Evaluate the following integrals analytically.

a. $\int xe^{-2x}\,dx$

Using integration by parts,

$$\int xe^{-2x}\,dx = -\tfrac{1}{2}xe^{-2x} - \int -\tfrac{1}{2}e^{-2x}\,dx = -\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} + C$$

Score: /3

b. $\int \cos^3(\theta)\sin^8(\theta)\,d\theta$

If $u = \sin(\theta)$, then $du = \cos(\theta)\,d\theta$, so
$\int \cos^3(\theta)\sin^8(\theta)\,d\theta = \int \cos^2(\theta)\sin^8(\theta)\cos(\theta)\,d\theta = \int (1-\sin^2(\theta))\sin^8(\theta)\cos(\theta)\,d\theta =$
$\int \sin^8(\theta) - \sin^{10}(\theta)\cos(\theta)\,d\theta = \int u^8 - u^1 0\,du = \tfrac{1}{9}u^9 - \tfrac{1}{11}u^1 1 + C =$
$\tfrac{1}{9}\sin^9(\theta) - \tfrac{1}{11}\sin^{11}(\theta) + C$
Alternatively, using integration by parts,
$\int \cos^3(\theta)\sin^8(\theta)\,d\theta = \int \cos^2(\theta)\left(\cos(\theta)\sin^8(\theta)\right)\,d\theta =$
$\cos^2(\theta)\left(\tfrac{1}{9}\sin^9(\theta)\right) - \int (-2\cos(\theta)\sin(\theta))\left(\tfrac{1}{9}\sin^9(\theta)\right)\,d\theta =$
$\tfrac{1}{9}\cos^2(\theta)\sin^9(\theta) + \tfrac{2}{9}\int \cos(\theta)\sin^{10}(\theta)\,d\theta = \tfrac{1}{9}\cos^2(\theta)\sin^9(\theta) + \tfrac{2}{99}\sin^{11}(\theta) + C$ (and these two results are actually equal).

Score: /3

c. $\int \dfrac{x+1}{x^2-2x-15}\,dx$

By the method of partial fractions, constants A and B exist such that

$$\frac{x+1}{x^2-2x-15} = \frac{x+1}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5},$$

so $x+1 = A(x-5) + B(x+3)$. Substituting $x = 5$ yields that $6 = 8B$, so $B = \frac{3}{4}$, and substituting $x = -3$ yields that $-2 = -8A$, so $A = \frac{1}{4}$.
Hence

$$\int \frac{x+1}{x^2-2x-15}\,dx = \int \frac{1/4}{x+3} + \frac{3/4}{x-5}\,dx = \tfrac{1}{4}\ln|x+3| + \tfrac{3}{4}\ln|x-5| + C$$

Score: /4

d. $\int e^{-3x}\cos(5x)\,dx$

Using integration by parts twice,

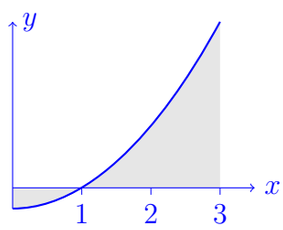
$$\begin{aligned}\int e^{-3x}\cos(5x)\,dx &= -\tfrac{1}{3}e^{-3x}\cos(5x) - \int -\tfrac{1}{3}e^{-3x}(-5)\sin(5x)\,dx \\ &= -\tfrac{1}{3}e^{-3x}\cos(5x) - \tfrac{5}{3}\int e^{-3x}\sin(5x)\,dx \\ &= -\tfrac{1}{3}e^{-3x}\cos(5x) - \tfrac{5}{3}\left(-\tfrac{1}{3}e^{-3x}\sin(5x) - \int -\tfrac{1}{3}e^{-3x}5\cos(5x)\,dx\right) \\ &= -\tfrac{1}{3}e^{-3x}\cos(5x) + \tfrac{5}{9}e^{-3x}\sin(5x) - \tfrac{25}{9}\int e^{-3x}\cos(5x)\,dx,\end{aligned}$$

so $(1+\frac{25}{9})\int e^{-3x}\cos(5x)\,dx = -\frac{1}{3}e^{-3x}\cos(5x) + \frac{5}{9}e^{-3x}\sin(5x)$, so
$\frac{34}{9}\int e^{-3x}\cos(5x)\,dx = -\frac{1}{3}e^{-3x}\cos(5x) + \frac{5}{9}e^{-3x}\sin(5x)$, so

$$\int e^{-3x}\cos(5x)\,dx = -\tfrac{3}{34}e^{-3x}\cos(5x) + \tfrac{5}{34}e^{-3x}\sin(5x) + C$$

Score: /4

Problem 2: Find the exact area of the region bounded between $y = 3x^2 - 3$ and the x-axis for x in the interval $[0, 3]$.



$$\begin{aligned}\int_0^1 0 - (3x^2-3)\,dx + \int_1^3 (3x^2-3) - 0\,dx &= \int_0^1 -3x^2+3\,dx + \int_1^3 3x^2-3\,dx \\ &= -x^3+3x\Big|_0^1 + x^3 - 3x\Big|_1^3 = (2-0)+(18-(-2)) = 22\end{aligned}$$

Score: /3

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Calculators permitted from here on.

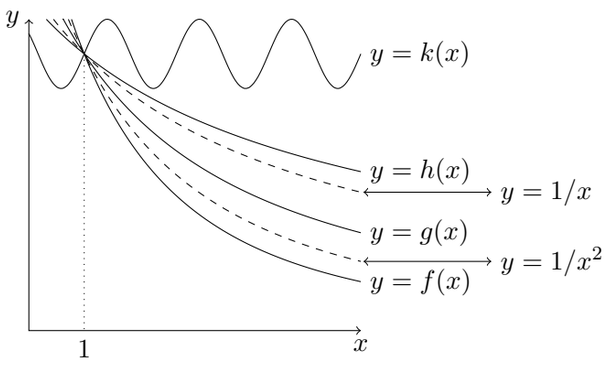
Problem 3: Determine analytically the convergence of the following integral.

$$\int_0^1 \frac{e^x}{x^2}\,dx$$

If $x \geq 0$, then $e^x \geq 1$. Therefore
$\int_0^1 \frac{e^x}{x^2}\,dx \geq \int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \int_b^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \frac{-1}{x}\Big|_b^1 = \lim_{b\to 0^+}(-1+\frac{1}{b}) = \infty$, so
$\int_0^1 \frac{e^x}{x^2}\,dx = \infty$.

Score: /4

Problem 4: The graph of $y_1 = 1/x$, $y_2 = 1/x^2$, and the functions f, g, h, and k are shown below.



a. Is the area between y_1 and y_2 on the interval $[1, \infty]$ finite or infinite? Explain.

The area under $y = 1/x$ is $\int_1^\infty \frac{1}{x}\,dx = \ln(x)\Big|_0^\infty = \infty$ while the area under $y = 1/x^2$ is $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_0^\infty = 1$.
The area between $1/x$ and $1/x^2$ is therefore **infinite** (as in $\infty - 1 = \infty$).

Score: /1

b. Using the graph, decide whether the integral of each of the functions f, g, h, and k on the interval $[1, \infty]$ converges, diverges, or whether it is impossible to tell. Reason must be provided for each correct answer.

The area under f is smaller than the finite area under $1/x^2$, so finite.
The areas under h or k are larger than the infinite area under $1/x$ so infinite.
Since all you know about the area under g is that it is larger than the finite area under $1/x^2$ but smaller than the infinite area under $1/x$, you cannot tell whether the area under g is finite or infinite.

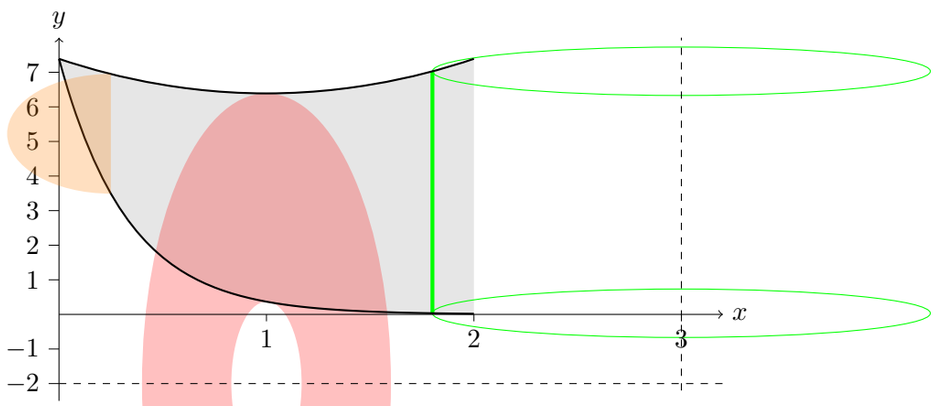
Score: /4

Problem 5: A 20 m tall parabolic arch spans 50 m. Suppose a supporting beam is to be constructed across the arch at its average height. How high is the cross beam placed?

The parabola has the form $y = 20 - ax^2$ and passes through $(\pm 25, 0)$. Filling in $(25, 0)$ yields that $0 = 20 - 625a$, so $a = \frac{4}{125}$.
The average height is then
$\frac{1}{50}\int_{-25}^{25} 20 - ax^2\,dx = \frac{1}{50}(20x - \frac{1}{3}ax^3)\Big|_{-25}^{25} = 20 - \frac{625}{3}a = \frac{40}{3} = 13.3\,\text{m}.$

Score: /3

Problem 6: Draw $y = e^{2-3x}$ and $y = (x-1)^2 + e^2 - 1$ on the grid. Shade the region bounded between these two graphs in the interval $[0, 2]$.



Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

$$\int_0^2 (x-1)^2 + e^2 - 1 - e^{2-3x}\,dx = \tfrac{1}{3}(x-1)^3 + e^2x - x + \tfrac{1}{3}e^{2-3x}\Big|_0^2 = -\tfrac{4}{3} + \tfrac{5}{3}e^2 + \tfrac{1}{3}e^{-4}$$

Score: /2

b. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the x-axis are semi-circles.

$$\int_0^2 \tfrac{\pi}{2}\left(\frac{(x-1)^2 + e^2 - 1 - e^{2-3x}}{2}\right)^2 dx = \tfrac{\pi}{8}\int_0^2 \left((x-1)^2 + e^2 - 1 - e^{2-3x}\right)^2 dx$$

Score: /2

c. The volume of the solid obtained by rotating the region around $y = -2$.

Using washers, the outer radius is $2 + (x-1)^2 + e^2 - 1 = (x-1)^2 + e^2 + 1$ and the inner radius is $2 + e^{2-3x}$, so the volume is

$$\int_0^2 \pi((x-1)^2 + e^2 + 1)^2 - \pi(2 + e^{2-3x})^2\,dx$$

Score: /2

d. The volume of the solid obtained by rotating the region around $x = 3$.

Using cylindrical shells, the height of each shell is $(x-1)^2 + e^2 - 1 - e^{2-3x}$, so the volume is

$$2\pi\int_0^2 (3-x)\left((x-1)^2 + e^2 - 1 - e^{2-3x}\right)\,dx$$

Score: /2

Problem 7: Suppose the force, F, required to compress a spring by 2 cm is 0.8 N. Find the work done in stretching the spring from equilibrium by 13 cm.

By Hooke's Law, $F = kx$, so $0.8\,\text{N} = k \times 0.02\,\text{m}$, so $k = 40\,\text{N/m}$. The required work is therefore $W = \int_0^{0.13} 40x\,dx = 20x^2\Big|_0^{0.13} = 0.338\,\text{J}$.

Score: /3

Problem 8: A tank is in the shape of an inverted circular cone with height 5 m and diameter 4 m. The tank is filled with water to 3 m deep. Find work done to pump all the water out a valve 2 m above the top of the tank.

A slice of water at height h from the bottom of the tank has radius $\frac{2}{5}h$ and needs to be lifted $7 - h$ meters. Hence the required work is $\int_0^3 (7-h) 9.8 \cdot 1000 \cdot \pi(\frac{2}{5}h)^2\,dh = 1568\pi \int_0^3 7h^2 - h^3\,dh = 1568\pi(\frac{7}{3}h^3 - \frac{1}{4}h^4)\Big|_0^3 = 67032\pi = 211\,000\,\text{J} = 211\,\text{kJ}$

Score: /5