Math 126
Fall 2016
Dr. Lily Yen

Test 1

Show all your work

Name: _____

Score: ___/46

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

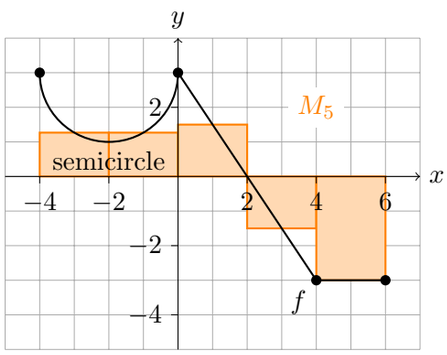
Problem 1: Find an explicit expression for $f(t)$ in the following definite integral.

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\sin\left(5+\frac{(\pi-3)k}{n}\right)\frac{1}{n}\right)=\int_{5}^{\pi+2}f(t)\,dt$$

$$f(t)=\frac{1}{\pi-3}\sin(t)$$

Score: /2

Problem 2: Consider the following function f defined for x in $[-4,6]$, and answer the questions below. Exact evaluations only.



a. $\displaystyle\int_{-4}^{0}f(t)\,dt$ $\quad\boxed{12-2\pi}$

b. $\displaystyle\int_{0}^{6}f(x)\,dx$ $\quad\boxed{-6}$

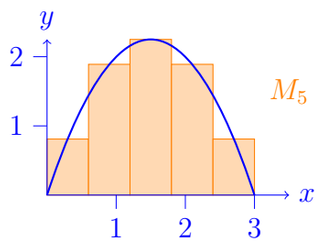
c. $\displaystyle\int_{0}^{6}|f(u)|\,du$ $\quad\boxed{12}$

d. $\displaystyle\left|\int_{-4}^{6}f(x)\,dx\right|$ $\quad\boxed{2\pi-6}$

e. Directly on the grid, draw M_5 for $\int_{-4}^{6}f(x)\,dx$.

Score: /7

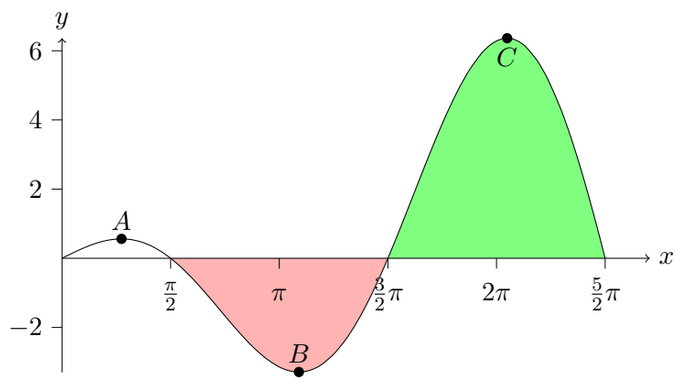
Problem 3: Find the expression of a Riemann sum with n intervals using the rectangular approximation method on the midpoints to estimate the area bounded by $f(x) = 3x - x^2$ and the x-axis. Draw the region of interest and M_n for a particular n.



$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{3}{n} f(\tfrac{3}{n}i + \tfrac{3}{2n}) = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{3}{n} \left(3(\tfrac{3}{n}i + \tfrac{3}{2n}) - (\tfrac{3}{n}i + \tfrac{3}{2n})^2\right)$$

Score: /4

Problem 4: The graph below shows $f(x) = x\cos(x)$. Let $F(x) = \int_0^x t\cos(t)\,dt$.



a. Locate the x-value(s) in $[0, 5\pi/2]$ where $F(x)$ attains local maxima.

By the FTC, $F'(x) = f(x)$. Therefore

x :	0		$\frac{\pi}{2}$		$\frac{3}{2}\pi$		$\frac{5}{2}\pi$
$F' = f$:	0	+	0	−	0	+	0
A :		↗		↘		↗	

, so F attains local maxima at $x = \frac{\pi}{2}$ and $\frac{5}{2}\pi$.

b. Determine the x-value(s) in $[0, 5\pi/2]$ where $F(x)$ attains the global maximum.

The area of the region between the x-axis and the graph of f over $[\frac{3}{2}\pi, \frac{5}{2}\pi]$ is much larger than the area of the region under $[\frac{\pi}{2}, \frac{3}{2}\pi]$, so $F(\frac{\pi}{2}) < F(\frac{5}{2}\pi)$, so F attains its global maximum at $x = \frac{5}{2}\pi$.
Actually, $F(\frac{\pi}{2}) = -1 + \frac{\pi}{2}$ while $F(\frac{5}{2}\pi) = -1 + \frac{5}{2}\pi$.

c. How many zeroes (roots) does $F(x)$ have in $[0, 5\pi/2]$?

Clearly $F(0) = \int_0^0 t\cos(t)\,dt = 0$. From there $F(x)$ increases to $F(\frac{\pi}{3})$, decreases even more to $F(\frac{3}{2}\pi)$, and increases still more to $F(\frac{5}{2}\pi)$. Therefore $F(\frac{\pi}{2}) > 0$, $F(\frac{3}{2}\pi) < 0$, and $F(\frac{5}{2}\pi) > 0$. Since F being an integral is continuous, F has a zero between $\frac{\pi}{2}$ and $\frac{3}{2}\pi$ and another zero between $\frac{3}{2}\pi$ and $\frac{5}{2}\pi$. Total **three** zeroes.

d. Label the inflection points of $F(x)$ with A, B, etc. on $[0, 5\pi/2]$ directly on the graph of f above. For each one, state whether the concavity changes from up to down or from down to up.

A: up to down; B: down to up; C: up to down.

Score: /5

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Calculators permitted from here on.

Problem 5: Integrate the following analytically.

a. $\displaystyle\int_{x/3}^{x/5} \sec^2(u)\,du$

$\int_{x/3}^{x/5} \sec^2(u)\,du = \tan(u)\big|_{x/3}^{x/5} = \tan(\frac{x}{5}) - \tan(\frac{x}{3})$.

Score: /2

b. $\displaystyle\int (t+9)^{-3}\,dt$

If $u = t+9$, then $\frac{du}{dt} = 1$, so $du = dt$. Therefore
$\int (t+9)^{-3}\,dt = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2(t+9)^2} + C$.

Score: /2

c. $\displaystyle\int \csc^5(\theta)\cos(\theta)\,d\theta$

If $u = \sin(\theta)$, then $\frac{du}{d\theta} = \cos(\theta)$, so $d\theta = \frac{1}{\cos(\theta)}\,du$. Thus
$\int \csc^5(\theta)\cos(\theta)\,d\theta = \int \frac{\cos(\theta)}{\sin^5(\theta)}\,d\theta = \int \frac{\cos(\theta)}{u^5}\,\frac{1}{\cos(\theta)}\,du = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C =$
$-\frac{1}{4\sin^4(\theta)} + C = -\frac{1}{4}\csc^4(\theta) + C$.

Score: /3

d. $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$

If $x = \frac{4}{3}x$, then $\frac{dx}{du} = \frac{4}{3}$, so $dx = \frac{4}{3}\,du$. Thus $\int \frac{1}{\sqrt{16-9x^2}}\,dx = \int \frac{1}{\sqrt{16-9(\frac{4}{3}u)^2}}\,\frac{4}{3}\,du =$
$\int \frac{1}{\sqrt{16-16u^2}}\,\frac{4}{3}\,du = \int \frac{1}{4\sqrt{1-u^2}}\,\frac{4}{3}\,du = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1}(u) + C = \frac{1}{3}\sin^{-1}(\frac{3}{4}x) + C$.

Score: /3

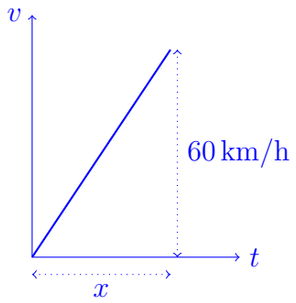
e. $\displaystyle\int \cot(x)\ln(\sin(x))\,dx$

If $u = \sin(x)$, then $\frac{du}{dx} = \cos(x)$, so $dx = \frac{1}{\cos(x)}\,du$. Therefore
$\int \cot(x)\ln(\sin(x))\,dx = \int \frac{\cos(x)}{\sin(x)}\ln(\sin(x))\,dx = \int \frac{\cos(x)}{u}\ln(u)\,\frac{1}{\cos(x)}\,du = \int \frac{1}{u}\ln(u)\,du$.
Now, if $v = \ln(u)$, then $\frac{dv}{du} = \frac{1}{u}$, so $du = u\,dv$. Therefore
$\int \frac{1}{u}\ln(u)\,du = \int \frac{1}{u}\,vu\,dv = \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}\big(\ln(u)\big)^2 + C = \frac{1}{2}\big(\ln(\sin(u))\big)^2 + C$.
You could also have substituted directly $w = \ln(\sin(x))$. Then $\frac{dw}{dx} = \frac{\cos(x)}{\sin(x)} = \cot(x)$, so $dx = \frac{1}{\cot(x)}\,dw$. Thus
$\int \cot(x)\ln(\sin(x))\,dx = \int \cot(x)w\,\frac{1}{\cot(x)}\,dw = \int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}\big(\ln(\sin(x))\big)^2 + C$.

Score: /4

Problem 6: Lily drives a distance of 20 km, accelerating uniformly from rest to 60 km/h. Graph her velocity versus time. How long does it take for her to reach her final speed?

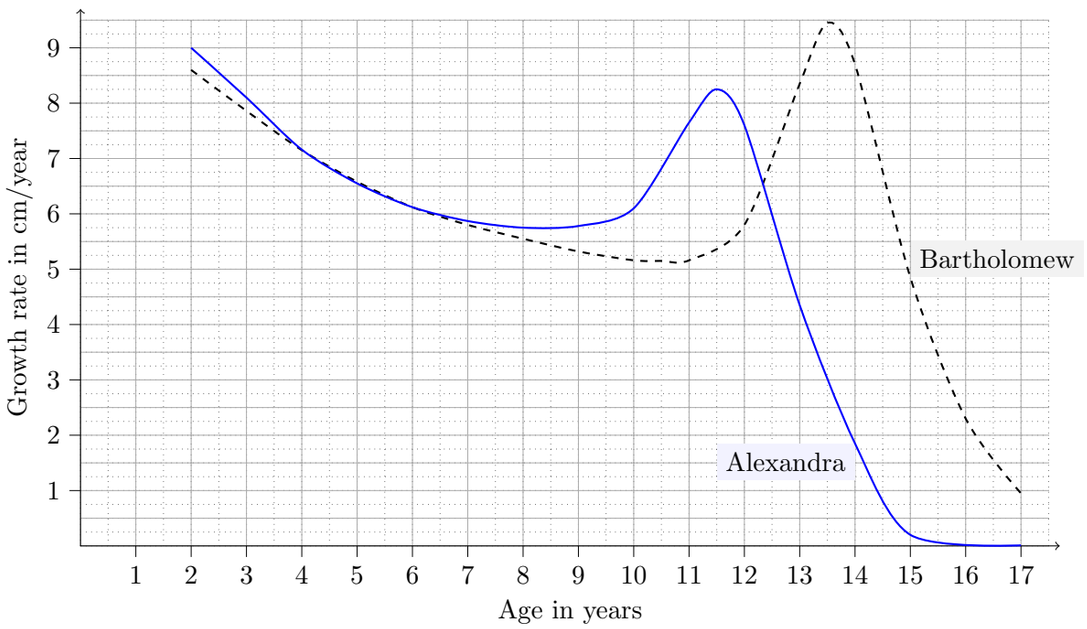
Since Lily's acceleration is uniform from rest, the v–t graph is a straight line through $(0, 0)$.



The area under the graph is the distance traveled, so 20 km. If the time it took is x, then the area is also $\frac{60x}{2} = 30x$, so $30x = 20$, so $x = \frac{2}{3}$ hour.

Score: /3

Problem 7: The graphs below show the height velocity curves of Alexandra and Bartholomew between ages 2 and 17. Assuming that Alexandra and Bartholomew attain average heights in Canada, answer the following questions.



a. Which curve represents Alexandra's? Explain how you can tell.

The solid one assuming Alexandra is a girl, that Batholomew is a boy, and that Alexandra therefore had her growth spurt first and gained less height.

b. Compare the start of Alexandra's growth spurt to the start of Bartholomew's. Give their ages before you compare them.

Alexandra's growth spurt began around age 10 while Bartholomew's began around age 12. Thus Alexandra began growing some two years before Bartholomew.

c. At the highest rate of growth, how many centimetres a year does Bartholomew grow?

About 9.5 cm/year.

Score: /3

Problem 8: The traffic flow rate past a certain point on a highway is

$$q(t) = 3000 + 2000t - 300t^2$$

(t in hours), where $t = 0$ is 8 AM. Use rectangular approximation method to estimate the number of cars passing by from 8 AM to 10 AM by filling the table below.

Approx	$n = 10$	$n = 100$	$n = 200$
L_n	8916	9172	9186
R_n	9476	9228	9214

What is the limit? Explicitly write out functions/program(s) you use or enter in your graphing calculator.

RAM program using $Y_1 = 3000 + 2000 * x - 3000 * x^2$, $A = 0$, $B = 2$, $n = 10, 100, 200$.
The total flow is $\text{FnInt}(Y_1, X, 0, 2) = 9200$ or
$\int_0^2 3000 + 2000t - 300t^2\,dt = 3000t + 1000t^2 - 100t^3\big|_0^2 = 9200$

Score: /4

Problem 9: Evaluate the following exactly using the Fundamental Theorem of Calculus.

a. Let $f(\theta) = \displaystyle\int_1^{2\theta} \cot(u)\,du$. Find $f'(\theta)$.

By the FTC, $f'(\theta) = 2\cot(\theta)$

Score: /1

b. Find $\dfrac{d}{dx}\displaystyle\int_{\sin(x)}^{\ln(x)} \frac{t}{1+t}\,dt$.

Let $f(x) = \int_0^x \frac{t}{1+t}\,du$. The $f'(x) = \frac{x}{1+x}$ by the FTC. Moreover, $\int_{\sin(x)}^{\ln(x)} \frac{t}{1+t}\,dt = \int_{\sin(x)}^{0} \frac{t}{1+t}\,dt + \int_0^{\ln(x)} \frac{t}{1+t}\,dt = -\int_0^{\sin(x)} \frac{t}{1+t}\,dt + \int_0^{\ln(x)} \frac{t}{1+t}\,dt = -f(\sin(x)) + f(\ln(x))$, so

$$\begin{aligned}\frac{d}{dx}\int_{\sin(x)}^{\ln(x)} \frac{t}{1+t}\,dt &= -f'(\sin(x)) \cdot \cos(x) + f'(\ln(x)) \cdot \frac{1}{x} \\ &= -\frac{\sin(x)}{1+\sin(x)} \cdot \cos(x) + \frac{\ln(x)}{1+\ln(x)} \cdot \frac{1}{x} \\ &= \frac{\ln(x)}{x(1+\ln(x))} - \frac{\sin(x)\cos(x)}{1+\sin(x)}\end{aligned}$$

Score: /2

c. $\dfrac{d}{dx}\displaystyle\int_e^{4\pi} \log(t^2+3t)\,dt$

Since $\int_e^{4\pi} \log(t^2+3t)\,dt$ does not depend on x (so it's a constant),

$$\frac{d}{dx}\int_e^{4\pi} \log(t^2+3t)\,dt = 0$$

Score: /1