Math 126
Fall 2015
Dr. Lily Yen

Test 3

Show all your work

Name: _____

Score: ___/44

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Determine the first five terms of the sequence starting with $n = 1$.

$$a_1 = 2, \quad a_2 = 1, \quad a_n = 2a_{n-1} - a_{n-2}.$$

$a_3 = 2a_2 - a_1 = 2 \times 1 - 2 = 0$, $a_4 = 2a_3 - a_2 = 2 \times 0 - 1 = -1$,
$a_5 = 2a_4 - a_3 = 2 \times (-1) - 0 = -2$:

$$2, 1, 0, -1, -2$$

Actually, $a_n = 3 - n$, but you were not asked or expected to find or prove that.

Score: /3

Problem 2: Solve the following initial value problem.

$$y'\sqrt{1-x^2} = y^2 + 1, \quad y(0) = 0.$$

If $\frac{dy}{dx}\sqrt{1-x^2} = y^2 + 1$, then $\int \frac{1}{y^2+1}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$, so $\tan^{-1}(y) = \sin^{-1}(x) + C$. Since $y(0) = 0$, $\tan^{-1}(0) = \sin^{-1}(0) + C$, so $0 = 0 + C$, so $C = 0$. Thus $\tan^{-1}(y) = \sin^{-1}(x)$, so

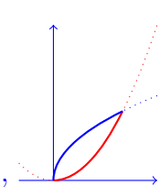
$$y = \tan(\sin^{-1}(x)).$$

If $\theta = \sin^{-1}(x)$, then $\sin(\theta) = x$. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, it follows that $\cos^2(\theta) = 1 - \sin^2(\theta) - 1 - x^2$, so $\cos(\theta) = \sqrt{1-x^2}$. Hence $y = \tan(\sin^{-1}(x)) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{x}{\sqrt{1-x^2}}$ is a simpler way of writing the same answer.

Score: /4

Problem 3: Express the arc length of $g(x) = \sqrt{x}$ over $[0, 1]$ as a definite integral. Is this integral an improper integral? If so, find another function with the same arc length whose integral is no longer improper. If not, explain why not. DO NOT EVALUATE.

The arc length is $\int_0^1 \sqrt{1 + (g'(x))^2}\,dx = \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \int_0^1 \sqrt{1 + \frac{1}{4x}}\,dx$, which is IMPROPER since the integrand is undefined at $x = 0$.



By symmetry, the arc length of $\sqrt{x}$ is the same as the arc length of x^2, That arc length is $\int_0^1 \sqrt{1 + ((x^2)')^2}\,dx = \int_0^1 \sqrt{1 + (2x)^2}\,dx = \int_0^1 \sqrt{1 + 4x^2}\,dx$, which is a proper integral which evaluates to $\frac{1}{2}\sqrt{5} - \frac{1}{4}\ln(\sqrt{5} - 2) \approx 1.479$.

Score: /4

Problem 4: Determine the limit of the sequence or show that it diverges.

a. $d_n = \sqrt{n+3} - \sqrt{n}$, for integers $n \geq 1$.

$d_n = \sqrt{n+3} - \sqrt{n} = \frac{\sqrt{n+3}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+3}+\sqrt{n}}{\sqrt{n+3}+\sqrt{n}} = \frac{\sqrt{n+3}^2-\sqrt{n}^2}{\sqrt{n+3}+\sqrt{n}} = \frac{n+3-n}{\sqrt{n+3}+\sqrt{n}} = \frac{3}{\sqrt{n+3}+\sqrt{n}} \to 0$ as $n \to \infty$.

Score: /4

b. $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

This is the well-known Harmonic Series. Now, since $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_1^b = \lim_{b\to\infty} \ln(b) = \infty$, it follows from Integral Test that h_n is divergent.

Score: /4

Problem 5: Determine whether the series converges conditionally, converges absolutely, or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$. Write out a few terms before you begin your analysis.

Since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \frac{1}{\sqrt{5}} - \cdots$$

is alternating and the terms decrease in size, the series converges.
Moreover, $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which is divergent by p-Test. Thus the original sum NOT absolutely convergent. Hence the series converges conditionally.

Score: /4

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{1}{3} + \frac{4}{9} + \frac{1}{3} + \frac{16}{81} + \frac{25}{243} + \cdots + a_n + \cdots$$

where $a_n = \frac{n^2}{3^n}$. Since, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2/3^{n+1}}{n^2/3^n} = \frac{(n+1)^2}{3n^2} = \frac{1}{3}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{3} < 1$ as $n \to \infty$, it follows from Ratio Test that the series converges absolutely.

Score: /4

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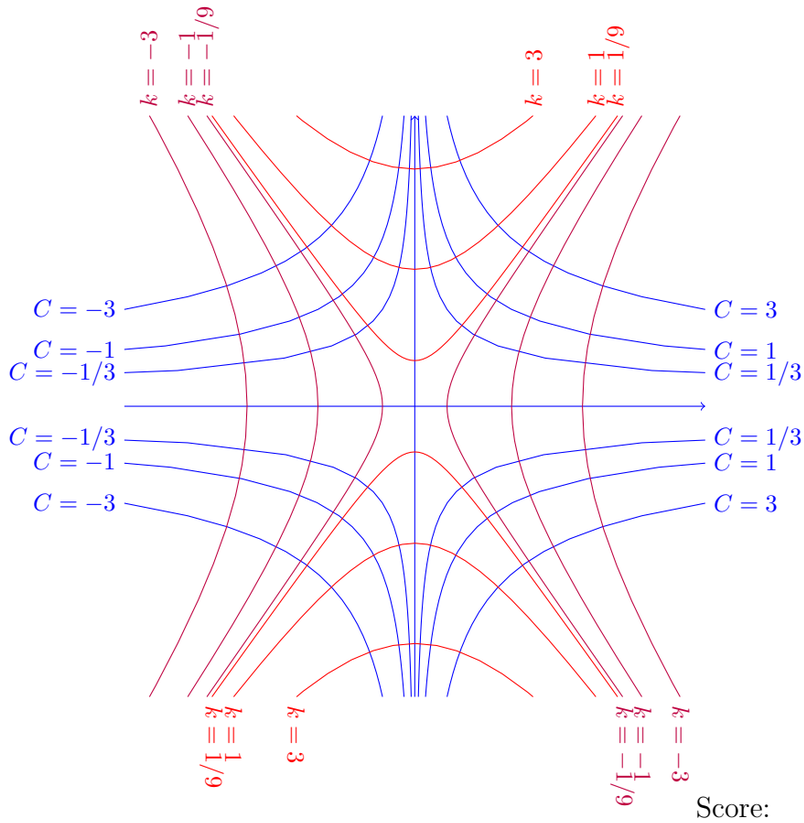
Calculators permitted in this part.

Problem 6: Find the orthogonal trajectories of the families of curves where C is a parameter. Sketch at least three members of each family on the same coordinate system. Label each curve with its corresponding parameter value.

$$xy^2 = C, \quad C \neq 0.$$



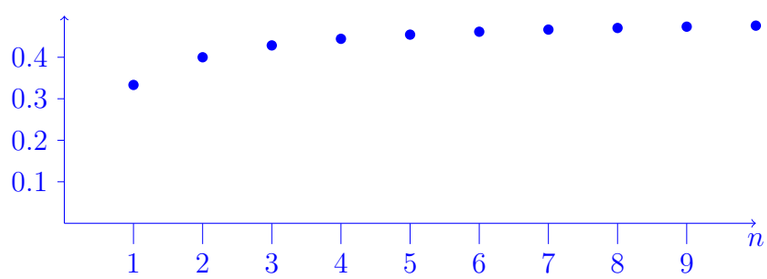
If $xy^2 = C$, then $y^2 + 2xyy' = 0$, so $y' = -\frac{y^2}{2xy} = -\frac{y}{2x}$. Therefore the orthogonal trajectories satisfy the equation $\frac{dy}{dx} = \frac{2x}{y}$, so $\int y\,dy = \int 2x\,dx$, so $\frac{1}{2}y^2 = x^2 + k$, so $y^2 = 2x^2 + 2k$, so $y = \pm\sqrt{2x^2 + 2k}$.

Score: /6

Problem 7: Find the sum of the series below by first graph the nth partial sum for the first 10 n's. State your function entry and window limits in your graphing calculator.

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \cdots$$

The denominator of each term in the summand is the product of consecutive odd integers.



Note that $\frac{1}{2i-1} - \frac{1}{2i+1} = \frac{2i+1}{(2i-1)(2i+1)} - \frac{2i-1}{(2i-1)(2i+1)} = \frac{2}{(2i-1)(2i+1)}$. Therefore $\frac{2}{1\times 3} + \frac{2}{3\times 5} + \frac{2}{5\times 7} + \frac{2}{7\times 9} + \cdots = \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{2n+1} = 1 - \frac{1}{2n+1} \to 1$ as $n \to \infty$. Thus the infinite sum $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \cdots = \frac{1}{2}$.

Score: /5

Problem 8: A pool holds 5000 litres of pure water. A tap is opened and a solution containing 0.2 g of chlorine per litre is fed into the pool for disinfection at a rate of 2 litres per minute. At the same time, a drain plug in the bottom of the pool is opened and the thoroughly mixed solution leaves at the same rate. How much chlorine is in the pool after two hours?

If $y(t)$ is the amount of chlorine in the pool after t minutes, then the concentration is $\frac{y}{5000}$. Thus 0.4 g of chlorine enters the pool per minute and $2 \times \frac{y}{5000} = \frac{y}{2500}$ grams leave. Thus

$$\frac{dy}{dt} = 0.4 - \frac{y}{2500} = \frac{1000 - y}{2500}$$

Therefore $\int \frac{1}{1000-y}\, dy = \int \frac{1}{2500}\, dt$, so $-\ln(1000 - y) = \frac{t}{2500} + C$, so $\ln(1000 - y) = -\frac{t}{2500} - C$, so $1000 - y = e^{-t/2500-C} = Ae^{-t/2500}$, where $A = e^{-C}$. Hence $y = 1000 - Ae^{-t/2500}$, and since $0 = y(0) = 1000 - Ae^0 = 1000 - A$, it follows that $A = 1000$.
The amount of chlorine after two hours is then $y(120) = 1000 - 1000e^{-120/2500} = 46.9$ g.

Score: /6