Math 126
Fall 2015
Dr. Lily Yen

Test 2

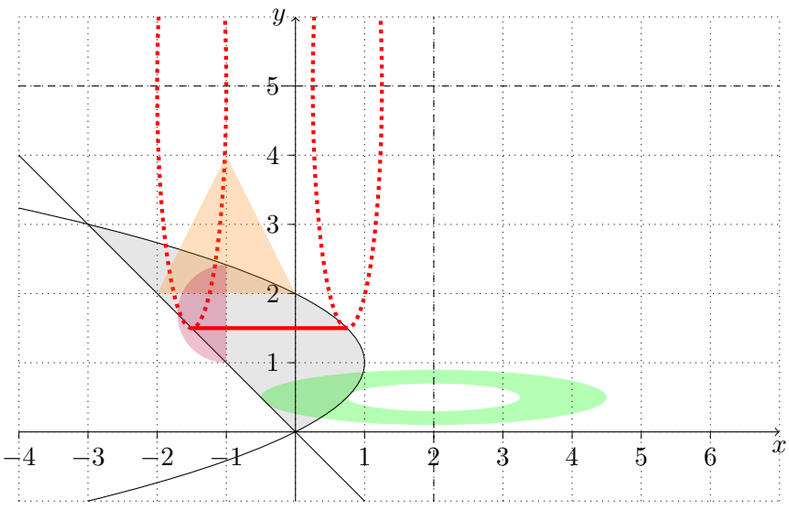
Show all your work

Name: _____

Score: ___/47

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: First draw $x + y^2 = 2y$ and $x + y = 0$. Shade the region enclosed by these two graphs.



Score: /2

Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

If $x + y^2 = 2y$ and $x + y = 0$, then $-y + y^2 = 2y$, so $y^2 = 3y$, so $y = 0$ or $y = 3$. Thus the intersection points are $(0, 0)$ and $(-3, 3)$. The area is then $\int_0^3 (2y - y^2) - (-y)\, dy = \int_0^3 3y - y^2\, dy$ which is easy to evaluate to $\frac{9}{2}$.

Score: /2

b. The volume obtained if the region is rotated about $y = 5$.

$\int_0^3 2\pi(5 - y)(3y - y^2)\, dy$ which is easy to evaluate to $\frac{63}{2}\pi$.

Score: /2

c. The volume obtained if the region is rotated about $x = 2$.

$\pi \int_0^3 (2 + y)^2 - (2 - 2y + y^2)^2\, dy$ which is easy to evaluate to $\frac{117}{5}\pi$.

Score: /2

d. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the y-axis are equilateral triangles.

$\frac{\sqrt{3}}{4} \int_0^3 (3y - y^2)^2\, dy$ which is easy to evaluate to $\frac{81}{40}\sqrt{3}$.

Score: /2

e. The volume of a solid with the shaded region as its base, and cross-sections perpendicular to the x-axis are semi-circles.

$\frac{\pi}{8} \int_{-3}^0 (1 + \sqrt{1 - x} + x)^2\, dx + \frac{\pi}{2} \int_0^1 1 - x\, dx$ which is easy to evaluate to $\frac{191}{240}\pi$.

Score: /3

Problem 2: Evaluate the following integrals exactly.

a. $\displaystyle\int t^3 e^{t^2}\,dt$

Use first a substitution $u = t^2$, then integration by parts:
$\int t^3 e^{t^2}\,dt = \int t^3 e^u \frac{1}{2t}\,du = \frac{1}{2}\int t^2 e^u\,du = \frac{1}{2}\int ue^u\,du = \frac{1}{2}ue^u - \frac{1}{2}\int 1\cdot e^u\,du =$
$\frac{1}{2}ue^u - \frac{1}{2}e^u + C = \frac{1}{2}(u-1)e^u + C = \frac{1}{2}(t^2-1)e^{t^2} + C$

Score: /3

b. $\displaystyle\int \sin(x)e^{3x}\,dx$

Using integration by parts twice,
$\int \sin(x)e^{3x}\,dx = \frac{1}{3}\sin(x)e^{3x} - \frac{1}{3}\int \cos(x)e^{3x}\,dx = \frac{1}{3}\sin(x)e^{3x} -$
$\frac{1}{3}\left(\frac{1}{3}\cos(x)e^{3x} + \frac{1}{3}\int \sin(x)e^{3x}\,dx\right) = \frac{1}{3}\sin(x)e^{3x} - \frac{1}{9}\cos(x)e^{3x} - \frac{1}{9}\int \sin(x)e^{3x}\,dx.$
Therefore $\frac{10}{9}\int \sin(x)e^{3x}\,dx = \frac{1}{3}\sin(x)e^{3x} - \frac{1}{9}\cos(x)e^{3x}$, so
$\int \sin(x)e^{3x}\,dx = \frac{3}{10}\sin(x)e^{3x} - \frac{1}{10}\cos(x)e^{3x} + C$

Score: /4

c. $\displaystyle\int \frac{x-3}{(2x-1)(x+5)}\,dx$

Use the method of partial fractions: If $\frac{x-3}{(2x-1)(x+5)} = \frac{A}{2x-1} + \frac{B}{x+5}$, then
$x - 3 = A(x+5) + B(2x-1)$. Substituting -5 for x yields that $-8 = -11B$, so
$B = \frac{8}{11}$. Substituting $\frac{1}{2}$ for x yields that $-\frac{5}{2} = \frac{11}{2}A$, so $A = -\frac{5}{11}$.
Thus $\int \frac{x-3}{(2x-1)(x+5)}\,dx = \int \frac{-5/11}{2x-1} + \frac{8/11}{x+5}\,dx = -\frac{5}{22}\ln|2x-1| + \frac{8}{11}\ln|x+5| + C.$

Score: /4

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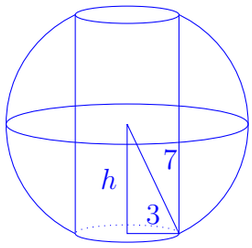
Calculators permitted from here on.

Problem 3: Find the volume of a solid remaining after drilling a cylindrical hole of radius 3 cm out of a sphere of radius 7 cm. Give 4 decimal place accuracy.

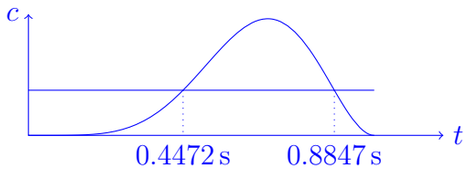


By the Pythagorean Theorem, $h = 2\sqrt{10}$ (see the figure). At height y, the horizontal cross section is an annulus with inner radius 3 and outer radius r, where $7^2 = y^2 + r^2$, so $r^2 = 49 - y^2$. The volume is therefore $\pi \int_{-h}^{h} r^2 - 3^2\, dy = \pi \int_{-h}^{h} 40 - y^2\, dy = \pi(40y - \frac{1}{3}y^3)\big|_{-h}^{h} = \pi(80h - \frac{2}{3}h^3) = \pi(80 - \frac{2}{3}h^2)h = \pi(80 - \frac{80}{3})2\sqrt{10} = \frac{320}{3}\pi\sqrt{10} \approx 1060.$

Score: /4

Problem 4: The cardiac output of the heart is the volume of blood pumped by the heart per unit time. Cardiac output, in litres per second can be modelled over a one-second heart beat as $c(t) = \dfrac{3t^3}{2}\sin^2(\pi t)$. Find the average cardiac output over a one-second heart beat accurate to 4 decimal places. Draw $c(t)$ and locate specific time(s) also accurate to 4-decimal places when the average cardiac output equals to the function value.

The average is $\frac{1}{1}\int_0^1 c(t)\, dt = \int_0^1 \frac{3t^3}{2}\sin^2(\pi t)\, dt = \frac{3(\pi^2-3)}{16\pi^2} \approx 0.1305\,\text{L/s}$.



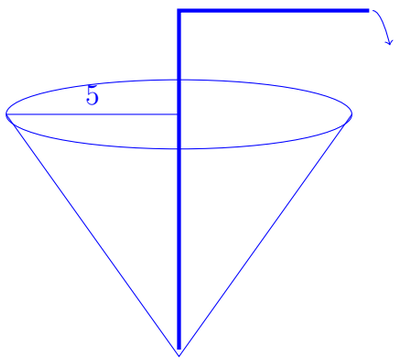
Score: /4

Problem 5: A spring has a natural length of 0.9 m. A force of 15 N stretches the spring to a length of 1.4 m. Find the work required to stretch the spring 2 m beyond its natural length.

Since $F = kx$, the spring constant is $k = \frac{15\,\text{N}}{1.4\,\text{m} - 0.9\,\text{m}} = 30\,\text{N/m}$. The work is $\int_0^2 F\, dx = \int_0^2 kx\, dx = \frac{1}{2}kx^2\big|_0^2 = 2k = 60\,\text{J}$.

Score: /3

Problem 6: An inverted conical tank with height 7 m and top diameter across 10 m is filled with water with density of 1000 kg/m^3. How much work is done to pump the water out a valve 3 m above the top of the tank until the tank has a depth of 2 m?



If the cone of water has radius r and height y, then $\frac{r}{y} = \frac{5}{7}$, so $r = \frac{5}{7}y$. The slice of water at height y then has volume $\pi r^2\,dy = \frac{25}{49}\pi y^2\,dy$. If the acceleration due to gravity is 9.81 m/s^2, then the required work is $\int_2^7 \frac{25}{49}\pi y^2 \cdot 1000g(10-y)\,dy = 8.18$ MJ.

Score: /5

Problem 7: Integrate the following exactly. $\displaystyle\int \cot^3(5\theta)\csc^2(5\theta)\,d\theta$

Let $u = \cot(5\theta)$. Then $\frac{du}{d\theta} = -\csc^2(5\theta)\cdot 5 = -5\csc^2(5\theta)$, so $du = -5\csc^2(5\theta)\,d\theta$, so $d\theta = \frac{-1}{5\csc^2(5\theta)}$. Therefore
$\int \cot^3(5\theta)\csc^2(5\theta)\,d\theta = \int u^3 \cdot \frac{-1}{5}\,du = -\frac{1}{20}u^4 + C = -\frac{1}{20}\cot^4(5\theta) + C$.

Score: /3

Problem 8: Circle improper integrals below. State a brief reason next to each item.

a. $\displaystyle\int_{2\pi}^{100\pi} x\cos(x)\,dx$ PROPER: always defined

b. $\displaystyle\int_1^{10} \frac{dx}{x^2 - x - 6}$ IMPROPER: undefined at $x = 3$

c. $\displaystyle\int_0^1 3x\ln x\,dx$ IMPROPER: undefined at $x = 0$

d. $\displaystyle\int_{-50}^{50} \frac{3x}{(x^2+100)^3}\,dx$ PROPER: always defined

Score: /4