Math 126
Fall 2015
Dr. Lily Yen

Test 1

Show all your work

Name: _____

Score: ___/50

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Express the limit as a definite integral.

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\sin\left(5+\frac{(\pi-3)k}{n}\right)\frac{\pi-3}{n}\right)=\int_0^{\pi-3}\sin(5+x)\,dx=\int_5^{\pi+2}\sin(x)\,dx$$

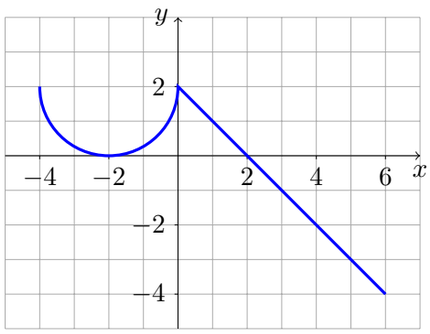
This is the Riemann sum evaluated at the right-hand endpoint.

Score: /3

Problem 2: Consider the following piece-wise defined function.

$$f(x)=\begin{cases}2-\sqrt{4-(x+2)^2}, & -4\le x<0\\ 2-x, & 0\le x\le 6\end{cases}$$

Draw f in the grid provided and answer the following questions.



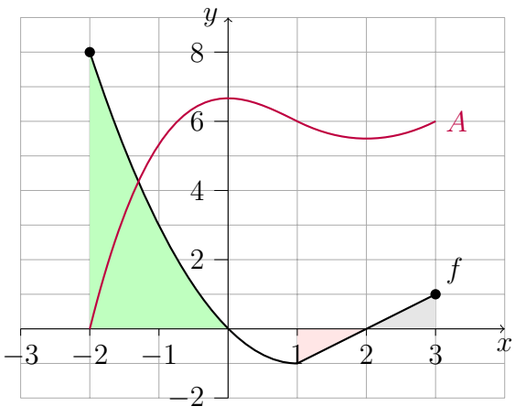
a. $\displaystyle\int_{-4}^{0} f(t)\,dt$ — $8-2\pi$

b. $\displaystyle\int_{-2}^{3} |f(x)|\,dx$ — $\frac{13}{2}-\pi$

c. $\displaystyle\left|\int_0^6 f(u)\,du\right|$ — 6

Score: /8

Problem 3: Define $A(x) = \int_{-2}^{x} f(t)\,dt$ for $x \in [-2, 3]$, where the function f is graphed below.



a. Evaluate $A(3) - A(1) =$ $\boxed{0}$.

The area of the region between the x-axis and the graph of f under $[1, 2]$ equals the area of the region over $[2, 3]$. These two areas therefore cancel each other and $A(1) = A(3)$.

b. Determine the x-value(s) in $[-2, 3]$ where $A(x)$ attains the global maximum.

By the FTC, $A'(x) = f(x)$. Therefore

x :	-2		0		2		3
$A' = f$:		$+$	0	$-$	0	$+$	
A :		$\nearrow$		$\searrow$		$\nearrow$	

. Since $A'(x) = f(x) \le 0$ when $x \in [0, 1]$, it follows that A is decreasing on $[0, 1]$, so $A(0) > A(1) = A(3)$. Thus A attains its absolute maximum at $x = 0$ and only there.

c. Determine the x-value(s) in $[-2, 3]$ where $A(x)$ attains the global minimum.

The area of the region between the x-axis and the graph of f over $[-2, 0]$ is much larger than the area of the region under $[0, 2]$, so $A(-2) < A(2)$, so A attains its global minimum at $x = -2$.

d. How do you classify $x = 2$ in the graph of A? Circle all that apply.

INFLECTION PNT. LOCAL MAX **LOCAL MIN** GLOBAL MAX GLOBAL MIN

The table above shows that A has a local minimum at $x = 2$.
Since $A' = f$ is decreasing on $[-2, 1]$ and increasing on $[1, 3]$, it follows that A has an inflection point at $x = 1$.

Score: /4

Problem 4: Evaluate the following exactly using the Fundamental Theorem of Calculus.

a. Let $g(x) = \int_{2}^{x^2} \frac{1-t}{1+t}\,dt$. Find $g'(x)$.

Let $f(x) = \int_2^x \frac{1-t}{1+t}\,dt$. Then $f'(x) = \frac{1-x}{1+x}$ by the FTC. Moreover, $g(x) = f(x^2)$, so

$$g'(x) = f'(x^2) \cdot 2x = \frac{2x(1-x^2)}{1+x^2}.$$

Score: /1

b. Find $\frac{d}{dx} \int_{\cos(x)}^{2x^3} \tan(u)\,du$.

Let $f(x) = \int_0^x \tan(u)\,du$. The $f'(x) = \tan(x)$ by the FTC. Moreover, $\int_{\cos(x)}^{2x^3} \tan(u)\,du = \int_{\cos(x)}^{0} \tan(u)\,du + \int_0^{2x^3} \tan(u)\,du =$ $-\int_0^{\cos(x)} \tan(u)\,du + \int_0^{2x^3} \tan(u)\,du = -f(\cos(x)) + f(2x^3)$, so

$$\begin{aligned} \frac{d}{dx} \int_{\cos(x)}^{2x^3} \tan(u)\,du &= -f'(\cos(x)) \cdot (-\sin(x)) + f'(2x^3) \cdot 6x^2 \\ &= -\tan(\cos(x)) \cdot (-\sin(x)) + \tan(2x^3) \cdot 6x^2 \\ &= \tan(\cos(x))\sin(x) + 6x^2 \tan(2x^3) \end{aligned}$$

Score: /2

Problem 5: Integrate the following analytically.

a. $\displaystyle\int \frac{(x^2-\sqrt{x})^2}{x^3}\,dx$

$\int \frac{(x^2-\sqrt{x})^2}{x^3}\,dx = \int \frac{x^4-2x^{5/2}+x}{x^3}\,dx = \int x - 2x^{-1/2} + x^{-2}\,dx = \frac{1}{2}x^2 - 4x^{1/2} - x^{-1} + C = \frac{1}{2}x^2 - 4\sqrt{x} - \frac{1}{x} + C.$

Score: /3

b. $\displaystyle\int (2-7x)^6\,dx$

If $u - 2 - 7x$, then $\frac{du}{dx} = -7$, so $du = -7\,dx$, so $dx = -\frac{1}{7}\,du$. Therefore
$\int (2-7x)^6\,dx = \int u^6(-\frac{1}{7})\,du = -\frac{1}{49}u^7 + C = -\frac{1}{49}(2-7x)^7 + C.$

Score: /2

c. $\displaystyle\int \cot^3(5\theta)\csc^2(5\theta)\,d\theta$

Let $u = \cot(5\theta)$. Then $\frac{du}{d\theta} = -\csc^2(5\theta)\cdot 5 = -5\csc^2(5\theta)$, so $du = -5\csc^2(5\theta)\,d\theta$, so $d\theta = \frac{-1}{5\csc^2(5\theta)}$. Therefore
$\int \cot^3(5\theta)\csc^2(5\theta)\,d\theta = \int u^3 \cdot \frac{-1}{5}\,du = -\frac{1}{20}u^4 + C = -\frac{1}{20}\cot^4(5\theta) + C.$

Score: /3

d. $\displaystyle\int_{-1}^{1} 3^x e^{7x}\,dx$

Since $3^x = e^{\ln(3^x)} = e^{x\ln 3}$, it folows that $3^x e^{7x} = e^{x\ln 3}\,e^{7x} = e^{(7+\ln 3)x}$. Therefore
$\int_{-1}^{1} 3^x e^{7x}\,dx = \int_{-1}^{1} e^{(7+\ln 3)x}\,dx = \frac{1}{7+\ln 3}\,e^{(7+\ln 3)x}\Big|_{-1}^{1} = \frac{1}{7+\ln 3}\,3^x e^{7x}\Big|_{-1}^{1} = \frac{1}{7+\ln 3}(3e^7 - \frac{1}{3e^7})$

Score: /3

Math 126
Fall 2015
Dr. Lily Yen

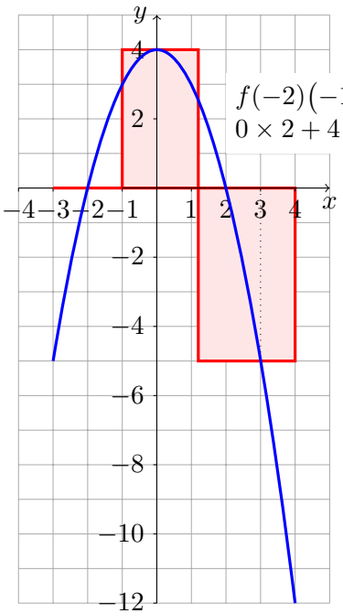
Test 1

Show all your work

Name: _____

Calculators permitted from here on.

Problem 6: Given the function $f(x) = 4 - x^2$ with a specific partition $P = \{-3, -1, 1.2, 4\}$ and its corresponding sample points $C = \{-2, 0, 3\}$, compute its Riemann sum. You need to draw the graph and its approximating rectangles, then write out each term in the sum before using the calculator to find the answer.



$f(-2)\big(-1-(-3)\big) + f(0)\big(1.2-(-1)\big) + f(3)\big(4-1.2\big) =$
$0 \times 2 + 4 \times 2.2 + (-5) \times 2.8 = -5.2$

Score: /3

Problem 7: Estimate the distance travelled from 2 seconds after the start of experiment to 6 seconds after the start by a particle with velocity (in m/s)

$$v(t) = \ln(t^2 + 1), \quad t \geq 0.$$

Fill the table below accurate to 4 decimal places.

Approx	$n = 10$	$n = 100$	$n = 200$
L_n	10.6370	11.0035	11.0236
R_n	11.4376	11.0836	11.0636

What is the limit of the distance (again accurate to 4 decimal places)?

The distance travelled is $\int_2^6 v(t)\,dt = \int_2^6 \ln(t^2+1)\,dt$. This can be approximated by left and right Riemann sums. The limit value is 11.0436 m.

Score: /4

Problem 8: If the acceleration of an object is given by $a(t) = 2t^2 - 3t$ (in m/s^2), with an initial velocity of 3 m/s. Find the displacement over the interval $[1, 4]$. Does it matter if you do not know the initial position? Explain.

If $\frac{dv}{dt} = a(t) = 2t^2 - 3t$, then $v(t) = \int 2t^2 - 3t\,dt = \frac{2}{3}t^3 - \frac{3}{2}t^2 + C_1$. Since $3 = v(0) = C_1$, it follows that $v(t) = \frac{2}{3}t^3 - \frac{3}{2}t^2 + 3$.
Thus the position at time t is $s(t) = \int v(t)\,dt = \int \frac{2}{3}t^3 - \frac{3}{2}t^2 + 3\,dt = \frac{1}{6}t^4 - \frac{1}{2}t^3 + 3t + C_2$.
Since $s(4) = \frac{68}{3} + C_2$ and $s(1) = \frac{8}{3} + C_2$, the displacement is $s(4) - s(1) = \frac{60}{3} = 20$ m.
As you can see the value of C_2 is irrelevant.

Score: /4

Problem 9: Find the following definite integrals analytically.

a. $\displaystyle\int_0^{\sqrt{2}} x\sqrt{4-x^4}\,dx$

If $x=\sqrt{2u}$, then $\frac{dx}{du}=\frac{2}{2\sqrt{2u}}=\frac{1}{\sqrt{2u}}$, so $dx=\frac{1}{\sqrt{2u}}\,du$. Therefore $\int_0^{\sqrt{2}} x\sqrt{4-x^4}\,dx=\int_0^1 \sqrt{2u}\sqrt{4-4u^2}\frac{1}{\sqrt{2u}}\,du=\int_0^1 2\sqrt{1-u^2}\,du=2\int_0^1\sqrt{1-u^2}\,du$, which is twice the area of the quarter circle with radius 1 (centred at $(0,0)$), so $2\times\frac{1}{4}\pi=\frac{1}{2}\pi$.

Score: /3

b. $\displaystyle\int_0^5 \frac{dx}{9x^2+25}$

If $x=\frac{5}{3}u$, then $\frac{dx}{du}=\frac{5}{3}$, so $dx=\frac{5}{3}\,du$. Thus $\int_0^5\frac{dx}{9x^2+25}=\int_0^3\frac{1}{9(\frac{5}{3}u)^2+25}\frac{5}{3}\,du=\frac{5}{3}\int_0^3\frac{1}{25u^2+25}\,du=\frac{1}{15}\int_0^3\frac{1}{u^2+1}\,du=\frac{1}{15}\arctan(u)\Big|_0^3=\frac{1}{15}\arctan(3)\approx 0.0833$

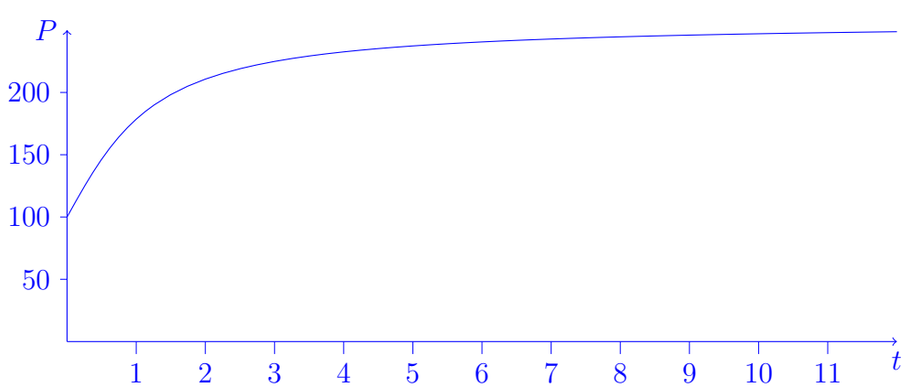
Score: /3

Problem 10: The population of squirrels grows at the rate of $\dfrac{100}{1+t^2}$ squirrels per month since the beginning of the population model.

a. Find the population increase, to the nearest squirrel, from month 3 to month 11.

The number of squirrels is $P(t)=\int\frac{100}{1+t^2}\,dt=100\arctan(t)+P_0$, where P_0 truly is the population at time 0 since $\arctan(0)=0$.
The increase is then $P(11)-P(3)=100\arctan(11)-100\arctan(3)=23.1091\approx 23$.

b. Draw the graph of the population of squirrels for the first year assuming 100 as initial population.



Score: /4