
Mathematics Between Friends

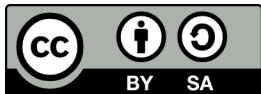
Lily Yen

Summer 2024 Edition

Capilano University

Copyright © 2022 by Lily Yen.

Mathematics & Statistics Department, Capilano University, 2055 Purcell Way, North Vancouver, BC, V7J 3H5, Canada



This work is licensed under the Creative Commons Attribution-ShareAlike 4.0 International License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-sa/4.0/>.

In memory of A-máⁿ
for her stories

To my mother
for sharing her love of mathematics



Contents

Contents	iii
List of Figures	ix
List of Tables	xi
Foreword	xiii
Preface	xvi
Acknowledgement	xviii
Introduction to the Student	xx
Introduction to the Instructor	xxiii
1 Solving Puzzles	1
1.1 Intuition and logic	1
Continuity—Introducing Anjali, Katharina, and Mei	1
Deductive reasoning—Meeting at orientation	5

1.2	Exploration	7
	Patterns—Shaking hands	7
	Variables—Picking berries	8
	Inductive reasoning—Sharing a chocolate bar	9
1.3	Tools	10
	Tree diagrams—Making a burger	10
	Patterns and connections—Building towers and shapes	11
	Organization—Seeing squares	12
	Permutations—Building a totem	13
1.4	Brave attempts	15
	Differences—Observing details	15
	Picture proofs—Visual reasoning	16
1.5	Exercises	24
	Deductive reasoning	24
	Inductive reasoning	29
	Variables	32
	Tree diagrams	33
2	Counting and Coding	37
2.1	Simple numeration systems	37
	Nonpositional versus positional—Introducing Fire Horse and her A-má ⁿ	37
2.2	Different systems, old and new—Meeting A-má ⁿ 's friends	38
	Egyptian numerals	39
	Roman numerals	40
	Incan quipu	42
	Mayan numerals	43
	Babylonian numerals	45
	Kaktovik numerals	47
2.3	Counting in other bases—Thinking deeply, alone	50
	Venturing into other bases	50
	Counting	51
	Converting back and forth	52
	Adding	53

	Subtracting	54
	Multiplying	55
	Converting between binary, octal, and hexadecimal systems	55
2.4	Coding—Encountering seven Draggoes in a Donegeon	57
	Silencegone	57
	Flightgone	58
	Rushgone	58
	Paingone	59
	Aquagone	60
	Tumblegone	61
	Beatgone	62
2.5	Tracking remainders—Accounting for seven Ladies with seven Draggoes	64
2.6	Exercises	67
	Counting rods	67
	Egyptian numerals	68
	Roman numerals	68
	Mayan numerals	69
	Babylonian numerals	71
	Kaktovik numerals	74
	Different bases	75
	Binary, octal, hexadecimal conversions	76
	Mixed systems	77
	Galley method	77
	Modular arithmetic	80
3	Measuring and tracking	84
3.1	Comparison of angles—Arrival of the stars	84
	Geometric elements—Introducing Pythagoras, Euclid, and Archimedes	84
	Euclid’s discourse on angles	85
3.2	Polygons and similarity—Welcoming party	91
	Pythagoras’s polygonal angle sums	91
	Pythagorean Theorem and beyond	92
3.3	Area and Volume—Archimedes’s proudest moment	94

	Areas	95
	Volumes	96
	Archimedes's proudest <i>Eureka!</i>	97
3.4	Origami symmetry—Archimedes at play	99
	Reflection from nature	99
	Rotation and translation on pots and flowers	100
3.5	GPS in two dimensions—Ancients with a modern toy	101
3.6	Exercises	103
	Angles	103
	Stars	110
	Polygons	111
	Pythagorean Theorem	114
	Areas	118
	Circles	125
	Volumes	127
	Symmetries	134
4	Saving, spending, borrowing, giving	141
4.1	Saving	141
	Fraction, percent, decimal—Introducing Brian, David, and Janette	141
	Percent/Decimal equation—Nomads Club	143
4.2	Spending	145
	Simple interest—Cost of living	145
	Compound interest—David's TFSA	148
4.3	Borrowing	153
	Loans or leases—chains and shackles	153
	Credit/Debit cards—plastic temptations	154
	Line of credit—a slow descent	158
	Mortgage or rent—saving or spending	160
4.4	Unintentional Giving	161
4.5	Exercises	163
	Fraction, decimal, and percent	163
	Currency conversion	170

Interest	171
Credit cards and loans	174
5 Predicting—Hamlet’s dilemma: To bet or not to bet	178
5.1 Probability tree—Hamlet’s games	178
Choosing one of two	178
Rolling two dice	180
Drawing from a bag	182
Tossing coins and rolling dice	184
5.2 Combinations of events—Hamlet’s and Samlette’s challenges	184
Complements	184
Unions and intersections	186
Conditionals	190
5.3 Tables and Trees—Hamlet and Samlette building	194
5.4 Phylogenetics to flu shot designs—Mutating viruses	197
5.5 Exercises	200
Theoretical probability	200
Empirical probability	212
Venn diagrams	216
Games	219
6 Understanding Data	221
6.1 Organizing and visualizing data	221
Introducing Florence and Hans	221
Creating the Virtualverse full of data	222
Tally and frequency	223
Stem-and-leaf plot and bar graphs	225
Line graphs	228
Pie charts	230
Multibar and stacked bar graphs	231
6.2 Describing data—centres and spread	232
Helping instructors in the Virtualverse	232
Centres—mean, median, and mode	233

Dispersion	237
6.3 Making sense of big data—taking advantage of open source	240
6.4 Exercises	241
Tally and frequency tables	241
Line graphs	244
Pie charts	245
Multibar and stacked bar graphs	246
Centres	250
Dispersion	255
A Solutions to Selected Exercises	263
Solutions for Chapter 1	263
Solutions for Chapter 2	273
Solutions for Chapter 3	286
Solutions for Chapter 4	302
Solutions for Chapter 5	311
Solutions for Chapter 6	321
Bibliography	331
Index	334

List of Figures

1.1	Many ways to slice the piece of ham	3
1.2	Sliced ham sandwich	4
1.3	Berries divided	8
2.1	Close up of the top of Galleria Giuseppe Siri	41
2.2	Quipu	42
2.3	Mayan corn	45
2.4	Conversion of 168 to base 2	53
2.5	Galley method for $234_5 \times 23_5$	55
3.1	Whitetail doe, Orchid blossom, and Cardinal	100
3.2	Otomipot, Tensionpot, and Hollow rose	101
3.3	GPS pinned by 4 satellites from GISGeography	102
3.4	Eulid's seven-point star	111
3.5	A heart shaped prism	133
3.6	Five cylinder parts reassembled	133
3.7	A grid for your 3d drawings	140
5.1	Two dice showing the same or not the same	185
5.2	A full deck of 52 cards	188

5.3	Drawing a \diamond or not a \diamond	190
5.4	A virus model	195
5.5	Probability tree for Lego blocks	197
5.6	Testing for the flu	200
6.1	A stem-and-leaf plot of ages	226
6.2	People with war trauma by age groups in 4 histograms	227
6.3	Line graph showing usage of the Counselling/Wellness Centre	229
6.4	Pie charts for types of counselling in 2019 (left) and 2020	230
6.5	Age/type visits to Counselling Service	232
6.6	Class age distribution table	235
6.7	Minimum wage age distribution histogram	236
6.8	Minimum wage by province effective October 2022	237
6.9	Income per month in US \$	239
6.10	Quiz score distribution	252
6.11	Class size distribution	253
6.12	A stem-and-leaf diagram of staff contacts	254
6.13	Visitors recorded for one week in May	257
A.1	Evenly divided Yin-Yang symbol	264
A.2	$(n - 1)(n + 1) = n^2 - 1$	266
A.3	The probability tree for Ham's game of red and teal balls	316

List of Tables

2.1	Kaktovik numerals	48
2.2	Base-8 counting	51
2.3	Kaktovik multiplication table	83
4.1	Earning 6% per year for ten years	149
4.2	Average daily balance	156
4.3	Fractions, decimals, and percent	166
4.4	Interest, principal, rate, and time	171
4.5	Future value, principal, rate, and time	172
5.1	The Sample space for a pair of dice showing 36 outcomes	181
5.2	Lego blocks by shape and colour	196
6.1	Frequency table for wars by continent	225
6.2	Wars and ages of trauma survivors	226
6.3	Visits to the Counselling/Wellness Centre per year	229
6.4	Breakdown of visits to Counselling Service	230
6.5	Age 25– and age 26+ visiting Counselling/Wellness Centre	231
6.6	Test scores	255
6.7	Quiz scores	256

6.8	Wait times at CapU's clinic	258
A.1	Fractions, decimals, and percent	305
A.2	Future value, principal, rate, and time	308
A.3	A frequency table for contacts	328

Foreword

“Mathematics” and “Friends”—a truly unique pairing! It takes a mathematician and mathematics educator of Dr. Lily Yen’s calibre to skilfully develop this pairing that is usually absent in a student’s study of mathematics. Mathematics can, and should, be accessible and shared between friends. *Mathematics Between Friends* is a most welcome addition to mathematics education, where Lily introduces and develops mathematical concepts via engaging narratives between friends, including you!

“Chance” is a major topic in Chapter 5, and it was “chance” that brought us together as colleagues at Capilano University. By chance Lily came by the Mathematics & Statistics Department in year 2000 to drop off her CV, and by chance we had just opened a one-class teaching position. Lily’s impressive mathematical background and demonstrated teaching skills easily resulted in her being successful in our competition for the position. Quickly her appointment status at Capilano expanded to include teaching the spectrum of our post secondary mathematics courses. Until my retirement as Coordinator of Mathematics & Statistics in 2005, I was constantly amazed by Lily’s creativity, dedication, and skill in her instructional assignments.

During my tenure at Capilano, Lily also volunteered to lead many

enrichment activities—from presenting talks and workshops for advanced high school students to serving as a Mathematics Olympiad coach and summer camp speaker for the Canadian Mathematics Society. Remarkably energetic, she challenged herself and others with new ideas and approaches in studying mathematics. As an accomplished classical musician, she enjoyed exploring topics that combined music and math.

In *Mathematics Between Friends*, Lily’s creativity and breadth of topic development is amazing. For example in Chapter 2 on *Counting and Coding*, Lily threads an engaging narrative of *Fire Horse* as she encounters different counting systems from diverse cultures of the world. Not just superficially engaging, Lily’s development in the narrative feels authentic, injecting at times current controversies relevant to the culture. The coding component of the chapter is equally engaging as *Fire Horse* navigates the domain of seven different Dragons, each posing a different coding scheme. The examples are interesting and the links provide a necessary, rich resource. The chapter concludes with exercises that are both accessible and challenging with direct linkage back to the narrative examples.

As the long time Coordinator of the Mathematics & Statistics Department, how could I not jump for joy as Pythagoras, Euclid and Archimedes chose Capilano for their time-travel sabbatical imagined by Lily as she introduces *Measures and Tracking* in Chapter 3. Again, Lily has used an engaging narrative to develop the central ideas of classical geometry with numerous links to explore the ideas more fully.

In Chapter 4, *Saving, Spending, Borrowing*, Lily begins a new narrative with three Capilano students exploring personal finance concepts. She skilfully localizes the concepts with appropriate references to current environmental issues and applications.

Probability is a fun and relevant mathematical topic, but even simple applications can be very challenging! Continuing with the narrative style in Chapter 5, *Predicting*, Lily strikes a balance, introducing the basic ideas informally in the body of the text, with the sidebar bringing in the terminology and a more formal treatment of the strategies to

calculate probabilities. The sidebar concepts and terminology do come quickly and require practice to master. The exercise sets provide ample opportunity with classic examples as well as current health related applications.

Data drives the world! In Chapter 6, *Understanding Data*, Lily once again provides a compelling narrative with rich examples to introduce basic statistical graphs and summary measures.

Throughout *Mathematics Between Friends*, Lily skilfully utilizes narratives involving fictional student learners, ancients like Pythagorus and Euclid, contemporaries like Greta Thunberg and Volodymyr Zelenskyy, to introduce and develop the basic ideas with the associated sidebars providing the details of the mathematical concepts. Accessible, engaging and truly reflective of the title, *Mathematics Between Friends*.

Well done Lily!

Ted Bentley
August 2022
Gibsons, BC

Preface

After teaching at Capilano University for nearly twenty years, I was very exhausted from commuting across the Second Narrows and the constant fear of a bridge accident causing me to be late for class. Too early to retire, I began fantasizing having a year of sabbatical to write a textbook on mathematics, not the usual one, but a fun book telling stories while teaching mathematics, one that makes students, many of whom have math anxiety, want to read and learn.

The first step in realizing this fantasy was to apply for a paid educational leave (PEL), so I got to work—first choosing the course to write for, Math 123, second, planning the chapter titles, third, listing headings of sections. Because Math 123 (Contemporary Mathematics) covers lots of interesting, current topics while also broadly satisfying numeracy requirement for many CapU’s degree programs, I had more fun than one should have had writing the PEL proposal to the committee—with chapter titles like ***Measuring and Tracking**—Archimedes, Euclid, and Pythagoras lost in town* and ***Predicting**—Hamlet’s dilemma: To bet or not to bet*.

Then my proposal was rejected—with an encouraging note for a resubmission. Both my division chair and my coordinator were surprised by the rejection, and also recommended resubmission the next year,

but I was ready to give up because the committee wanted to see some sample chapters to be reassured that I would do what I promised.

Then Covid lockdown happened to CapU. All my activities out of home were cancelled, thus plenty of time to write another proposal, in particular, the sample chapters. Right before the deadline, I resubmitted my proposal, and was granted a three-quarter equivalent sabbatical.

A year into the Covid lockdown with a non-teaching spring semester at hand, I decided to start writing a section on my favourite subject—geometry—to see how the book would develop. After a few weeks of hard work writing one section, I put it aside for a week, then returned to it pretending that I was a student reading it for the first time. To my great horror, the story could not have been more boring. If I, who love math, would never read such a book, how about the students most of whom do not like mathematics?

At that moment, the small panic which began initially from the back of my mind now stared straight at me in flashing neon signs—*You don't know how to write stories*—never mind if it was about math. In distress, I reached out to the English Department at Cap requesting the privilege of auditing a creative writing course. The English Department coordinator, Leah Bailly, replied with an offer for me to audit her creative writing course focusing on the narrative style. In the short number of weeks with Leah, I was introduced to many contemporary writers with diverse writing styles; in addition, I was taught to experiment with different techniques, like embedding dialogues in story-telling. In short, Leah's course fundamentally changed my writing.

Once the sabbatical began, I learned to work like a real writer—regardless of whether I had ideas or not, I sat in front of my computer, gathering information, learning, planning, and—if lucky—writing. The first chapter began forming, followed by the second, and so on; every chapter became my favourite chapter up to that point—I fell in love with the characters, new things to learn, and the writing. All the joys and adventures of writing helped me realize what a privilege it was to be awarded this PEL to learn and write.

Acknowledgement

Many people helped me with this book, and to whom I am extremely grateful. My husband, Mogens Lemvig Hansen, created and taught me how to code many diagrams in `TeX` as my personal tech support, proofread the book for mathematics, searched the type of creative commons license best suited, and regularly reminded me that I did not write “like someone with twenty novels under her belt”, but was “pulling off teaching mathematics through story-telling pretty well”. For his support and encouragement, I owe him my gratitude.

My daughter, Justine Hansen, drew all the cartoons and vector drawings. I am grateful for how her pictures enlivened the stories.

My oldest child, Kristian Hansen, inspired many games throughout the book.

Leah Bailly, my English professor and Capilano colleague, taught me how to write using dialogues.

Kate Wright, the host of Good Shepherd Lutheran Church book club, proofread and asked questions which led to more details in the development of mathematical ideas.

Danielle Wills, my colleague who also teaches Math 123 regularly, met me several times to discuss her impression of every chapter. Her

super positive comments helped carry me through to the end of the project.

Lily Ligocki, my first junior high school friend in Canada, shared pictures of her pottery totem.

Jonathan Shapiro, my friend from 1987 Budapest Semester in Mathematics, shared photos of his travels to provide examples of Roman numerals in the chapter on numeration systems.

Andy Liu and Richard Hoshino who worked with me as the coaching team for Team Canada in preparation for 2003 International Math Olympiad inspired some of the characters from the chapter on geometry.

Richard Hoshino read the first five chapters and suggested a rewrite of the first chapter to match the style of the other chapters.

Ted Bentley, a statistician and long-time Coordinator of the Department of Mathematics and Statistics at Capilano University (then College) hired me in 2000 and taught me to teach mathematics better. His comments on Chapters 5 and 6 helped improve the book. Ted also wrote the Foreword.

Jonathan Jedwab, my research collaborator at Simon Fraser University, inspired me to be patient when writing mathematics. Once he shared a story about the misuse of probability in courts. The story is in the sidebar of Chapter 5, ***Predicting**—Hamlet's dilemma: To bet or not to bet*. Jonathan also read Chapter 1 carefully and made suggestions that improved the book.

Good Shepherd Lutheran Church book club and movie nights provided many inspirations for the sidebar material.

Many people in the online community helped make the book richer.

Introduction to the Student

Mathematics entered my life at age six as arithmetic, doing addition/subtraction drills on large sheets of brown pattern paper while my mother sewed for her children and for teaching in a southern Taiwanese national vocational school. By grade five/six, many word problems were so convoluted that my mother could not help me without introducing x as a variable in an algebraic equation. By the time junior high started in grade seven, I was totally lost in solving the simplest type of linear equations for the ever-evasive x . My mother taught me algebra the way she was taught—moving variables and numbers left and right of the equal sign without mathematical logic. A good memory saved me from hating mathematics for the rest of my life because I just memorized how to perform a newly taught technique and executed it precisely. For the next two and a half years in junior high, mathematics became one of those subjects—something for me to work on without much interest.

This all changed when my family immigrated to Canada. Jumping into the second half of grade nine as a non-English speaker, I could only participate in mathematics out of all required subjects. Even when the descriptions and explanations of exercises were totally incomprehensible, the mathematical expressions were logical and clear for me to understand. With Taiwanese high school mathematics being two years ahead of

Canadian mathematics, I became a nuisance to every math teacher, from Lord Byng Secondary School, Moscrop Junior High, to McPherson Junior High, completing math assignments in five minutes while others struggled. By the time I settled in McPherson, Mr. Mendez gave me old Waterloo Math Contests (now Canadian Mathematics Competition) to entertain me while other students learned the day's lesson. Up to that point, I had never known mathematics to be so much fun, mixing simple math techniques of different topics to solve fresh and engaging problems. With English still a huge obstacle, every contest problem became a source of new vocabulary. I soon fell in love with contest problems, from multiple choice to short answer to full solution Math Olympiad questions.

As a Far East Asian immigrant burdened with cultural expectations, university majors were quite limited—either pre-med or pre-law, but I managed to convince my parents that I would be able to survive independently after university with a math major. Little did I know that mathematics would develop into a life-long passion through the mentorship of many wonderful professors and supervisors. Owing so much to those who opened the door to the wonder and beauty of mathematics for me, I would like to share such beauty with you via this book.

The story in each chapter is accompanied by mathematical definitions and examples in the sidebar for a formal development of the subject. If reading both the story and the sidebar material at the same time is unfamiliar, I recommend focusing on the story, one section at a time. Once the plot of the section is no longer foreign, one can read the section again, this time, paragraph by paragraph while matching sidebar material. The symbol \triangleleft marks the end of a mathematical development. The exercises at the end of each chapter help build mastery of the topic. Each chapter is enriched by many internet links (in magenta) both mathematical and otherwise; students are encouraged to check out each link to follow their interests. Red texts are links within the book.

As one who has benefited greatly from life-long learning, I hope

to inspire you to become life-long learners, thus in the process, fully engaged citizens of the world, to better social ills like racism, economic inequity, political and climate crises—yes, even through mathematics. May the stories and sidebar materials of the book open doors for many!

Some solutions are provided for exercises. Those marked [IM] are solutions in the Instructor's Manual to this book.

Introduction to the Instructor

Many students registered in Math 123 [24] at Capilano University for their program requirement are anxious about mathematics. In addition, some of them may carry deep within them the misconception of what it means to be **gifted** in mathematics, consequently, claim that they are not *gifted*, thus cannot learn mathematics. If not gifted in mathematics means not being able to carry out mental arithmetic with lightning speed, this book debunks the myth by encouraging students to use technology from calculators to spreadsheets, for mathematics is not about fast arithmetic computation at all.

Mathematics between Friends assumes very little mathematical background [30], and develops most of what is required either through conversations in the narrative portion or in the sidebar via formal development. If reading both the story and the sidebar material at the same time is unfamiliar, I recommend reading the story in each section first. Once the plot of the story is no longer foreign, one can read the section again, this time, paragraph by paragraph with relevant sidebar material. The symbol \triangleleft marks the end of a mathematical development. Each chapter is enriched by many internet links (in **magenta**) both mathematical and otherwise: Students are encouraged to check out each link to follow their interests. **Red** texts are links within the book.

The goal is to help students understand everything they are doing and to engage them in an **authentic mathematical experience**. For those students fearful of mathematics, I hope that the stories and explorations of each chapter will convince them that mathematics is not about crunching numbers or memorizing formulas, but loving how their brains take on challenges and stay focused until a tantalizing problem is fully solved or understood.

Because the world our students live in is more intricately connected than a generation ago, I have also tried either to apply some of the mathematics in this book to current issues, or use everyday concerns to motivate mathematical ideas. When an instructor feels that a particular section in some chapter is not easily examinable, the section may inspire a student project.

Chapters in this book are mostly independent from other chapters; instructors may teach in any order. The exercises for the last chapter, *Understanding Data*, include questions which require adequate digital literacy from students to find statistical information on the Internet. Since the information may be time sensitive, with new data replacing the old, the solutions to these questions are based on data available in May of 2022.

Answers to exercises labelled [IM] are found only in the Instructor's Manual to this book.

Solving Puzzles—Adventures of Anjali, Katharina, and Mei in Capilano University

1.1 INTUITION AND LOGIC

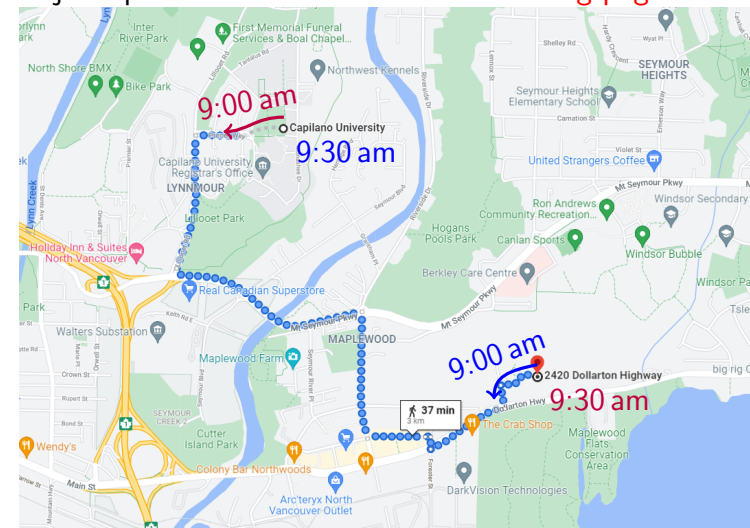
Continuity—Introducing Anjali, Katharina, and Mei

Anjali just arrived in Vancouver from Punjab, India. Grateful to get a spot in Capilano University’s residence on Dollarton Highway, she takes full advantage of her location and enjoys her half-hour power walk from Grouse Hall to campus when it’s not raining, and when she is not late for class.

On the day of New International Student Orientation, Anjali left Grouse Hall at 9 am while talking to her mother on the phone learning all about her whole extended family taking turns looking after her great-grandmother who very recently lost all strength in her legs and could no longer walk or stand. When Anjali arrived at the Centre for Sport and Wellness (CSW), it was 9:30 am.

The next day, she finished a workout in the CSW early, and started walking back at 9 am, again talking to her mother on the phone. This

Anjali’s path from Puzzle 1 on the following page:



Map data ©2021 Google Canada

time she learned that her great-grandmother fell many times because she forgot her sudden loss of mobility. Anjali's stronger cousins needed to be around to lift the great-grandmother up, not an easy task because she was built like a tree, totally solid even at age 95. Arriving at Grouse Hall at 9:30 am, along exactly the same path, Anjali realized, to her surprise, that at exactly the same time as the day before, she was at exactly the same spot along Mount Seymour Parkway. "How could that be?" Anjali pondered.

To answer her question, Anjali enlisted her roommate, Inderpreet, in an experiment. "Inderpreet, could you start walking from Grouse Hall at 9 am tomorrow towards CSW on the path shown on this map? At the same time, I will leave CSW for Grouse Hall on the same path. Both of us must get to our destinations in 30 minutes."

Inderpreet had no patience for such experiments, especially after listening to Anjali's reason. "According to your map, an average person would take 37 minutes to complete this route. If I have to make it in 30 minutes, I will have to be watching the time constantly, totally ruining the walk. You don't need me to walk to solve your puzzle. Of course, we will meet somewhere at the path—thus, same time, same spot. Period."

Anjali was speechless.

Katharina was an exchange student from Germany living in Surrey, where housing was more affordable. On her long commute to CapU's North Vancouver campus, Katharina usually texted with her sister back home. Even in a crowded bus during rush hour, texting in German felt private enough to discuss serious family problems, like her sister suspecting that their father was having an affair with a lady in the same church council. Katharina could not decide whether to believe her sister because their mother was adamant that there was no affair between them.

On the day of New International Student Orientation, Katharina wanted to make a ham open sandwich in order to explore around CapU afterwards. "Look at how irregular this slice of ham is," Katharina remarked to herself. "Can I split the slice equally anyway regardless of

PUZZLE 1 Is there a spot along Anjali's path that she will be in both trips at precisely the same time?

This problem was inspired by puzzle 50 in *My Best Mathematical and Logic Puzzles* by Martin Gardner [12].

the angle the knife makes with the bottom of the cutting board?”

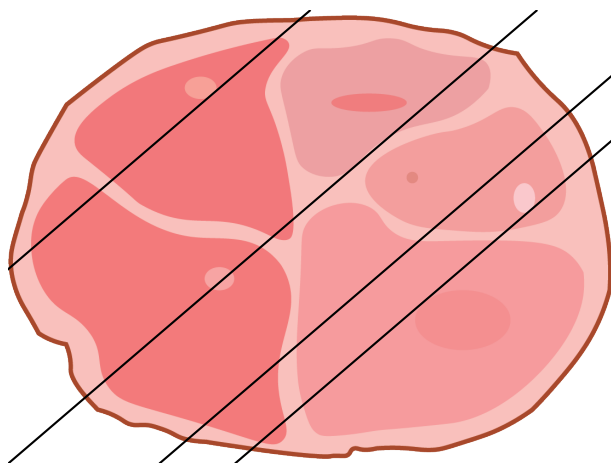


Figure 1.1: Many ways to slice the piece of ham

The question intrigued Katharina enough to make her test out different angles at which to hold her knife with respect to an edge of the cutting board. “When I fix the slant of my knife and sweep it from the left to the right, the amount of ham on the left of my knife grows from nothing to the entire ham, so during this sweeping of the knife, there must be a unique position where the knife splits the ham equally.” Figure 1.1 shows one such angle.

After Katharina put the slice of ham on top of a slice of dark sourdough rye bread, her mind was further intrigued. “With ham on bread, can I still find a way to cut the open sandwich to split *both* the ham and the bread equally?” Puzzles like these offered her a safe escape from worrying about her family back home, so Katharina often immersed herself in creating more and more variations of the initial puzzle. Staring at the knife first, then the open sandwich, Katharina tried many different positions for a possible cut. “I know that there are infinitely many possible slants I may hold my knife to achieve an

PUZZLE 2 Can Katharina split the slice of ham equally regardless of how her knife is held with respect to the edge of the cutting board?

Continuity is an important concept in the theory of functions. In our everyday language, we use **continuous** to mean having no breaks or interruptions. Most natural phenomena are continuous, like Anjali’s position varies continuously with time. Once a slant in which Katharina holds the knife is fixed, as the knife sweeps from the left to the right, the difference between the ham to the left of the knife minus the ham to the right of the knife varies continuously with the position of the knife. Similarly, the temperature difference of diametrically opposite places on any circle varies continuously with the angle the diameter forms with the horizontal diameter.

The **Intermediate Value Theorem** (IVT) says that a continuous phenomenon cannot skip a value. In both Katharina’s open sandwich division problem and Mei’s temperature difference problem, the value guaranteed to exist is 0 because at one extreme was a positive number while the other, negative. The number 0 is between positive and negative numbers. This style of reasoning is **deductive** reasoning where we apply accepted facts and general principles (in our case, the IVT) to arrive at a specific conclusion (the existence of a position, a cut of equal division, or diametrically opposite points of the same temperature). ◀

equal split of the slice of ham. Expressed as an angle relative to an axis system, if the knife at position θ (that splits the ham equally) has more bread on the left of the knife, then as the knife rotates 180° , resting at position $180^\circ + \theta$, there will be more bread on the right. Somewhere in between, I will have an angle for an equal division of the slice of bread also.” Figure 1.2 shows many possible angles for the placement of the knife, each of which splits the ham equally.

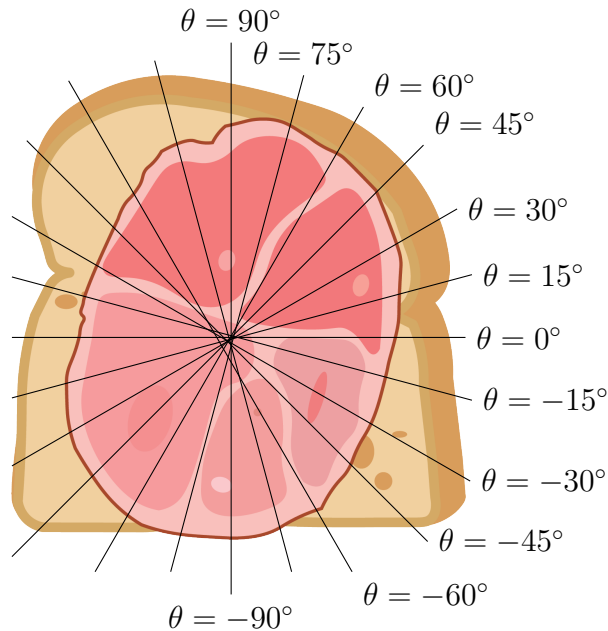
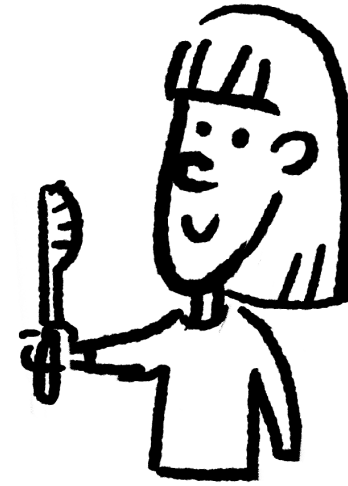


Figure 1.2: Sliced ham sandwich

Mei grew up in Burnaby, and still lives with her family while attending university. Having access to a family car, Mei loves driving to green spaces of the Lower Mainland for bird watching, berry picking, or mushroom hunting. However, temperature variation in the Lower Mainland is a concern. One day, she noticed that when she drew a



PUZZLE 3 Suppose a slice of ham is on top of a slice of bread, can Katharina cut the open sandwich with one cut to divide both the ham and the bread equally?



circle anywhere on a map of the Lower Mainland, at any moment, there existed a pair of diametrically opposite points P and Q on that circle corresponding to locations where the temperatures were equal. For example, her circle could be the one with Capilano University and Kwantlen Polytech as points P and Q . Mei wondered, “Could this be because the Lower Mainland has small temperature variations at any moment, or that this region is too small?”

To answer her own question, one day in February, Mei took out a map and drew a circle with Vancouver and Saskatoon diametrically opposite from each other. “Is it possible to find diametrically opposite points on this circle where the temperatures are equal? Really? During a Canadian winter when everywhere in Canada has double digit temperatures with the West Coast being the only region whose temperature is above zero?”

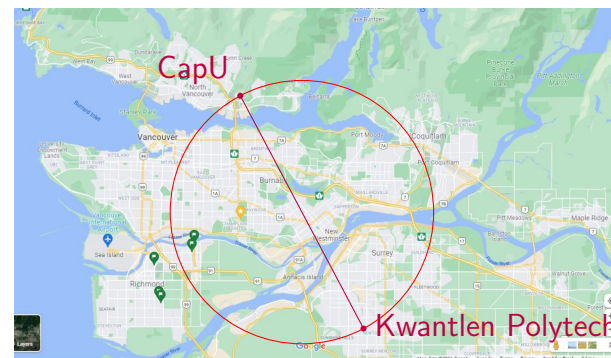
Mei usually talks to herself while solving puzzles. “My circle has a diameter whose end points are Vancouver and Saskatoon. Then I take Vancouver’s temperature minus Saskatoon’s temperature (a big positive number). As I rotate the diameter around the centre of the circle, the temperature difference of its end points gradually changes. After half a turn of the circle, or 180° , the temperature difference is negative because it is Saskatoon’s temperature minus Vancouver’s. Somewhere in the rotation from 0° to 180° , the difference has to be zero at least once. Bingo! At that angle with a zero difference, the temperatures of the end points of the diameter are equal.”

Mei smiled at herself, “Talking to myself helps me think so much better!”

Deductive reasoning—Meeting at orientation

Anjali and Katharina who met at New International Student Orientation just became friends with Mei in a guided tour of STEM. Being extremely curious, both Anjali and Katharina wanted to know when Mei’s birthday was. Ever capturing any opportunity to play games, Mei gave them a list of 10 possible dates:

A map of Greater Vancouver with Capilano University and Kwantlen Polytech as diametrical opposites.



Map data ©2021 Google Canada

PUZZLE 4 If Mei draws a circle on a map, is it possible to find diametrically opposite points on the circle where the temperatures are equal?

April 15 16 19
June 17 18
July 14 16
August 14 15 17

Mei then told Anjali and Katharina separately the month and the day of her birthday, respectively. Unbeknown to Mei, Anjali and Katharina loved playing party games as much as Mei. Immediately, Anjali began, “I don’t know when Mei’s birthday is, but I know that Katharina doesn’t know, either.”

Katharina jumped in, “At first I didn’t know when Mei’s birthday is, but I know now.”

Anjali looked at the dates again, and concluded, “Then I also know when Mei’s birthday is.”

“What pure luck to bump into people as crazy about games as I am!” Mei thought aloud. “Tell me how you figured it out, Anjali.”

“Well, every statement in our conversation provided new information about your birthday. When I was told the month, and was certain that Katharina could not figure out your birthday either, it was because I was given a month without unique dates, namely July or August. Since both April and June had unique dates 19th and 18th, if I had had either, I would not have been able to claim that Katharina did not know your birthday, either.” While Anjali caught her breath, Katharina nodded away.

Feeling encouraged, Anjali continued, “When Katharina heard that I knew Katharina did not know the date, Katharina deduced that I was told either July or August. With these two months left, Katharina was able to determine your birthday because Katharina was not given the 14th which appeared with both July and August. When I learned that Katharina figured out your birthday, I concluded that your birthday was not the 14th. I then claimed that I knew your birthday because I was given July which had only the 16th left. On the other hand, if I had been given August, I would still not know your birthday.”

PUZZLE 5 With the dates given by Mei and the three exchanges between Anjali and Katharina, how can one determine Mei’s birthday?

Puzzle 5, known as *Cheryl’s Birthday*, was written by Dr. J.Y.B. Woi for the Singapore and Asian Schools Math Olympiad in 2015.

All three new friends looked at one another and knew instantly their year at CapU would be filled with games and adventures whenever they gathered.

1.2 EXPLORATION

Patterns—Shaking hands

The first time Anjali, Katharina, and Mei met at the guided tour of STEM, they shook hands with one another.

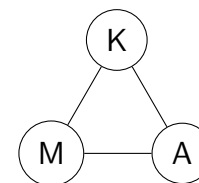
A senior from CapU led a group of 10 new students including Anjali, Katharina, and Mei to tour the labs in the Fir building and the engineering lab in the Willow building. Katharina whispered to Mei and Anjali, “We just counted 3 handshakes among us. Now with 10 new students in our group, how many handshakes will take place?”

The expert organizer, Anjali, began first. “Let me organize the number of students and the corresponding handshakes in a table starting with 1 student who has no one to shake hands with, or 0 handshake. As I add one student at a time to the room, the number of handshakes grows: +1, +2, +3, ...Ah, 45 handshakes among all ten of them.”

Students s	Handshakes h	Selected graphs
1	0	
2	1	—
3	3	△
4	6	⊠
5	10	⬠
6	15	
7	21	
8	28	
9	36	
10	45	

PUZZLE 6 How many handshakes are there when three people shake hands with one another?

We can represent the people in Puzzle 6 with a **graph** where **vertices** represent the people and **edges**, handshakes. Three edges represent 3 handshakes.



PUZZLE 7 How many hand shakes are there when 10 students all shake hands with one another?

When the information is organized as a table for Puzzle 7, it becomes easier to see a pattern.

Variables—Picking berries

After touring STEM, Mei showed Anjali and Katharina a good patch of late blackberries near the parking lot at the end of Monashee Drive. Katharina, 186 cm tall, brought down a lot of high branches for Anjali and Mei to feast on. Anjali loved berry picking, and picked twice as many blackberries as Mei. To show their appreciation for Katharina's help, both Anjali and Mei gave Katharina the same number of blackberries. If Katharina did not pick any berry on her own, and had 30 berries in total from her friends, and suppose Mei gave three quarters of the berries she picked to Katharina, then how many berries did each keep?

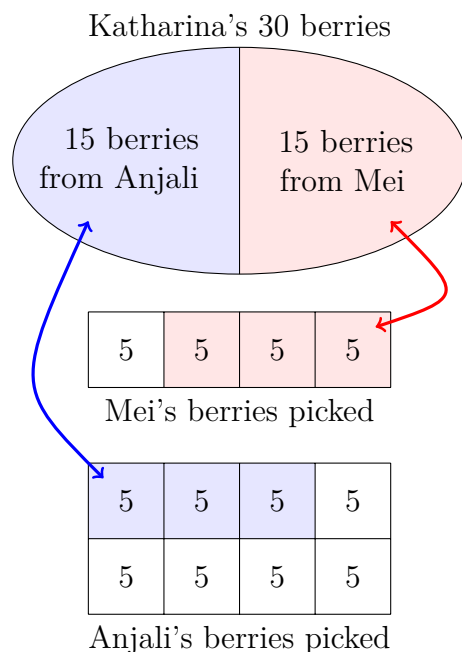


Figure 1.3: Berries divided

PUZZLE 8 Given that Anjali picked twice as many blackberries as Mei, and both Anjali and Mei gave Katharina the same number of blackberries, if Katharina without picking any berry of her own had 30 berries in total from her friends, and suppose Mei gave three quarters of the berries she picked to Katharina, then how many berries did each keep?

Drawing a diagram for Puzzle 8 helps one understand the problem.

One may define new **variables** to translate the problem into **equations** to solve. It is important to define the variables clearly. For example, “let m be Mei's blackberries” is ambiguous because one could understand m as the number of berries Mei gave to Katharina, or those Mei picked, or those Mei kept. For the sake of simplicity, let m be the berries Mei kept to eat, or $1/4$ of the berries she picked. We know that Mei gave away 15 berries which is $3/4$ of what she picked, so $15 = 3m$, or $m = 15/3 = 5$. The total of what Mei picked is $4m = 4 \times 5 = 20$. Therefore, Anjali (who picked twice as many as Mei) picked $2 \times 20 = 40$ berries, and got to keep $40 - 15 = 25$ berries. ◀

In addition to talking to herself while solving puzzles, Mei also liked

drawing. “Let me start with a bubble containing Katharina’s total of 30 berries: 15 from Anjali and 15 from me. If 15 berries I gave to Katharina constituted $\frac{3}{4}$ of all the berries I picked, then the rectangular fraction box here shows that I must have picked 20 in total—four equal parts with three parts shaded pink to denote three quarters.” Mei continued drawing more boxes. “Similarly, I draw the second rectangle to represent Anjali’s berries. Since Anjali picked twice as many as I, she had 40 in total, 15 of which spread equally over three blue squares. The parts not shaded represent what we got to keep, i.e. I ate 5, Anjali, 25, and Katharina, 30.”

Inductive reasoning—Sharing a chocolate bar

Once satisfied with blackberry picking and eating, they headed toward the Bosa Centre for Film and Animation to line up for the President’s barbeque. Any free food on campus attracts long lines. While the three were waiting in the end of a long line, Katharina took out a Milka Alpine Milk chocolate bar from her backpack, and began breaking her chocolate piece by piece. The breaking of the chocolate bar prompted Mei to ask, “What is the smallest number of breaks Katharina needs to make at the indent of the 4×6 bar to separate the entire bar into twenty-four 1×1 pieces?”

Anjali shared her observation, “Can you see that Katharina started with one whole bar, and each break she made produced an extra piece? For example, the first break resulted in two smaller pieces. Then she took one of the pieces and broke it, and we saw three pieces in total. With a 4×6 chocolate bar, breaking only at the indent, Katharina must perform 23 breaks to get 24 pieces.”

“That is so clear. Thank you, Anjali.” Mei remarked enthusiastically.

“Here are the 24 pieces of chocolate! Dive in!” Katharina offered to the other two.

Mei wondered aloud again, “Are we going to ruin our appetite by eating chocolate before lunch?”



PUZZLE 9 When Katharina took a 4×6 chocolate bar and broke the bar at the indent of the bar, what is the smallest number of breaks she would need to separate the entire bar into twenty-four 1×1 pieces?

When a problem involves a big number, it is sometimes helpful to change the problem to a smaller setting. For example, one can begin with a 1×1 Swiss chocolate square which requires no break. Next, a 1×2 bar, one break, and so on, to a 2×4 bar. This style of reasoning is **inductive** reasoning where one draws a general conclusion by observing a pattern in specific instances.

A table helps with pattern building for Puzzle 9. Let the horizontal entries record the length, while the vertical, the width. For example, the bold **7** represents the smallest number of breaks a 2×4 bar requires.

	1	2	3	4	5
1	0	1	2	3	4
2	1	3	5	7	9
3	2	5	8	11	14
4	3	7	11	15	19

◁

“Come on, Mei!” Chuckling, both Anjali and Katharina sighed simultaneously.

1.3 TOOLS

Tree diagrams—Making a burger

At the President’s barbeque, a buffet table displayed

- 3 choices of buns: white, whole wheat, and gluten free,
- 5 choices of condiments: mayonnaise, pickles, roasted peppers, mustard, and ketchup,
- 4 choices of vegetables: lettuce, tomatoes, sweet onion, cucumbers, and
- 2 types of patties: beef or vegan.

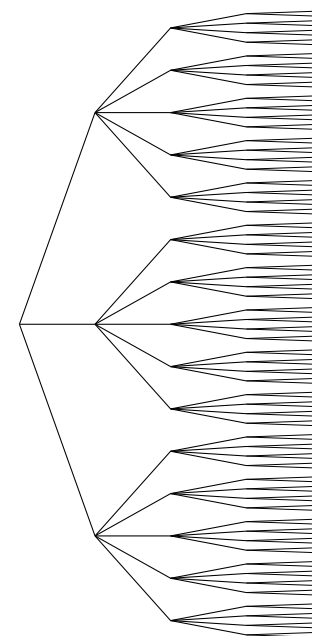
Anjali turned to her friends for help. “What are we to do with all this food?”

“Let me show you, Anjali.” Katharina went ahead of Anjali. “We make a burger according to the order food groups are displayed. First pick a bun, then some condiments, some vegetables, and one kind of patty.”

“I see,” Anjali continued, “If you make your burger with only one item from every category, then there are many ways of building one’s own burger. I remember from math in high school a tree we could draw to represent the choices for each category. For example, beginning at one node, three edges are drawn to indicate the 3 choices of buns. Once a type of bun is chosen, it is followed by 5 choices of condiments, represented by five edges. Continuing this way, we see a tree ending with $3 \times 5 \times 4 \times 2 = 120$ nodes representing 120 different ways of making a burger. Truly individualistic!”

PUZZLE 10 Suppose Katharina would like to make a cheese burger using only one item from every category, how many ways could she make her burger?

A tree diagram helps display all choices in **Puzzle 10**: first node branches into 3 choices of buns (white, whole wheat, and gluten free). Once a bun is chosen, 5 choices of condiments follow. Upon settling on a condiment, 4 choices of vegetables present themselves. Finally, once a vegetable is chosen, one must choose one of the 2 available patties.



◁

Mei tried to clarify, “Remember that you are not restricted to just one condiment or one vegetable, so even more ways are available. Unfortunately, in my case, I just became gluten intolerant recently, and have been a vegetarian since grade 10. The number of ways I can build my burger using only one item from each food group is much smaller.”

“Let me figure this out using Anjali’s counting tree,” Katharina began drawing edges to try counting with the tree diagram. “For the bun, Mei has to choose a gluten free bun (only one edge for the buns) with a vegan patty (also only one edge for the patties) while still having all the other choices for condiments and vegetables. This means Mei will have $1 \times 5 \times 4 \times 1 = 20$ ways of making her vegetarian gluten free burger.”

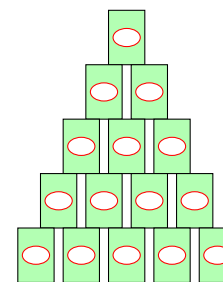
“I want to make a burger and eat now. All this talk is making me really hungry. And look, more people are lining up still.” Anjali suddenly became self conscious, and made her friends finish making their burgers quickly and left the buffet table.

Patterns and connections—Building towers and shapes

“Finally, we got our burgers and juice,” Anjali let out a sigh of relief. “It’s so nice to see that neither of you chose a can of pop. I don’t drink anything out of aluminum cans because of the suffering of the displaced tribal people in Orissa province in my country due to bauxite mining. Bauxite is used in the making of aluminum.” In response, Mei began picking up empty pop cans to recycle. “Let’s have a competition building pop can towers,” suggested Katharina, who began building a triangular tower from Mei’s stash.

“On the topic of reducing and recycling, my mom taught me to travel with a fruit knife and toothpicks so that I can eat any kind of fresh fruit without contributing to more packaging waste” Mei shared in passing while taking out her pack of toothpicks. She began forming

PUZZLE 11 Mei just became a gluten intolerant vegetarian. How many ways could she make her burger, also choosing one item from every category?

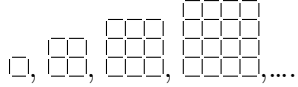


PUZZLE 12 If a triangular tower has one can on top, two on the second level, consecutively to ten on the bottommost level, how many cans are used? Can Mei build such a tower using all of 7 cans?

Figurate numbers describe sequences where each sequence counts the number of dots in a given shape. For example, the **triangular numbers** are represented by dots forming triangles: \bullet , $\bullet\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet$, **Square numbers** count the number of dots forming squares: \bullet , $\bullet\bullet$, $\bullet\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet\bullet\bullet$,

The number of cans in each triangular tower seems familiar. Indeed, in the handshake problem, one encountered the same sequence: 1, 3, 6, 10, Since 7 is not one of the numbers, it is impossible for Mei in Puzzle 12 to build a triangular tower using all of 7 pop cans. ◀

a square with four toothpicks. Anjali took some to form a 2×2 square.



Katharina followed her friends and started building a 3×3 square with the toothpicks. As she built, she asked Mei, “Do you have enough toothpicks for me to make a 3×3 square?”



Mei thought aloud again, “Let’s see...First for the vertical toothpicks: 4 columns of 3 toothpicks contribute 12 in total. If we just rotate the structure 90° to account for the horizontal ones, we get another 4 rows of 3 toothpicks. So, all together Katharina will need $2 \times (4 \times 3)$, or 24 toothpicks. I think there is plenty here for that.”

Organization—Seeing squares

“Katharina’s square reminded me of a game my sisters and I used to play at home,” reminisced Anjali, “on the square tiles in our living room. We would each choose a different number greater than 1 and count squares of that size to see who won with the most number of squares.”

Mei needed more clarification. “What do you mean?”

“Let me use Katharina’s 3×3 square as an example.” Anjali began explaining. “Let’s count squares of different sizes here: How many 1×1 ’s, 2×2 ’s? Remember to include the largest square, the 3×3 .”

Mei began counting. “We have nine 1×1 squares, four 2×2 squares, and only one 3×3 square. So a total of $9 + 4 + 1 = 14$ squares. When I colour the upper left hand corner’s 1×1 square of each 2×2 square in orange, I end up with an orange 2×2 square, accounting for four 2×2 squares, . Similarly, I colour the upper left hand corner square of the 3×3 in blue. There is only one such blue square, .

Katharina looked impressed, “Wow, Mei. You just simplified drawing lots of overlapping squares into a neat pattern, so much easier to count!”

PUZZLE 13 How many toothpicks are needed to form a 4×4 square? How about an $n \times n$ square for any positive integer n ?

When a pattern in Puzzle 13 is being established, one way of knowing the pattern is to find a formula giving the number of toothpicks in an $n \times n$ square. A table of initial values helps one see what is added to the previous value, especially to decompose the structure into smaller, easier to express parts. Even though the patterns built so far begin with $n = 1$ as the smallest case, sometimes, it can be insightful to start the pattern with $n = 0$.

Katharina’s reasoning extends to an $n \times n$ square— $n + 1$ rows of n toothpicks and the same for vertical ones, resulting in $2 \times (n + 1) \times n$ toothpicks. It is important to check the formulas. In this case, when $n = 1$, we have $2 \times (1 + 1) \times 1 = 4$; when $n = 2$, we have $2 \times (2 + 1) \times 2 = 12$, as seen in the figures. \triangleleft

PUZZLE 14 How many squares of different sizes are there in a 3×3 square? How about an $n \times n$ square?

The organization of information often reduces the complexity of a problem. In Puzzle 14, when one organizes counting of squares into 1×1 squares, 2×2 squares, and so on, a pattern emerges and helps lead to the general formula for the total number of squares in an $n \times n$ square.

Note that 1, 4, and 9 are the first three **square numbers**.

Permutations—Building a totem

After Anjali, Katharina, and Mei finished their burgers, the three started walking towards the library. A little forest greeted them on their way with tall spruce trees filling the skies, mature rhododendrons lining the path, and hellebores and primulas covering the ground. The sweet fragrance of fall enticed them to linger longer in the forest. “I wish I had my pottery from home. With a one-metre piece of rebar, I could build a totem of pottery,” exclaimed Katharina, “to decorate the forest.”

Mei was curious about Katharina’s hobby, “Can you show me some pictures of your creation?”

“Let me find my pottery folder...” Katharina took out her phone.

Anjali looked over Katharina’s shoulder, “I like these five pieces. How many ways can we make a totem with these five?”

Mei also wanted to test out Anjali’s counting tree diagram, “Anjali’s tree diagram can help us here if we build the totem from the bottom up. First, all 5 pieces are available to us as the bottommost piece, so five choices, or five edges out of the first node. Once the first piece is picked, there are only 4 pieces left to choose from as the second piece. As we build the totem piece by piece, one fewer piece is available at each level. This gives us $5 \times 4 \times 3 \times 2 \times 1 = 120$ different arrangements for the totem.”

“I forgot to say that I often cover the end of the rebar with a special piece. Suppose I put the raccoon at the top of five pieces, how many ways can the totem be arranged?”

Anjali started, “We can keep Mei’s counting of five pots to arrange below the top piece. This results in the same $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways below the top piece, leaving us only one way to top the totem with the end piece, still 120 ways in total. If we use only five pots in total with a designated top piece, then we will have a different number.”

Katharina wanted a turn in counting, too. “Let us label the pots G , C , A , and T , then the corresponding tree diagram for arranging four

This way of colouring only the upper left hand corner of different squares of the same dimension with the same colour shows the relationship between the coloured shape relative to the original shape. One can extend this idea to rectangles and deduce a similar sequence counting rectangles of different dimensions. ◀



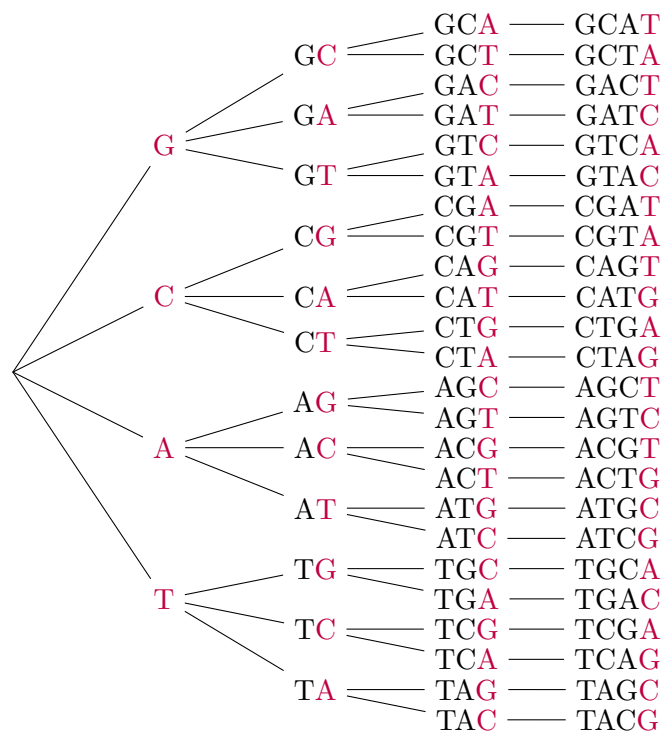
PUZZLE 15 How many ways can Katharina arrange 5 different pieces on a totem?



Creation of TigerLilyPottery

A **permutation** of a collection of objects is an arrangement of some of the objects in a line. For example, five different

pots below the top piece is like this. I will use purple to indicate the pot added at each level.



Sorry to take so much time drawing all the arrangements!”

“Oh, dear! I am going to hit bridge traffic soon if I don’t get going,” Mei suddenly realized and started running, “but I still want to visit the Math Learning Centre.”

“We will come back to the forest another time. I heard that there were fun games at the Math Learning Centre. Let us join you, Mei.” Katharina called out after Mei.

“Do you remember where the Math Learning Centre is?” asked Anjali, “I only remember that it is on the first floor of the Library.”

pieces are arranged in a line. Counting the number of **permutations** is helped greatly with a **tree diagram**. <

PUZZLE 16 If a special top piece is chosen from one of the five pieces, like the raccoon in the photograph, how many ways can the totem be arranged?

Since the top piece has its place pre-determined, one only has 4 pieces to arrange, so $4 \times 3 \times 2 \times 1 = 24$ ways. <

PUZZLE 17 The rooms in the Library are numbered with three digits. Knowing that the first digit is 1 for rooms on the first floor, how many rooms does one need to check?

Katharina was eager to help. “I noticed that rooms in the Library were numbered with three digits. Since we know that the first digit is 1, and there are 10 digits to choose from for each of the ten’s and the one’s places, we would have 10×10 , or 100 choices. However, some of the rooms have a letter at the end of the three digits. This is totally impossible: 100×26 choices! I am going to look up the **MLC** on my phone instead of checking all the rooms.”

“Thank you, Katharina. I don’t have time for games anymore.” Mei sighed.

1.4 BRAVE ATTEMPTS

Differences—Observing details

By the time Anjali, Katharina, and Mei rushed into LB126, they were out of breath.

“Welcome to the MLC!” An instructor cheerfully welcomed the three, “Would you like to try to find ten differences in the pictures of a park near a bridge?”

“I don’t have time for games,” Mei whispered to her friends while anxiously checking the time on her phone, “I bet this game is just a warm-up for something.”

“Come and see,” called out Anjali, “This is the real purpose—to help us review our notations,” pointing at the next game full of math notations.

PUZZLE 18 Describe the difference between each set of notations. Extra points will be given for correct usage of each notation.

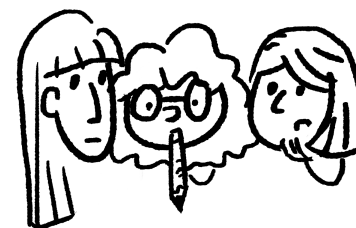
1. $>$ and \geq
2. μ , U and u
3. \cup , \vee
4. \emptyset , o , O , and 0
5. \in and \subset
6. $=$, \equiv , and \cong
7. $=$, \approx , and \sim

“Let’s give it our best shot in the shortest time possible.” Anjali began. “The first set I remember from school: ‘greater than’ versus



NeoK12 covers many subjects from kindergarten to grade 12, including games and puzzles.

Attention to detail is essential in the study of mathematics. You may recall the **famous example**: “Pardon impossible to send to Siberia”. Depending on where you place the comma, whether after “Pardon” or after “impossible”, the resulting statement has opposite meaning.



The following statements are true: $3 > 2$, $3 \geq 2$, $2 \geq 2$.

‘greater than or equal to’. The extra line at the bottom of $>$ tells us that ‘equal to’ is also OK.”

Katharina continued, “We had to learn some Greek letters in school. I can see that μ is the Greek letter ‘mu’, and lower case u can be confused with μ . I remember that in math, capital letters and lower case letters represent different things and should not be confused with each other.”

Mei reluctantly tried the third, “I don’t know if you had to learn set theory and logic at school. The third one has the **set union** notation and the disjunction, ‘OR’, from logic which we had to learn in computer science in high school. Here, I think \cup tries to remind us that it is a set ‘Union’, and \vee looks similar because of a similar function.”

Katharina took over from Mei, “Germany’s neighbour, Denmark, has a funny letter \emptyset , ‘o slash’. From my math class back home, we used the notation \emptyset , inspired by the Danish letter to denote the ‘empty set’. The rest of them are just the letter o in lower and upper cases and the numeral, zero.”

Anjali tried to speak as fast as possible, “Number 5 is from set theory again: ‘an element of (a set)’ versus ‘a subset of (a set)’.”


Finally, Mei combined two items in one, “Questions 6 and 7 are about different types of similarity: ‘equal to’, ‘equivalent to’, versus ‘congruent to’, and ‘equal to’, ‘approximately equal to’, and ‘similar to’. The last notations from both we used in geometry for congruent and similar triangles or other geometric shapes. What is next?”

Picture proofs—Visual reasoning

Katharina saw a table displaying nine pictures. As the three friends arrived, the cheerful instructor handed them each a handout. “You need to match each formula on the handout to a picture on the table. The same formula may be used more than once to describe different pictures.”

However, it is *not* true that $-3 \geq -2$. We can write $-3 \not\geq -2$ to mean that -3 is neither greater than nor equal to -2 . \triangleleft

In statistics, μ stands for population mean (average). \triangleleft

Let A and B be two sets. The union of sets A and B , , is denoted by $A \cup B$. Let p and q be two statements. The disjunction “ p or q ” is denoted by $p \vee q$.

In set theory, to denote a set with no elements, or the empty set, we use \emptyset . Sometimes, to distinguish the number 0 from the letter “o” or “O”, one may write \emptyset to indicate the number 0.

Let a be an element or member of set A . Then one writes $a \in A$. Let every element of set B be an element of set A . When at least one element of A is not in B , then B is a proper subset of A , written $B \subset A$. \triangleleft

The equal sign $=$ should be used precisely—only when the left side equals to the right side. In modular arithmetic or in the study of predicate logic, one may see the symbol \equiv , like $17 \equiv 10 \pmod{7}$ or in logic $\text{True} \vee \text{False} \equiv \text{True}$.

“Approximately equal to”, denoted by \approx , is used when an evaluation is not exact, like $\frac{3}{7} \approx 0.4857$. If triangle A is similar to triangle B , we use $A \sim B$. \triangleleft

Anjali took a quick look at her handout, “Hmm—some of these formulas look familiar; for example, the first one describes the triangular numbers, and the next two, the square numbers. I wonder what the pictures look like.”

Formulas on the handout:

$$1. \quad 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1) \text{ for any positive integer } n.$$

For example:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \mathbf{9} = \frac{1}{2} \times \mathbf{9} \times (\mathbf{9} + 1) = 45$$

$$2. \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for any positive integer } n.$$

For example:

$$1 + 3 + 5 + \underbrace{(2 \times \mathbf{4} - 1)}_{=7} = \mathbf{4}^2 = 16$$

$$3. \quad 1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2 \text{ for any positive integer } n.$$

For example:

$$\begin{aligned} 1 + \mathbf{2} + 1 &= \mathbf{2}^2 \\ 1 + 2 + \mathbf{3} + 2 + 1 &= \mathbf{3}^2 \\ 1 + 2 + 3 + \mathbf{4} + 3 + 2 + 1 &= \mathbf{4}^2 \end{aligned}$$

$$4. \quad 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) + (2n - 1) + \cdots + 5 + 3 + 1 = n^2 + (n + 1)^2 \text{ for any positive integer } n.$$

For example:

$$1 + 3 + (2 \times \mathbf{2} + 1) + 3 + 1 = \mathbf{2}^2 + (\mathbf{2} + 1)^2$$

$$\text{or } 1 + 3 + 5 + 3 + 1 = 4 + 9 = 13$$

One can also express these formulas in words: **1**: The sum of the first n consecutive positive integers is the n th triangular number. For example, the first triangular number is 1; the second is $1 + 2 = 3$; the 3rd is the sum of the first 3 consecutive whole numbers $1 + 2 + 3 = 6$, and so on.

2: The sum of the first n consecutive odd whole numbers is the n th square number.

3: The sum of the first n consecutive whole numbers and the first $n - 1$ consecutive whole numbers is the n th square number.

4: The sum of two consecutive square numbers is formed from two sums of consecutive odd numbers.

$$5. \frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \dots$$

More precisely,

$$\frac{1+3+\dots+(2n-1)}{(2n+1)+(2n+3)+\dots+(4n-1)} = \frac{1}{3}$$

$$6. \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{1}{3}$$

$$7. \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

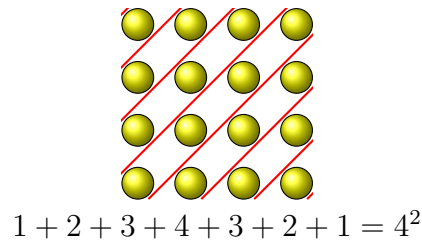
$$8. \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \dots + \frac{1}{\binom{n(n+1)}{2}} + \dots = 2 \text{ for any positive integer } n.$$

Mei quickly shuffled over to the display table for the pictures. While she was scanning the pictures, Anjali and Katharina joined her.

“Hmm—what a novel idea!” Anjali looked at the shapes and colours of the pictures, some with accompanying equations. “I haven’t seen anything like this before in math. Have you?” Anjali asked.

Katharina thought a little and said, “Maybe in some math enrichment talks in high school. It was so much fun taking pictures like these to design different ways of partitioning the whole, like the first picture here is already partitioned into anti-diagonal rows of balls.”

1. Square numbers



<

5: The first even number, say $2n$, of consecutive odd whole numbers is divided into the first half, or the first n consecutive odd numbers, and the second half. Then form a fraction with the sum of the first half as the numerator while the sum of the rest of the odd numbers is the denominator. According to [Stillman Drake's Galileo Studies \[6\]](#) this formula was first discovered by Galileo in 1615.

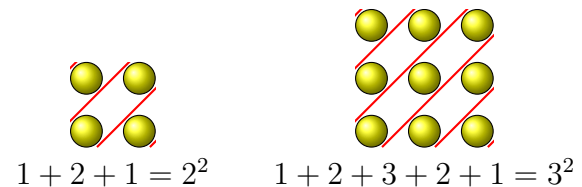
6 & 7: An infinite sum of numbers resulting in a small fraction. The numbers, $\frac{1}{4}, \frac{1}{16} = \frac{1}{4^2}, \frac{1}{4^3}$, form a [geometric sequence](#) because each subsequent number is the product of a fixed number and the previous; in this case, $\frac{1}{4}$ is the fixed number repeatedly multiplied to produce the next term.

The sum of an infinite geometric sequence is a [geometric series](#).

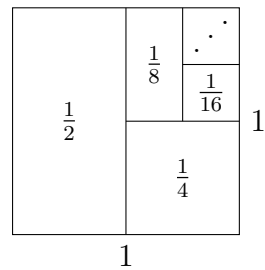
8: The last formula is an infinite sum of the reciprocals of the triangular numbers.

The pictures are selected from Roger Nelsen's [Proofs without Words: Exercises in Visual Thinking \[22\]](#).

Looking at smaller examples of the pictures can be helpful. For example for Picture 1:

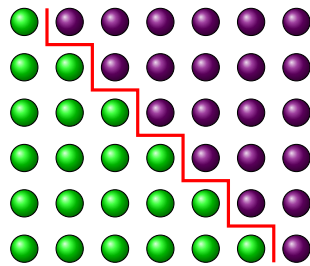


2. Area of a square subdivided



Picture 3 and the formula it proves was known to “the ancient Greeks” (as cited by Martin Gardner in *Mathematical Games*, volume 229 of Scientific American) [11].

3. Sum of 2 triangular numbers.



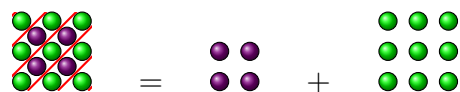
4. Think of each triangle’s area as a fraction of the big triangle’s area.



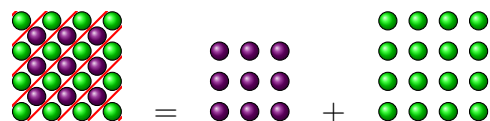
5. Two interlacing consecutive square numbers



$$1 + 3 + 1 = 1^2 + 2^2$$

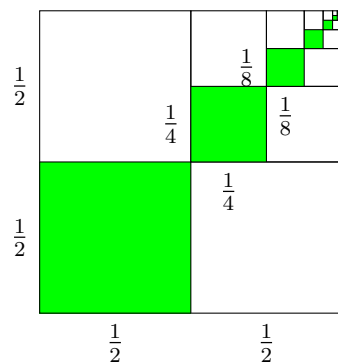


$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$

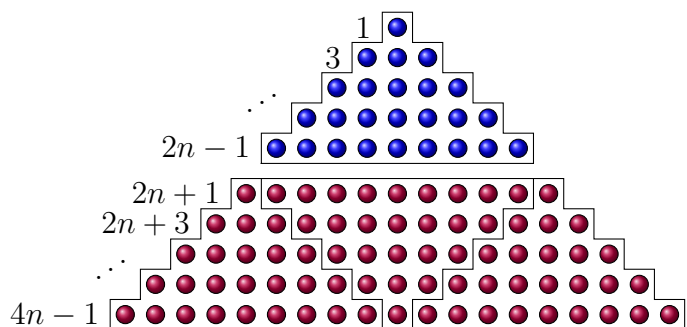


$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

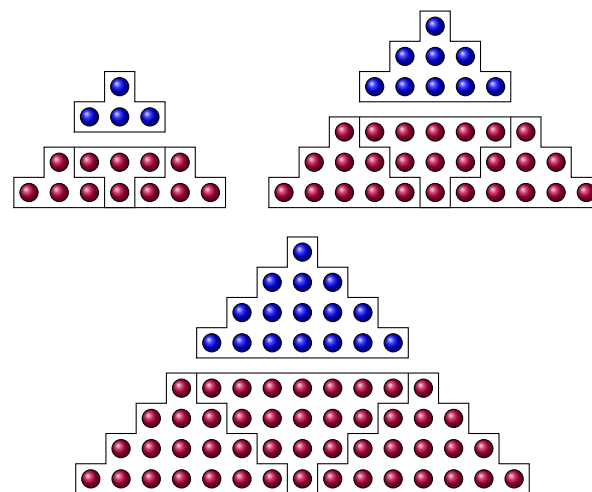
6. Fractions of the 1×1 square area.



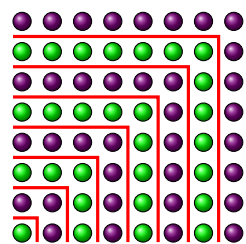
7. Form fractions with the number of blue balls as the numerator and red, the denominator.



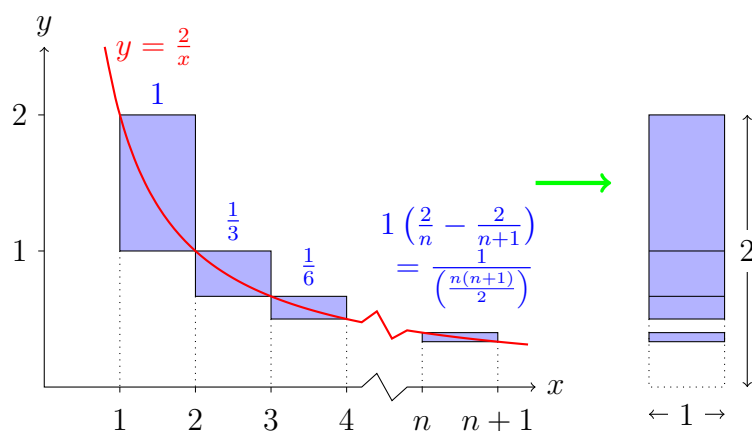
Smaller cases of Picture 7:



8. Sum of consecutive odd numbers




9. Telescoping sum displayed



Picture 8 proves one of the sums of odd integers from *Introduction to Arithmetic* by *Nicomachus of Gerasa* [14] (circa C.E. 100), a Greek mathematician from modern day Jerash, Jordan.

After looking over her handout carefully, Anjali whispered to her friends, “I don’t know about you, but my phone’s free spreadsheet app from Google, ‘Sheets’, can help me check some of these formulas. With n ’s all over the place, I can’t get a feel for the smaller cases. Let me show you how I get a whole column of numbers very quickly.



1. Click on the icon 
2. To generate consecutive positive integers, type 1 in cell A1 and push ‘enter’ or ‘return’. Now the blue highlighted box surrounds cell A2. Type “=A1+1” and push ‘enter’. You will see 2 in cell A2.
3. To copy and paste the formula to all cells below A2, first hover on A2 and choose ‘copy’. To copy the formula to many cells at once, hover on cell A3 then continue swiping down to cell 31, say. Now all cells should be shaded with the A3 boxed. Then go back to hover on cell A3. Choices will reappear for you to choose ‘paste’. Magically, all cells will be filled with whole numbers in ascending order, identical to the leftmost column.
4. To see what the first formula $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$ gives, we need to prepare two columns, one for the left hand side of the equal sign, and one for the right. Type 1 in cell B1. For B2, we want to obtain $1 + 2$ in a clever (recursive) way. If the n th cell of column B represents $1 + 2 + 3 + \dots + n$, then the previous cell, or $B(n-1)$, has the number $1 + 2 + 3 + \dots + (n - 1)$. To get B_n , we just add $A_n (= n)$ and $B(n - 1)$. Specifically, for B2, we type “=A2+B1”.
5. Now, copy the formula and paste it into B3 and on to see the result. We have the triangular numbers again.

On a computer, click on , then  Apps to find 

On a computer, right click on the mouse to choose ‘copy’.

Push ‘shift’ and the down arrow to highlight as many cells as you want, say to 31, and right click on the mouse to choose ‘paste’.

The cell contents in black are what you type while the gray is the result of pasting. Notice the equal sign, ‘=’, which must be typed.

	A	B	C	D
1	1	1	=A1*(A1+1)/2	=B1-C1
2	=A1+1	=A2+B1	=A2*(A2+1)/2	=B2-C2
3	=A2+1	=A3+B2	=A3*(A3+1)/2	=B3-C3
4	=A3+1	=A4+B3	=A4*(A4+1)/2	=B4-C4
⋮	⋮	⋮	⋮	⋮

Since the right hand side of the formula is a quadratic function in n , it is enough to check three distinct values.



6. For the right hand side of the formula, we start a new column, say C. Type in cell C1, “=A1*(A1+1)/2”, and copy and paste the formula onto all cells from C2 on.
7. We see that columns B and C are identical which confirms the formula at least to $n = 31$.
8. If we want to be extra careful in checking the values from columns B and C, we can start a new column, say D, and type “=B1-C1” in cell D1. Copy and paste the formula from D1 onto D2 and below to see a whole column of 0’s. If there is a non-zero value in column D, it would be easy to spot.”

“What a great idea!” responded Katharina and Mei in unison. They immediately took out their phones to find a spreadsheet program. Mei continued, “You just saved us either from tedious calculations by hand or boring substitutions on our calculators. I might even be able to get on the Second Narrows Bridge before it’s completely jammed.”

“Let us divide and conquer,” suggested Anjali, “since I already did the first formula, if we each take one, we will cover the first four formulas. The rest are fractions which might be more easily seen from the pictures.”

Mei cheerfully reported, “I have an idea for the sum of consecutive odd whole numbers. Since Anjali generated consecutive positive integers for us already in Column A, I will let cell E1 be ‘=2*A1 -1’ and copy and paste to other cells in Column E, thus producing all consecutive odd numbers. Then I borrow Anjali’s idea from Column B to get the sum, namely, in cell F1, I type 1, then cell F2 should be ‘=E2+F1’, copy and paste to other cells in Column F. To get the right hand side, namely, the square numbers, simply enter in cell G1 ‘=A1*A1’, copy and paste to other cells in Column G.”

Katharina shook her head, “I think the two of you are overthinking this. I am going to assume that the formulas on the sheet are correct and don’t need to be checked like you were doing with your spreadsheets.

Building entries to find the sum of consecutive odd whole numbers:

	A	E	F	G
1	1	=2*A1-1	1	=A1*A1
2	=A1+1	=2*A2-1	=E2+F1	=A2*A2
3	=A2+1	=2*A3-1	=E3+F2	=A3*A3
4	=A3+1	=2*A4-1	=E4+F3	=A4*A4
⋮	⋮	⋮	⋮	⋮

<

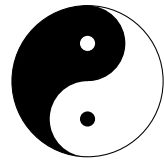
Exercise 1.39 on page 36 asks you to finish the task of matching formulas to pictures.

Instead, I am just going to look at each picture and see which formula expresses the picture's breakdown of the formula."

The three friends very quickly identified the formulas and said goodbye to a great day of orientation. Mei dashed for her car on Purcell Way; Anjali walked towards the Bookstore in the Birch building; and Katharina headed back towards Monashee Drive for her long walk to Lower Seymour Conservation Reserve on Lillooet Road.

1.5 EXERCISES

1.1 (CONTINUITY) Mei: The Yin-Yang symbol from Taoism is used in traditional Chinese medicine to explain opposites. I wonder if there is a line splitting regions of the same colour equally so that on either side of the line, I will see the same black (and white) area.

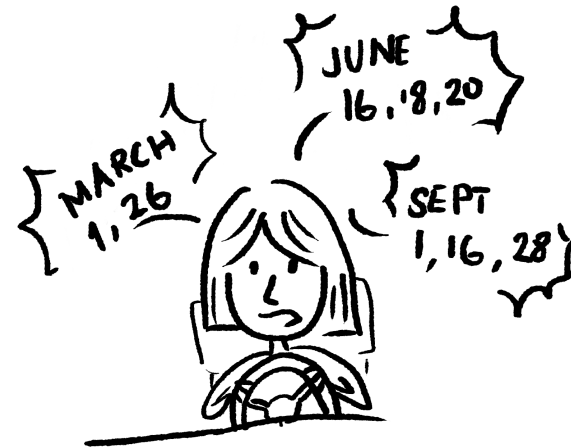
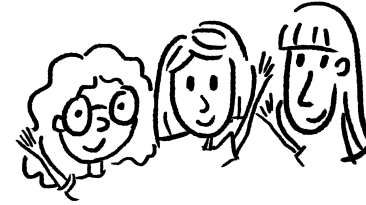


Deductive reasoning

Mei was indeed too late to cross the bridge smoothly. While stuck in her car, she began designing more birthday questions similar to the one she gave Anjali and Katharina regarding her birthday.

1.2 Nicole and Adrienne just became friends with Harmony, and they want to know when her birthday is. Harmony gives them a list of ten possible dates:

March 1 26
 May 18 28
 June 16 18 20
 September 1 16 28



Harmony then tells Nicole and Adrienne separately the month and the day of her birthday, respectively.

Nicole: I don't know when Harmony's birthday is, but I know that Adrienne doesn't know either.

Adrienne: At first I didn't know when Harmony's birthday is, but I know now.

Nicole: Then I also know when Harmony's birthday is.

When is Harmony's birthday?

1.3 Nicole and Adrienne just became friends with Harmony, and they want to know when her birthday is. Harmony gives them a list of ten possible dates:

March 1 26

May 18 28

June 16 18 20

September 1 16 28

Harmony then tells Nicole and Adrienne separately the month and the day of her birthday, respectively.

Nicole: I don't know when Harmony's birthday is, but I know that Adrienne doesn't know either.

Adrienne: At first I didn't know when Harmony's birthday is, but I know now.

Nicole: I STILL DON'T know when Harmony's birthday is.

What month is Harmony's birthday?

1.4 Ken and Jacqueline just became friends with Howard, and they want to know when his birthday is. Howard gives them a list of ten possible dates:

January 9 15 27

October 9 14

November 14 15 29

December 3 29

Howard then tells Ken and Jacqueline separately the month and the day of his birthday, respectively.

Ken: I don't know when Howard's birthday is, but I know that Jacqueline doesn't know either.

Jacqueline: At first I didn't know when Howard's birthday is, but I know now.

Ken: Then I also know when Howard's birthday is.

When is Howard's birthday?

1.5 Lucas and Florence just became friends with Kelly, and they want to know when her birthday is. Kelly gives them a list of ten possible dates:

January 2 15

June 3 9

August 3 12 21

October 2 9 21

Kelly then tells Lucas and Florence separately the month and the day of her birthday, respectively.

Lucas: I don't know when Kelly's birthday is, but I know that Florence doesn't know either.

Florence: At first I didn't know when Kelly's birthday is, but I know now.

Lucas: Then I also know when Kelly's birthday is.

When is Kelly's birthday? [IM]

1.6 Lucas and Florence just became friends with Kelly, and they want to know when her birthday is. Kelly gives them a list of ten possible dates:

January 2 15

June 3 9

August 3 12 21

October 2, 9 21

Kelly then tells Lucas and Florence separately the month and the day of her birthday, respectively.

Lucas: I don't know when Kelly's birthday is, but I know that Florence doesn't know either.

Florence: At first I didn't know when Kelly's birthday is, but I know now.

Lucas: I STILL DON'T know when Kelly's birthday is.

What is Kelly's birth month?

1.7 Oliver and Vicky just became friends with Yang, and they want to know when his birthday is. Yang gives them a list of ten possible dates:

January 11 17 28
 March 7 17
 July 15 25
 December 7 11 25

Yang then tells Oliver and Vicky separately the month and the day of his birthday, respectively.

Oliver: I don't know when Yang's birthday is, but I know that Vicky doesn't know either.

Vicky: At first I didn't know when Yang's birthday is, but I know now.

Oliver: Then I also know when Yang's birthday is.

When is Yang's birthday? [IM]

1.8 James and Janice just became friends with Beatrice, and they want to know when her birthday is. Beatrice gives them a list of ten possible dates:

March 10 16
 May 15 24
 July 1 15 29
 November 1 16 24

Beatrice then tells James and Janice separately the month and the day of her birthday, respectively.

James: I don't know when Beatrice's birthday is, but I know that Janice doesn't know either.

Janice: At first I didn't know when Beatrice's birthday is, but I know now.

James: I STILL DON'T know when Beatrice's birthday is.

What month is Beatrice's birthday in? [IM]

Mei suddenly looked up, "I must have been stuck here for a long time. Look at how many birthday questions I made. When am I going to get off the bridge?"

1.9 Solve this KenKen puzzle:

3-	2÷		4+
	2	4+	
6×			2÷
4+		4	

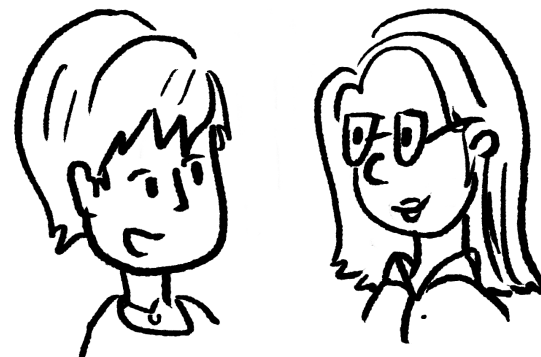
1.10 While walking to have lunch together, two CapU mathematicians had this conversation:

Linda: I hear that you have three daughters. What are their ages?

Joanne: The product of their ages (whole numbers) is 36.

Linda: I cannot determine their ages from that.

A **KenKen** puzzle of size $n \times n$ has each box filled with one of the numbers $1, 2, 3, \dots, n$, where each number must be used, but may not be repeated within the same row or column. The clues for each cage (outlined shapes) are given on the upper left hand corners. For example, $4+$ means that the sum of the numbers in the cage is 4, $3-$ means that the difference is 3, $6\times$: the product is 6, $2\div$: the quotient is 2.



Joanne: The sum of their ages is the same as the address number on that house across the street. (They both see the house number.)

Linda: I still cannot determine their ages.

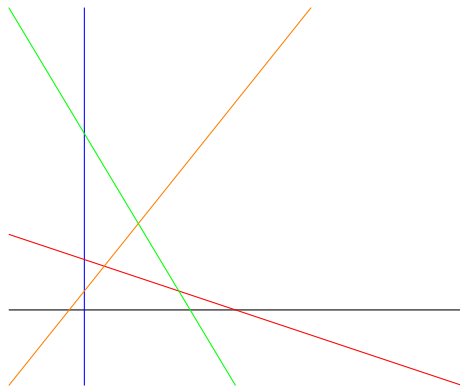
Joanne: My oldest daughter has red hair.

Linda: I know their ages now.

Find the ages of the three daughters.

Inductive reasoning

1.11 Anjali draws a straight line, then a second one crossing the first line at one point, then a third line, crossing each of the previous lines at different points. Continuing this way, how many crossing points will there be after Anjali draws 10 lines?



1.12 Anjali takes a sheet of blank paper and considers that as having one region. She then draws a straight line splitting the paper into two regions, then a second line crossing the first line at one point and splitting the paper into four regions. She continues drawing straight lines in general position, that is, no three lines meet at one point. How many regions would she have after drawing 6 lines?

The puzzle about Joanne's daughters' ages uses the same mathematical reasoning as the puzzle from Ted-Ed, *Can you solve the passcode riddle?*



1.13 Observe the following pattern: $1 \times 1 = 1$, $11 \times 11 = 121$, $111 \times 111 = 12321$, $1111 \times 1111 = 1234321$. Can you write down the product of 11111×11111 without the calculator? Why does it work? Does this still pattern extend to a string of 10 ones? How would you modify it to make it work?

1.14 Observe the pattern produced by the powers of 11: Recall that $11^0 = 1$. Try to predict 11^5 ? Check it with your calculator.

1
11
121
1331
14641

1.15 Establish the pattern produced by the powers of 101: Recall that $101^0 = 1$. Compare it with the pattern generated by the powers of 11. When you predict 101^5 , how does it differ from predicting 11^5 ? [IM]

1.16 Establish the pattern produced by the powers of 1001: Recall that $1001^0 = 1$. Compare it with the pattern generated by the powers of 11. When you predict 1001^6 , how does it differ from predicting 11^6 ? [IM]

1.17 Remember the square numbers 1, 4, 9, 16, 25, Observe the following pattern: $4 \times 6 = 24$, $9 \times 11 = 99$, $19 \times 21 = 399$, $29 \times 31 = 899$. Can you describe the pattern and write down the product of $899\,999\,999 \times 900\,000\,001$?

1.18 Richard P. Feynman, winner of the Nobel Prize in physics in 1965 and professor at Caltech for many years, wrote an interesting book about his career, *“Surely You’re Joking, Mr. Feynman!”* [10]

In it, he tells of working on a secret project for the US government at Los Alamos where the atomic bomb was tested during World War

II. In a letter to a friend at that time, he wrote that he had noticed something very peculiar. If you perform the division $1 \div 243$, you get

$$0.004\ 115\ 226\ 337\ \dots$$

According to the pattern you observe, write as many digits as you can following 337. Use your computer to check and explain whether the pattern breaks down after more than 20 digits or if it continues further.

1.19 The numbers in this list are all **prime**.

31
331
3331
33331
333331
3333331
33333331

Write the next number in this list according to the pattern you see. Do you think this number is prime? Start dividing your proposed number by 2, 3, 5, 7, 11, 13, 17, 19, ..., that is, the first few prime numbers.

1.20 The numbers in this list are all prime.

61
661
6661

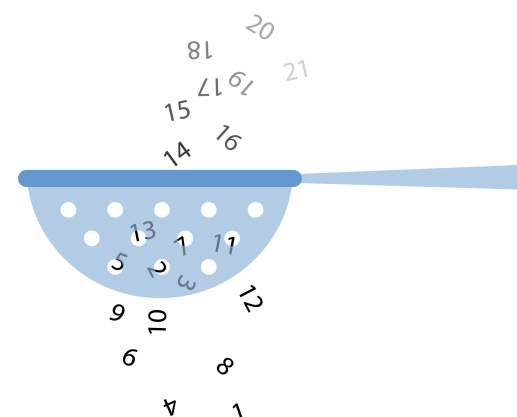
Write the next number in this list according to the pattern you see. Do you think your proposed number is prime? Start dividing the number by 2, 3, 5, 7, 11, 13, 17, 19, ..., that is, the first few prime numbers. [IM]

The story in Exercise 1.18 on the previous page was reported in Chapter 21 of *Penrose Tiles to Trapdoor Ciphers* by Martin Gardner [13], also at the end of Chapter 1, Lesson 4 of *Mathematics A Human Endeavor* by Harold R. Jacobs [18].

A **prime** number is a whole number greater than 1 that is not a product of two smaller whole numbers. A positive integer that is not **prime** is **composite**.

A **composite** number is a positive integer with a factor greater than 1 but less than itself. For example, 91 is composite, or not prime, because $91 \div 7 = 13$ where both 7 and 13 are factors of 91, and they are also greater than 1 and less than 91.

The **Sieve of Eratosthenes** is an **algorithm** or procedure for finding all primes. Only the primes are caught inside the sieve while the unit, 1, and **composite** numbers are sifted out.



1.21 The numbers in this list are alternating **composite** and prime.

$$91 = 7 \times 13$$

991

$$9991 = 97 \times 103$$

99 991

$$999 991 = 17 \times 58 823$$

9 999 991

Write the next two numbers in this list according to the pattern you see. Do you think the pattern of composite and prime continues? [IM]

Variables

1.22 Mei made some fruit leather roll ups for snacks. Suppose she had a 25 cm long piece that she wanted to share with Katharina, giving Katharina 5 cm more than herself, how much would Mei have?

1.23 Anjali and Inderpreet would like to share a roll of ribbon. If the roll contains 6 m, and Anjali got 50 cm more than Inderpreet, how many centimetres of ribbon did each get?

1.24 Katharina brought some German gummy bears to share with Anjali and Mei. If Katharina had 30 gummy bears to begin with, and Anjali and Mei both received twice as many gummy bears as Katharina kept for herself after sharing, how many did each get?

1.25 Katharina brought some German liquorice to share with Anjali and Mei. If Katharina split 72 pieces of liquorice into three piles in the ratio of 3 : 2 : 1, three parts for herself, two parts for Anjali, and one part for Mei, how many pieces would each get?

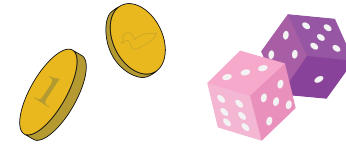
Large numbers may be difficult to check especially when the smallest prime factor is large. The following [link](#) offers a primality test from Chris K. Caldwell of the University of Tennessee at Martin.



1.26 In a recent CapU Terry Fox fundraiser, Anjali, Katharina, and Mei ran a total of 72 km. If Anjali ran one more kilometre than Katharina and 23 km more than Mei, how far did Anjali run?

1.27 Anjali shared a long strip of fruit leather with her sisters, Simran and Gurpreet. The strip was 60 cm long. Anjali's strip was three times as long as Simran's; and Gurpreet's strip was twice as long as Simran's. Find the length of each person's strip of fruit leather.

1.28 Mei's mom bought treats consisting of M&M's, Skittles, and Coffee Crisps to give out for Halloween. If she had twice as many M&M's as Skittles, and 30 more Coffee Crisps than M&M's, how many of each would she have knowing that she had 200 treats all together?



Tree diagrams

1.29 Katharina flips a one-Euro coin and a loonie (one Canadian dollar coin). Draw a tree to show all possible results of flipping these two coins. Use 'H' to indicate a 'head', and 'T' for a 'tail'.

1.30 Mei flips a nickel, a dime, a quarter, a loonie, and a toonie (a two-dollar Canadian coin). How many different results can she expect from flipping these five different coins?

1.31 Anjali rolls a pink die and a purple die, both ordinary six-sided dice labelled with 1, 2, 3, 4, 5, 6. How many different results (numbers landing uppermost) can she expect? A pink 1 and a purple 3 is different from a pink 3 and a purple 1.

1.32 Mei invited Katharina, Anjali, and her roommate, Inderpreet, to go on a road trip to Banff.

- a. Suppose all four could drive Mei's car which had two front seats and two back seats. How many seating arrangements were there?

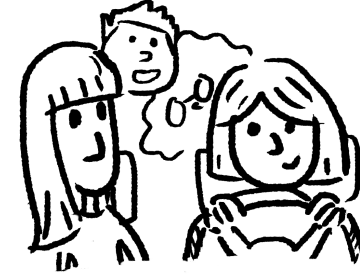
- b. Suppose Mei had to drive due to insurance regulations, how many seating arrangements were there?

1.33 For a project in one of Katharina's courses, every student had to write a project proposal choosing three different topics out of 25 listed by the professor. The professor required that the proposed topics be listed in priority order: first, second, and third choice. How many different project proposals containing 3 topics are possible?

1.34 Even though Anjali does not have access to a kitchen, she would still like to prepare some garam masala for Mei and Katharina to try. In her garam masala recipe, she needs to lay on a tray the following spices for roasting: coriander seeds, cumin seeds, black peppercorns, curry leaves, whole cloves, black cardamom pods, cinnamon sticks, and bay leaves. In how many orders could Anjali lay these spices on a tray? For example, putting cumin seeds on the tray followed by whole cloves is different from whole cloves first, then cumin seeds.

1.35 To thank Anjali for the delicious garam masala, Mei organized a Thanksgiving dinner for her university friends showcasing the versatility of this new found spice mix. Mei offered button mushrooms, red bell pepper squares, red onion pieces, pineapple cubes, cauliflower florets, zucchini cubes, all marinated in garam masala vinaigrette. Everyone invited was to make one or two veggie skewers from a selection of the vegetables. Supposing a skewer could fit 5 pieces of vegetables comfortably, how many different 5-piece skewers could be made?

1.36 Katharina has a coin purse with exactly five different coins: a nickel, a dime, a quarter, a loonie, and a toonie. When she reaches into the purse and randomly takes out two coins, how many different ways are there to get two coins?



1.37 Katharina had a test matching 4 formulas to 5 picture proofs where all 4 formulas must be used at least once. If a computer generated answers randomly for this test abiding by the rules, how many ways were there?

1.38 (DIFFERENCES) Anjali: Can we make up True/False questions for one another to review some notations? Katharina: That is a great idea. I don't study so well on my own, but having someone quizzing me is so much more fun. Mei: Let me start. Suppose the x 's in my statements are all numbers on the number line.

a. $x + 3 > x + 2$

b. $x + 3 \geq x + 2$

c. $x + 3 \leq x + 2$

d. $x + 3 < x + 2$

e. $x + 3 = x + 2$

f. $x < x + 2$

g. $x \leq x + 2$

Katharina: You've got to stop now, Mei. You are giving us too many too fast.

Anjali: I will try a few—definitely not as many as Mei's—and specify my variables.

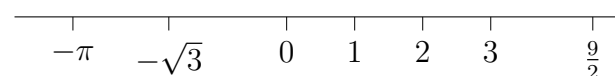
h. $a \geq -300$, for all whole numbers a .

i. $a^2 + b^2 = (a + b)^2$, for some real numbers a and b .

j. $\sqrt{a^2 + b^2} = a + b$, for some real numbers a and b .

k. $\frac{a + b}{2} \geq \sqrt{ab}$, for non-negative real numbers a and b .

The number line contains all real numbers.



<

The square root of 16, denoted by $\sqrt{16}$, is 4 because $4^2 = 16$.

<

Katharina: That is enough, Anjali. You are not much better than Mei. These questions need some experiments. I will need to use the spreadsheet. I have a few of my own for you two to try.

1. I know we were taught not to add fractions this way:

$$\frac{s}{t} + \frac{u}{v} = \frac{s+u}{t+v}$$

It is not hard to choose numbers for s , t , u , and v to show that the left hand side is not equal to the right hand side in general. However, do you know when this way of adding fractions is OK for integers s , t , u , and v ?

- m. Inspired by Anjali's square root questions, I wonder when this equation is true.

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b}$$

- n. How about this one? For integers n and m , when is $n^3 + m^3 \leq (n+m)^3$?

Mei: Katharina, you shouldn't have complained about my questions which only involved one variable. You two posed questions with two variables that require spreadsheet experiments. We are going to have a lot of fun with your questions.

1.39 (VISUAL REASONING) Match all nine figures in *Picture proofs—Visual reasoning* on page 16 with the formulas given in that section.

Counting and Coding—How does Fire Horse count?

2.1 SIMPLE NUMERATION SYSTEMS

Nonpositional versus positional—Introducing Fire Horse and her A-máⁿ

Fire Horse was the ninth grandchild and third granddaughter to her A-máⁿ. In a traditional Asian culture where daughters could not carry on the family name, to be born in the year of the fire horse was an even worse omen than being born a girl. Luckily for Fire Horse, however, her A-máⁿ loved her as much as she loved her oldest grandson.

The grandparents of Fire Horse were long gone, but Fire Horse occasionally saw them in her dreams. One night, for no apparent reason, Fire Horse dreamed of herself as a little girl back in her grandparents' large factory courtyard, chasing chickens and ducks under shaded hanging pots of orchids her A-Kong cultivated. A-máⁿ called out to Fire Horse, "Come and sit next to me. Let me show you something."

"Wow, so many baby chicks," Fire Horse sat on a low bamboo stool and looked with wonder at what A-máⁿ was showing her.

"Do you know how many? Let us scratch a stroke in the dirt for each chick, and when we have five, we will put the fifth stroke across



A-Kong means grandfather in Taiwanese, a variant of Hokkien.

The Chinese Zodiac contains 12 animals, one for each year, where the Horse is the seventh of the cycle. Combined with five metaphysical elements: metal, water, wood, fire, and earth, one derives a 60-year cycle, Fire Horse being

the other four, like this |||| for 8.”

Fire Horse was totally engrossed in tracking which baby chicks were counted, and which not yet, when A-máⁿ interrupted her, “May I show you how to write the number using these chopsticks?” A-máⁿ began putting down old chopsticks vertically, one at a time, while teaching Fire Horse the name of each number:

one	six T	eleven — 	sixteen —T
two 	seven TT	twelve — 	seventeen —TT
three 	eight TTT	thirteen — 	eighteen —TTT
four 	nine TTTT	fourteen — 	nineteen —TTTT
five 	ten —○	fifteen — 	twenty =○
<hr/>			
twenty-one = 	twenty-two = 	twenty-three = 	

After counting 23 baby chicks, Fire Horse ran around the factory counting everything with her counting rods—those old chopsticks from A-máⁿ. She started with A-Kong’s ≡TT pots of orchids, followed by A-máⁿ’s —TTT chickens, TTTT ducks, then ⊥||≡TTTT glass bottles in the factory, T huge wooden barrels, —|||| factory workers, her ||| aunts, ||||| uncles, and TT cousins, ...

Before she was done counting everything, Fire Horse was picked up by her mother on the scooter and waved good-bye to A-máⁿ.

2.2 DIFFERENT SYSTEMS, OLD AND NEW—MEETING A-MÁ^N’S FRIENDS

Fire Horse dreamed of her A-máⁿ again, this time in A-máⁿ’s gathering with her celestial friends. “Fire Horse, I have to introduce you to my friends in the Blessings Club,” A-máⁿ greeted Fire Horse with her usual warm smile. “I know that my daughter, your third Aunt, is terribly worried about her baby granddaughter’s eye. My friends in the Blessings Club are joining forces with me in blessing the baby girl, especially after her operation for the removal of the cancerous eye.”

one of the 60 ($= 12 \times 5$) combinations. The latest year of the Fire Horse was 1966 when Japan saw a sudden **drop in births**.

The oldest grandson was the most favoured grandchild in a traditional East Asian family from which Fire Horse came.

For a deeper understanding of the power of superstition from a Chinese Canadian girl’s perspective, watch *Eve and the Fire Horse*, a movie by [Julia Kwan](#).



The **tally marks** or **hash marks** are used in counting ongoing results. For example, 23 would be written as |||| |||| |||| ||| . This is a **unary** numeration system. \triangleleft

Counting rods are positional with digits 1 to 9 using the same place values as our **Hindu-Arabic** system, that is, 1’s, 10’s, 100’s, 1000’s, and so on. From the 1’s place with one vertical rod representing 1, in the 10’s place, it switches to a horizontal rod for 1. The alternating arrangement of horizontal and vertical rods helps track

“I didn’t know that you got help from so many friends to bless us,” replied Fire Horse. “I would love to meet your friends in the club.”

Egyptian numerals

“Actually, more than just meeting them. My friends learned about how much you loved counting with me as a pre-schooler, and all of them are eager to show you how to count in their own ways. Here is Lady Khepri from Egypt. She prepared this drawing of ancient Egyptian numerals for you.”

1	10	100	1000	10 000	100 000	1 000 000
	∩	∩	⊥	∩	∩	⊥
Stroke	Heel bone	Scroll	Lotus flower	Pointing finger	Fish or tadpole	Astonished person

“As you can see, Fire Horse, our numerals can be written anywhere and still mean their corresponding values. However, for ease of reading, one often organizes the numerals in descending or ascending order. Do you know that more than $\perp \perp \cap \cap \cap$ blocks were used to build the **Great Pyramid of Giza**?” Lady Khepri drew two astonished persons and three tadpoles.

“Lady Khepri, this number must be two million and three hundred thousand. May I know why you belong to the Blessings Club?” Fire Horse asked timidly.

“My $\cap \cap \cap \cap \cap \cap \cap \cap \cap \cap$ descendants in Alexandria and $\cap \cap \cap \cap \perp$ in Cairo are exhausted trying to find adequate work to provide for their families. With little hope for the future, your grandmother and my friends help encourage my descendants with small signs of hope to keep them going,”

positions. For example, 36 is $\equiv T$ versus 306, $\equiv T$. \triangleleft

Yang Hui’s triangle is the Chinese version of Pascal’s triangle, published in 1303, more than 300 years before Blaise Pascal was born.

Pascal’s triangle is shown using counting rods on the top page of the picture taken from a Japanese mathematics textbook.

Fire Horse counted 37 hanging pots of orchids, 18 chickens, 9 ducks, 7255 glass bottles, 6 wooden barrels, 14 factory workers, 3 aunts, 5 uncles, and 7 cousins. \triangleleft

A **number** is an abstract concept conveying how many objects are counted; a **numeral** is a symbol representing a number. Different cultures use different numerals to represent the same number.



Egyptian numerals are **nonpositional**.

sighed Lady Khepri.

Roman numerals

A-máⁿ noticed that Lady Lucia just joined the group. “Here is Lady Lucia! Her numerals use letters from the alphabet,” greeted A-máⁿ.

Lady Lucia was too preoccupied with her own thoughts to talk about her numerals. “You must help me first to bless my family living on the coast. With so many asylum seekers arriving from the south and east, my family is so worried about not having enough space to house the asylum seekers, or enough jobs for them, or even enough food to feed them. I need us to bless them with wisdom and open mindedness for solving this difficult challenge. I know it is possible to do because some of my descendants in one coastal town managed beautifully—for a few years anyway—until another party was elected.”

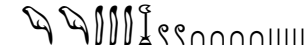


A-máⁿ comforted Lady Lucia, “We will definitely bless your family with wisdom and open mindedness so that the asylum seekers will be a blessing to their community, too. Fire Horse, my granddaughter, visits rarely, and would like to know how your numerals work.”

How many of Lady Khepri’s descendants live in Alexandria? 245.

How many in Cairo? 231 000.

How many descendants are there in both Alexandria and Cairo written in Egyptian hieroglyphs?



◀

Mayor Lucano of Riace saved a dying village by welcoming asylum seekers until a new party took over.

Roman numeral	Hindu-Arabic numeral
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

“Oh, yes. The Roman numerals can be puzzling; sometimes we add, and sometimes we subtract, depending on where the letters are placed. First, only 7 letters are used: **I** for 1, **V** for 5, **X** for 10, **L** for 50, **C** for 100, **D** for 500, and **M** for 1000. We write from the left to the right, starting with the largest numeral. For example, **MDCXVI** equals $1000 + 500 + 100 + 10 + 5 + 1 = 1616$.”

Fire Horse was very excited to learn this simple system, “With only 7 letters, I just need to add all the numbers together. Hmm, but you mentioned subtraction...”

“That’s right, Fire Horse,” replied Lady Lucia, “when you see a smaller value—no more than two levels smaller—immediately to the left of a larger value, then you need to subtract the smaller value from the larger. For example, **IV** means $5 - 1 = 4$, and **IX** means $10 - 1 = 9$. See if you can figure out what this means, **MCMXLIV**.”

Fire Horse started breaking the string of letters, “Let’s see. I am going to leave the first **M** by itself, followed by **CM** which need to be together because 100 is smaller than 1000, then **XL** because 10 is smaller than 50, and finally **IV** which is 4 as you showed me before. This is $1000 + (1000 - 100) + (50 - 10) + (5 - 1) = 1944$. Wow, now I can read the years in the end of film credits.”

Lady Lucia smiled and added, “Not just the years films are made, Fire Horse, you can also read years of statues and buildings in Europe now. Can you read the year on top of Galleria Giuseppe Siri in Genoa, Italy?” (See Figure 2.1.)



Figure 2.1: Close up of the top of Galleria Giuseppe Siri

$$\begin{array}{c} \mathbf{MDCXVI} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ 1000 + 500 + 100 + 10 + 5 + 1 = 1616 \end{array}$$

William Shakespeare died in **MDCXVI**.

$$\begin{array}{c} \mathbf{M \quad CM \quad XL \quad IV} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ 1000 + (1000 - 100) + (50 - 10) + (5 - 1) = 1944 \end{array}$$

The movie, *Casablanca*, won the Oscar for Best Picture in **MCMXLIV**.

Galleria Giuseppe Siri, Genoa, Italy.



Photographed by Jonathan Shapiro, Cal Poly State University.

MDCCCXXVII requires only addition, so 1827.

Incan quipu

“Not all cultures need numerals to record numbers, you know.” An elegant lady suddenly appeared in their midst.

“Fire Horse, here is another member of the Blessings Club. Meet my good friend, Lady Bachue.” A-máⁿ quickly introduced them, turning towards Lady Bachue, “Please show Fire Horse how the Incas kept numerical records.”

“The **quipu** consists of many cords with knots. Not long ago, a **university student** just deciphered how different coloured cords accounted for different names, so our quipu was used more like a spread sheet than just numerical records.”



Figure 2.2: Quipu

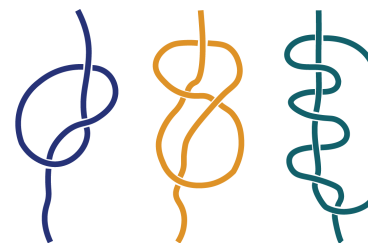
By Pi3.124 - Own work, CC BY-SA 4.0,
<https://commons.wikimedia.org/w/index.php?curid=69539294>

“This is so beautiful!” Fire Horse marvelled at the complex patterns on each cord. “I can see three different types of knots on a cord: a **single knot**, a **figure-eight knot**, and a **long knot**. The single knots are always above the long knot or the figure-eight knot.”

“Good for you, Fire Horse! When we tie knots, we begin with the biggest place using single knots, like 3 single knots together may mean 30, 300, 3000, and so on, depending on where they are relative to the bottom most knot. For the 1s place, we use a **figure-eight knot** for 1, and for 2 to 9, a **long knot** with 2 turns for 2, 9 turns for 9 and all the

The **mathematics of the Incas**, though without written records, supported an impressive system of roads, architecture, agriculture, and administration.

A **single knot**, a **figure-eight knot**, and a **long knot**:



numbers in between. If you start reading the cord from the dangling end which shows the 1s place, then all the single knots in clusters will have their places well defined.” Lady Bachue explained patiently.

“What if I have no tens and no hundreds, but 3000?” Fire Horse wondered.

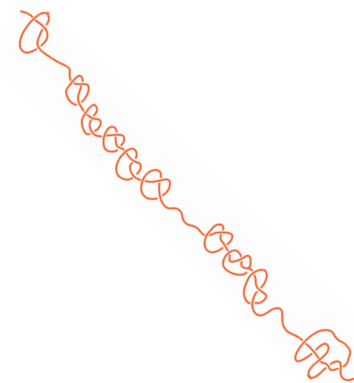
“Notice how the knots are clustered and spaced out to match neighbouring cords’ place values. When you see no knots where other cords’ 10s or 100s place shows knots, you know that you have nothing in the corresponding places. We don’t have many original quipus left for you to see, but people are making them now to revive the ancient art. I hope you try to make some for yourself.” Lady Bachue commented kindly, “and when you do, remember **quinoa farmers** of my people. They might have been helped by the price of quinoa increasing six fold some years ago, but now they have competitors all over and are cutting diversity and rejecting traditional farming practices in exchange for larger yields. We in the Blessings Club continue to bless the health and well being of my people, to be inspired by what the **Incas built** and to remember who they are.”

Mayan numerals

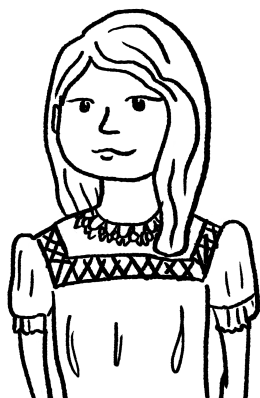
“I also would like my descendants to remember who they are,” a new voice joined their discussion. Lady Bachue brightened up as she introduced her friend from the north, Lady Chimalmat, to Fire Horse, “Lady Chimalmat came from a rich culture of writing, art, and science, especially in the study of astronomy which resulted in the Mayan calendar consisting of 365 days, that is, 18 months of 20 days per month plus 5 bad days. You have to see how Lady Chimalmat count with Mayan numerals.”

Lady Chimalmat proudly added, “Compared to other ancient calendar systems of the world, the Maya had the most complex, intricate, and accurate systems. Our numerals are simple, just dots and horizontal bars: We use a dot, ●, for one and a horizontal bar, —, for five. Thus ●●● is 13. For larger numbers, we combine these in a base-20 system

What is the numerical value expressed by the cord?



◁



written vertically with the most significant digit at the top. For example $\overset{\bullet}{\bullet}\overset{\bullet}{\bullet}$ is 2 twenties and 4, so 44. As you can see, this is a positional system. We use the special symbol, a shell, $\text{\textcircled{0}}$, for zero. For example, reading from the top, a dot with a bar is $1 + 5 = 6$, so $\overset{\bullet}{\text{\textcircled{0}}}$ is 6 twenties and no 1s, so $6 \times 20 = 120$.

There is an exception in our base-20 place values due to the significance of 360. The third place counts the $20 \times 18 = 360$ s (not 400s place as many might conclude from 20×20), so $\overset{\text{\textcircled{0}}}{\bullet}$ is $5 \times 360 + 0 \times 20 + 11 = 1811$, ten years before we were freed from the **Spanish rule**.”

Fire Horse was amazed by such an elegant system as Lady Chimalmat showed her more and more numerals, but suddenly remembered what Lady Chimalmat said when she joined the group. “What would you like your descendants to remember?” Fire Horse asked.

Lady Chimalmat started sharing, “Their food, their land, their culture—so much lost to **NAFTA**, the so called ‘North American Free Trade Agreement.’ Instead of local corn, bean, and squash, cheap sugar laden corn fills their dinner table, with more sugar drinks—I

Examples of converting Mayan numerals to Hindu-Arabic numerals:



1. $\overset{\bullet}{\bullet}\overset{\bullet}{\bullet}$ In one place, at most 4 dots can be seen; therefore, when 4 dots are placed directly above 3 dots, the top 4 dots must be in the next larger place. Also, in each place, horizontal bars are below the dots. When 4 dots are below a horizontal bar, the dots must be in the next smaller place. Beginning from the 1s place at the bottom, one sees 4 1s, followed by 13 20s, and finally, 4 360s, in total, $4 + 13 \times 20 + 4 \times 360 = 1704$.



2. $\overset{\bullet}{\bullet}\overset{\bullet}{\bullet}$ Here, one begins with 16, followed by 4 in the 20s place, and 10 in the 360s place, all together, one gets 3696.



3. $\text{\textcircled{0}}$ This begins with no 1s, 12×20 , and 12×360 , so 4560.

The power of Mayan numerals is seen best in addition and subtraction. Try to add the first two examples and notice the ease in which one re-bundles into the next larger place. Similarly, notice the ease of unbundling for subtractions. \triangleleft

The comparison in corn production, government subsidies, and prices from **Corn in Mexico and the US** by Philip

mean, pop—to boot. My descendants are getting sicker and sicker from unhealthy food. The loss of their land to growing strawberries and lettuces for rich northern countries also meant further loss of healthy food source. Our wonderful food culture is reduced to fast junk food. Oh, how tirelessly we bless them in our club to help them regain their healthy way of living!”



Figure 2.3: Mayan corn

Babylonian numerals

Out of the blue, a lady burst into their midst, breathless. “My friends, please help me bless my descendants who depend on the **Euphrates River**. Years of drought, construction of hydro-electric dams, wars, and mismanagement have dried up so much of the river that in some areas, animals are dying and people are starving. So much history is associated with the Euphrates River; I simply can’t imagine the drying up of such a mighty river.”

“We have been blessing that river and the people living along the river for a long time,” sighed A-má” and her friends, “but the Euphrates River just gets drier and drier. How desperate are the poor farmers! We wish we could do more.”

Martin highlights the challenges faced by Mexican farmers.

The epidemic of diabetes in **Chiapas**, Mexico, is caused by the introduction of sugar drinks to the local population while at the same time depleting fresh water source for the production of said drinks.

The plight of the Zapotecs of Oaxaca, Mexico is shown vividly on pages 69–71 in **Moe-Lobeda**’s book: *Resisting Structural Evil* [21].

Mayan corn was central to the culture. GMO-contaminated corn threatens biodiversity of the local corn; fortunately, public awareness has helped to start turning the tide.



“Lady Shala, meet my granddaughter, Fire Horse,” said A-máⁿ, “who is here to learn about your numerals.”

“Hi, Fire Horse, before we talk about the Babylonian numerals, you have to promise me that you will learn about the Euphrates River and help the people who depend on the river in whatever way you can,” Lady Shala urges.

“Sorry, Lady Shala, that I know so little about the history of Euphrates River other than the **Hanging Gardens of Babylon** which exhibited an engineering feat of irrigation almost three thousand years ago. I promise to learn more about the river and its people,” answered Fire Horse.

“Now, let me show you how elegant our numeration system is,” Lady Shala began, “like Mayan numerals, Babylonian numerals used only two symbols, a stroke, Υ for 1, and an angle, \sphericalangle for 10. Imagine holding a wedge-tipped reed stylus to make a mark on a soft clay tablet. When you mark the stroke for one, it starts wider then tapers to a thin line. For a 10, you start an angle with a thin line, then at the turn, broadens the mark before tapering to a thin line again. That is what you can see in some **Babylonian clay tablets**,” continued Lady Shala. “Our system is also positional, like Mayan numerals, but instead of base-20, we use base-60. Because 60 has so many factors, working in base-60 made fractions easier for us and allowed us to advance in astronomy and mathematics.”

“Instead of starting from the bottom as in Mayan, we begin from the right for the 1s place where we can write up to 59 in a base-60 system. The angles denoting 10s are written to the left of the strokes, or 1s. When there are lots of strokes, we organize them into three in a line to make them easier to read. Can you tell me what number this is?”



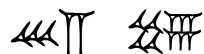
The **Babylonian clay tablet** from Yale Peabody Museum of Natural History shows an approximation of $\sqrt{2} \approx 1 + 24/60 + 51/3600 + 10/216000$. A fraction of $25/60 + 35/3600$ is seen following 42 on the tablet to show that the diagonal of a square with side length 30 is $42 + 25/60 + 35/3600 \approx 42.42638888$ compared to $30\sqrt{2} \approx 42.42640687$.



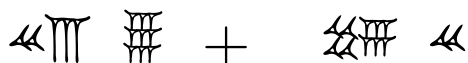
“Since there are 4 angles and 7 strokes, I think it is $4 \times 10 + 7 \times 1 = 47$.” Fire Horse ventured. “I am interested in larger numbers, like the last year of the Fire Horse, 1966, in Babylonian numerals.”

Lady Shala smiled with a twinkle in her eye and continued, “I like your enthusiasm. If the place values from the left are the 1s (up to 59), the 60s, followed by the 3600s place, you will need to begin with the 60s place since there isn’t enough for any 3600s. How many 60s does 1966 have?”

Fire Horse responded quickly, “I see—since $1966 \div 60 = 32$ with a remainder of 46, I must have 3 angles and 2 strokes in the 60s place and 4 angles and 6 strokes in the 1s place. That’s it.” Fire Horse beamed, “Like this.”



“I think you will appreciate how easy addition and subtraction are performed in base-60,” Lady Shala wants to show more to Fire Horse. “Try to add these two numbers.”



“Thank you very much, Lady Shala. You have given Fire Horse lots to try now. She should continue with some examples of subtraction herself. I still need to find Lady Aua before Fire Horse leaves because Lady Aua has something exciting to show my granddaughter.” A-máⁿ cordially bowed to the ladies before ushering Fire Horse away.

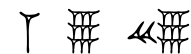
Kaktovik numerals

Lady Aua had been expecting A-máⁿ and Fire Horse. “Welcome to my little Alaskan Iñupiat corner! I understand that you, Fire Horse, have been learning a lot of numeration systems from my friends in the Blessings Club, all of which, ancient systems. What I have here for you

In the beginning of learning any numeration system, it is helpful to convert both ways, from 1966 to Babylonian numerals, and back: $32 \times 60 + 46 = 1966$. ◁

A brief history of [the influence of a sexagesimal \(base-60\) system](#) is given by Michael A. Lombardi of the National Institute of Standards and Technology in Boulder, Colorado.

The answer to the addition question is $1389 + 2780 = 4169$. Notice how the 1s place simply combines 2 angles and 9 strokes, similarly for the strokes in the 60s place. With 6 angles in the 60s place, one bundles them into 1 in 3600s place.



Again, to check that the Babylonian numerals shown is 4169, one uses expansion: $1 \times 3600 + 9 \times 60 + 29 = 4169$. ◁

is a new invention from 1994, by students of Kaveolook Middle School on **Barter Island**. My descendants there could not fit our language to the decimal system because Inupiat has a base-20 system. To address the deficiency, they invented a set of simple numerals for a base-20 system and started using them to do arithmetic. It caught on so quickly that even in the local community college, one can now take a course in Inuit mathematics using this new system, the Kaktovik numerals.

“We use \times for zero to symbolize arms crossed, or nothing in this place, \setminus for 1, and \checkmark for 5.”

0	1	2	3	4	5	6	7	8	9
\times	\setminus	\vee	\wedge	\mathbb{W}	\checkmark	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle
10	11	12	13	14	15	16	17	18	19
$>$	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle	\sphericalangle

Table 2.1: Kaktovik numerals

Fire Horse remembered from Lady Chimalmat’s Mayan numerals and thought out loud, “This is very similar to Mayan numerals, not just because they are both base-20 systems, but within each place value, the numerals are organized into base-5. The Kaktovik numerals are much easier to operate, though, like $2 + 2 = 4$ looks like $\vee + \vee = \mathbb{W}$.”

“Good for you, Fire Horse!” Lady Aua clapped. “Wait until you see long division in Kaktovik numerals. It is so natural to think in terms of up-down strokes and side-way lines. My descendants used to struggle with math; however, after being taught Kaktovik numerals, they now score above the national average in standard mathematics assessment. What a difference it makes to learn mathematics in numerals related to our own language! Your grandmother and my friends in the Blessings Club are working hard to comfort the First Peoples of North America due



Many indigenous numeration systems in the American Continent are **penta-vigesimal**, that is, base-20 with a sub-base-5, like the Mayan numerals.

Some European names for numbers are also organized in groups of 20 like the French 90 is *quatre-vingt-dix*, or four twenties and ten. In Danish, numbers from 50 to 90 also use groups of 20 in their names.

A table of Kaktovik numerals can be found on page 13 of **Sharing Our Pathways: A Newsletter of the Alaska Rural Systemic Initiative** featuring an article by William Clark Bartley: *Making the Old Way Count*. Mr. Bartley was the teacher who led the students of his class to invent Kaktovik numerals.

For some examples of long division using Kaktovik numerals, please watch a **4-minute video** by Artifexian.

to the recent discoveries of unmarked graves of children on Residential School grounds in Canada. Fire Horse, be a good listener when you meet people of different cultures. We have so much to learn from one another.”





“Thank you for your wisdom and passionate sharing of Kaktovik numeration system. I will try my best to listen to people of all cultures and learn from them.” Fire Horse held Lady Aua’s hands while saying good-bye.

Fire Horse thought about all these numeration systems she just learned from the ladies in her A-máⁿ’s Blessings Club and wondered how big a number one could express using just four distinct digits from each positional system. In the system she was taught, 9876 is the biggest such four-digit number, and according to its place values, it is nine thousand eight hundred and seventy-six.

Place value:	10^3	10^2	10^1	$10^0 = 1$
Digits:	9	8	7	6

Expansion: $9 \times 10^3 + 8 \times 10^2 + 7 \times 10^1 + 6 \times 10^0 = 9876$

For Mayan numerals,

Allowed digits	Place value		Expansion
0–19	$20 \times 18 \times 20$		$19 \times 20 \times 18 \times 20$
0–19	18×20		$+ 18 \times 18 \times 20$
0–17	20		$+ 17 \times 20$
0–19	1		$+ 16 \times 1$
			$= 143\,636$





Ethnomathematics in mathematics education is the study of the relationship between mathematics and culture. Respecting the language of a culture by inventing a numeration system to match the counting words (and its base-20 organization) is the first step towards engaging students for the study of mathematics.

Indian Boarding Schools in the US functioned like Canadian Residential Schools.

The exponential notation a^n means $a \times a \times a \times \dots \times a$, that is, n copies of a multiplied together. We call n the exponent and a the base in the notation. For example, in the base-10 system, the fifth place from the right has the value, $10^4 = 10 \times 10 \times 10 \times 10 = 10\,000$. When the exponent is 0, then $a^0 = 1$ for all non-zero base- a ; in particular, $10^0 = 1$. ◁





much bigger than 9876.

For Babylonian numerals,

Place value:	60^3	60^2	60^1	$60^0 = 1$
Digits:				
Expansion:	$59 \times 60^3 + 58 \times 60^2 + 57 \times 60^1 + 56 \times 60^0$			
	$= 12\,956\,276$			

even bigger than Mayan numerals.

For Kaktovik numerals,

Place value:	20^3	20^2	20^1	$20^0 = 1$
Digits:				
Expansion:	$19 \times 20^3 + 18 \times 20^2 + 17 \times 20^1 + 16 \times 20^0$			
	$= 159\,556$			

slightly larger than Mayan numerals because of the 20s instead of 18s.

“Thank you for sharing your ways of counting with me, ladies.” Fire Horse shook hands with everyone before bidding them farewell. Just at the last moment, A-máⁿ gave Fire Horse a big bear hug while whispering into her ear, “Bless everyone wherever you go!”

2.3 COUNTING IN OTHER BASES—THINKING DEEPLY, ALONE

Venturing into other bases

Fire Horse awoke from her dream of visiting A-máⁿ and her friends from the Blessings Club and couldn’t wait to start exploring new ways of counting. “Why do I restrict myself to just base-20 or base-60? Let

me see what happens when I count in base-8. Since the place values will be powers of 8, I only have 8 digits available in each place, namely, $\{0, 1, 2, 3, 4, 5, 6, 7\}$, because eight will be written 10_8 for having 1 eight and no 1s. I will use the subscript 8 on the lower right hand side to remind me that this is in base-8,” explained Fire Horse to herself. Similar to her comparison of positional numeration systems, Fire Horse constructed the following for a 4-digit base-8 number. Since the largest digit possible in base-8 is 7, she wrote $7654_8 = 7 \times 8^3 + 6 \times 8^2 + 5 \times 8^1 + 4 = 4012_{10}$.

Place value:	8^3	8^2	8^1	$8^0 = 1$
Digits:	7	6	5	4
Expansion:	$7 \times 8^3 + 6 \times 8^2 + 5 \times 8^1 + 4 \times 8^0 = 4012_{10}$			

Counting

Fire Horse remembered that learning a new number system meant learning how to count; therefore, she made herself Table 2.2 to see the pattern of progression from the largest digit 7 to the next number by changing the neighbouring place.

0_8	1_8	2_8	3_8	4_8	5_8	6_8	7_8
10_8	11_8	12_8	13_8	14_8	15_8	16_8	17_8
20_8	21_8	22_8	23_8	24_8	25_8	26_8	27_8
30_8	37_8
\vdots							\vdots
70_8	71_8	72_8	73_8	74_8	75_8	76_8	77_8
100_8	101_8	...					

Table 2.2: Base-8 counting

Tom Lehrer composed the song, *New Math*, where, among other things, he also explained subtraction in base-8.

With the Mayan, Babylonian, and Kaktovik numerals, you learned new symbols for the digits. Here in Section 2.3, we use the familiar digits but interpret them in a new way, $7654_8 = 4012_{10}$.

The goal was to continue using distinct digits to make the largest possible 4-digit number, here in base-8. ◀

A spreadsheet can help establish patterns of counting in different bases. Recall our work with Google Sheets on pages 22–23. The function DEC2BIN converts a decimal number to binary (base 2). Type “=DEC2BIN(A1)” into cell B1, then copy and paste that cell into column B, which will show the binary numbers from 1 in B1 to 11 001 in B25. Similarly, one can use DEC2OCT or DEC2HEX for converting to octal (base 8) and hexadecimal (base 16), respectively.

	A	B
1	1	=DEC2BIN(A1)
2	=A1+1	=DEC2BIN(A2)
3	=A2+1	=DEC2BIN(A3)
\vdots	\vdots	\vdots

◀

To convert from decimal to other base- b for $b < 10$ for the sake of counting patterns, a combination of the **floor function** (for the quotient) and the **modular function** (for the remainder) will help for *two-digit* counting patterns.

Fire Horse tried the next extreme, that is, using only 2 digits, $\{0, 1\}$ to count in base-2. “0, 1—hmm, I am stuck because there are only two digits allowed; I can’t use 2, but in base-2, the place values are powers of 2,” thought Fire Horse, “so I can just use 10_2 to express 2_{10} . Now I see the pattern, $11_2, 100_2, 101_2, 110_2, 111_2, 1000_2, 1001_2$, and so on. This also means that I can use my fingers to count up to 1023 if I do **binary counting** instead of the usual base-10!”

“With the place values on top of each corresponding digit of a binary number, I can easily convert from binary to Hindu-Arabic base-10 numerals, say 10100111_2 to base-10.” Fire Horse constructed the table for the binary number.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	$2^0 = 1$
1	0	1	0	0	1	1	1
$1 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$							
							$= 167$

Converting back and forth

“How about going the other way? Suppose I want to convert 168 to binary. Since I know how to count in binary, it should be one more than 167 which is 10100111_2 , so $10101000_2 = 168$. Like I did converting 1966 to Babylonian, where I needed to see the number of 60s in 1966, for binary, I would need to see how many 2^1 s are in 168, then $2^2, 2^3$, and so on. **Successive division** by 2 will give me the digits beginning from $2^0 = 1$ s place.” Fire Horse started to see her way through as shown in Figure 2.4 on the following page. To read the resulting binary number, Fire Horse found the largest place in the conversion with a non-zero digit, namely, 128s place, and copy the binary number, 10101000_2 , which is identical to her prediction from counting.

Fire Horse was excited by this algorithm which could be easily adapted to the conversion of Hindu-Arabic numerals to Mayan numerals by dividing the right number at the right time.

A more complex code is needed for larger numbers. For example, in base-7, the first 25 counting numbers are shown by entering “=FLOOR.MATH(A1/7)*10+MOD(A1,7)” in cell C1, and copy and paste the cell to the rest of Column C. (Why will this formula fail at 49?) ◀

0_2	1_2	10_2	11_2
100_2	101_2	110_2	111_2
1000_2	1001_2	1010_2	1011_2
1100_2	1101_2	1110_2	1111_2
10000_2	10001_2	10010_2	10011_2
10100_2	10101_2	10110_2	10111_2
11000_2	11001_2	11010_2	11011_2
\vdots			\vdots
Binary counting			

For an introductory video on **binary numbers** (base-2), please visit Khan Academy. ◀

Computer scientists think of binary numbers as a string of ON’s and OFF’s (1 for ON and 0 for OFF). For the sake of conversion from binary to Hindu-Arabic, when 1 is under the place 2^7 , it translates to “Take 2^7 ”. On the other hand, a 0 under the place 2^6 translates to “Skip 2^6 ”. In short, the expansion is $2^7 + 2^5 + 2^2 + 2^1 + 2^0 = 167$, namely, pick out the places with 1s under them in order to add the corresponding place values. ◀

An example of converting 13373 into Mayan numerals using **successive division**:

2 168	0	1s digit
2 84	0	2s digit
2 42	0	4s digit
2 21	1	8s digit
2 10	0	16s digit
2 5	1	32s digit
2 2	0	64s digit
2 1	1	128s digit
2 0	0	256s digit

Figure 2.4: Conversion of 168 to base 2

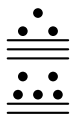
Adding

After learning how to convert numerals back and forth between different bases, Fire Horse began adding them. For example, addition in base-5 would require one to bundle fives into the next bigger place, like in the 1s place of $12_5 + 34_5$, $2 + 4 = 6 = 1 \times 5 + 1$; therefore, Fire Horse put 1 in the 1s place and a small 1 (for one 5) in the 5s place. Similarly for the 5s place, $1 + 1 + 3 = 5 = 1 \times 5 + 0$ yielded a 0 in the 5s place and a 1 in the 5²s place.

$$\begin{array}{r} 1 \\ 12_5 \\ + 34_5 \\ \hline 101_5 \end{array}$$

“I want to try adding two binary numbers to see how bundling works in groups of 2.” Fire Horse began writing her example.

$$\begin{array}{r} 111 \\ 10110_2 \\ + 1111_2 \\ \hline 100101_2 \end{array}$$

20 13373	13	1s digit	
18 668	2	20s digit	
20 37	17	360s digit	
20 1	1	7200s digit	

Turning 1, 17, 2, 13 into Mayan numerals from the top is straight forward. ◁

The reverse of successive division from the conversion algorithm is **successive multiplication**, or **Horner’s method**. In the example for 13373, Horner’s method would

1. take the result of the largest place, namely 1, and multiply by 20: 20×1 ,
2. add 17 to the previous result (now, 37), then multiply by 18: $18 \times (17 + (20 \times 1))$,
3. add 2 to the previous result (now 668), then multiply by 20: $20 \times (2 + 18 \times (17 + (20 \times 1)))$,
4. finally, add 13 to the previous result to get 13373 as required. ◁

Complete the following examples of addition by finding X:

$$1. \begin{array}{r} 1 \\ X7_8 \\ + 34_8 \\ \hline 103_8 \end{array}$$

“It seems that as long as I know single digit addition: $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$, $1 + 1 = 10_2$, and $1 + 1 + 1 = 11_2$, I am all set for adding in binary.” Fire Horse felt ready to move on to subtraction.

Subtracting

“Let me see if I still remember what my grade two teacher taught me about subtraction starting from the **1s** place, say $25 - 8$. Since 5 is smaller than 8, I need to go to the **10s** place and unbundle one **10** into 10 **1s**, thus changing 2 **10s** into 1 **10**.” Fire Horse started writing the subtraction algorithm.

$$\begin{array}{r} \text{10} \\ \text{2 } 5 \\ - \quad 8 \\ \hline 1 \text{7} \end{array} \quad \text{where the } 7 = 10 + 5 - 8$$

“As long as I unbundle the size of the base (instead of 10) whenever I am short, then the same subtraction algorithm will work. Let me try $101_5 - 34_5$. For the sake of counting backwards for subtraction, I will count in base-5 first: 1, 2, 3, 4, 10_5 , 11_5 , 12_5 , 13_5 , 14_5 , 20_5 , 21_5 , Starting from the **1s** place, where it is not enough to subtract 4 from 1, I look in the **5s** place—hmm, but 0 doesn’t help, so I look at the **25s** place also, and take 1 from 10_5 **5s** to get $10_5 - 1_5 = 4_5$ in the **5s** place. Now in the **1s** place, I have $11_5 - 4_5 = 2_5$, simply from counting backwards 4 steps.”

$$\begin{array}{r} \text{45} \\ \text{101}_5 \\ - \quad 34_5 \\ \hline 12_5 \end{array} \quad \text{where the } 2 = 5 + 1 - 4$$

Fire Horse found subtraction in different bases routine and started playing with multiplication.

$$2. \quad \begin{array}{r} \text{1} \\ \quad 5 \text{2}_6 \\ + \quad 3 \text{4}_6 \\ \hline 1 \text{X} \text{0}_6 \end{array}$$

$$3. \quad \begin{array}{r} \text{1} \\ \quad 2 \text{3}_4 \\ + \quad \text{X} \text{2}_4 \\ \hline 1 \text{2} \text{1}_4 \end{array}$$

4. This addition is missing the base. Find the base to

make the following addition correct.

$$\begin{array}{r} \text{1} \\ \quad 45_X \\ + \quad 34_X \\ \hline 112_X \end{array}$$

◁

Multiplying

“The complication with multiplication is tracking numbers for different places,” thought Fire Horse to herself, “so using the **Galley Method** will solve this problem in any base. Let me try to multiply $234_5 \times 23_5$. Wow, this is amazing! When all the answers in the grid are expressed in base-5, the triangles lying to the left of the same diagonal belong to the same place, so I get a product of 12042_5 .” (See Figure 2.5.)

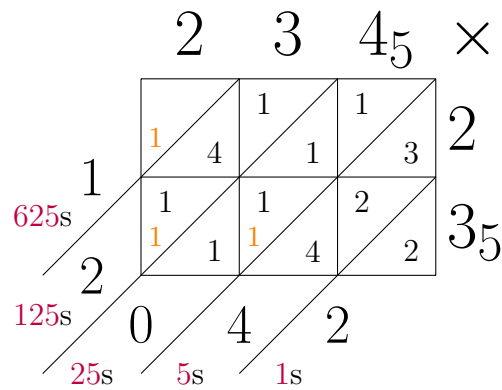


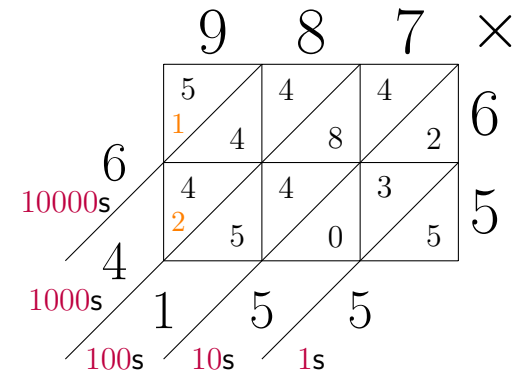
Figure 2.5: Galley method for $234_5 \times 23_5$

Converting between binary, octal, and hexadecimal systems

Fire Horse finished her exploration of multiplication and felt ready to take a break from counting in so many different bases when she started wondering what would happen if a base was greater than 10. “With only 10 digits available, I would need other symbols for *digits* like 10, 11, 12, 13, and so on. Perhaps, I can borrow capital letters from the alphabet to let A = 10, B = 11, C = 12, D = 13, E = 14, F = 15 in the case I am computing in base-16 and need six more *digits*.”

The **Galley Method** or **Lattice Method** of multiplication uses a rectangle subdivided into triangles to record digit-by-digit multiplication while tracking places. This method of multiplication and **division** using counting rods was known in ancient China since the first century.

An example of multiplication using the Galley Method is shown for $987 \times 65 = 64155$. Notice that the upper left hand corner triangle occupies the largest place, and only needs to be added to the number (if any) carried from the next smaller place.

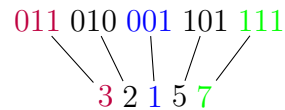


Making a multiplication table in base-5 is useful for those products using more than one digit:

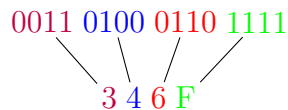
$$2 \times 3 = 11_5, 2 \times 4 = 13_5, 3 \times 4 = 22_5, 4 \times 4 = 31_5. \quad \triangleleft$$

Base-8 and base-16 have special significance in computer science and are also called **octal** and **hexadecimal** respectively.

Suddenly, Fire Horse became very excited about binary, octal, and hexadecimal systems. “Wow, 8 and 16 are both powers of 2, namely, $8 = 2^3$, and $16 = 2^4$. Therefore, converting back and forth between binary and either octal or hexadecimal should be easy when digits are organized into groups of the appropriate sizes. For example, let me see about $11\ 010\ 001\ 101\ 111_2$ converted to base-8 and base-16. For base-8, I will begin grouping the binary number from the smallest place in groups of 3 because the largest number expressed by three digits in base-2 is $111_2 = 7_{10}$, precisely the largest digit allowed in base-8. The resulting base-8 number is 32157_8 .”



Fire Horse was delighted to see how fast conversion works the other way, each digit in base-8 just expanded to a 3-digit binary number. “In the case of base-16, I will group the binary digits from the right into 4 digits per group to get $346F_{16}$. Converting from base-16 to binary is similar, where each digit in base-16 expands to a 4-digit binary number. I think I have done enough converting between numbers of different bases. Could someone please tell me why they are so useful in digital technology?”



And with that question, Fire Horse took a break from counting and converting, and hiked up the mountain to bathe in B.C.’s magical rain forests.

Binary base-2	Octal base-8	Hexadecimal base-16
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000		8
1001		9
1010		A
1011		B
1100		C
1101		D
1110		E
1111		F

The 3-digit binary numbers (in blue) are matched with octal numbers whereas 4-digit binary numbers are with hexadecimal ones. ◀

2.4 CODING—ENCOUNTERING SEVEN DRAGGONES IN A DONEGEON

Not all Fire Horse's dreams were happy and sweet visits with people she loved. Sometimes, Fire Horse had panic stricken, cold sweat drenched nightmares, classic ones like being chased by a faceless villain, running faster than she could down an infinite flight of stairs leading into an abyss, or being unable to scream for help when she really needed to. But, these nightmares would usually occur one at a time, at most once a month.

Silencegone

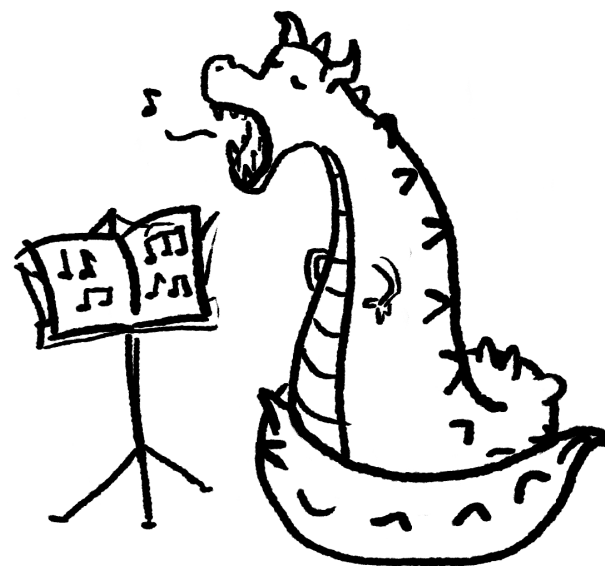
One night, however, Fire Horse found herself lost in a Donegeon inhabited by seven Draggones. To get out of the Donegeon, Fire Horse had to encounter each Draggone individually and face their fiery breath. At first, Fire Horse arduously adapted to the darkness of the Donegeon. Once she could see slightly with the help of small shafts of light filtering through the crevices, she saw a scaleless, menacing Draggone in whose presence Fire Horse lost her voice. She wanted to scream for help hoping that sound might travel through the crevices and be heard because the presence of the Draggone brought back floods of memories where she wanted to speak out, even to scream, but was voiceless. The Draggone bellowed, "Do not disturb me when I am working on my scales to break the curse of being scaleless! Don't you know that I am Silencegone, the one who can grant you your voice back? You must decipher a code involving two words: The first sounds the same as note number 11₂ of the French musical scale; the second, 'bunny' in Mandarin."

Fire Horse remembered her do-re-mi scale from the children's choir she used to belong to. Her mother was born in the year of the bunny, so she could still say *tu* for bunny in Mandarin. Immediately, Fire Horse picked up a stick and wrote *Me Too* on the dirt. Silencegone breathed their tongue loosening fire that swooshed over Fire Horse to give her a strong voice not only to speak up for herself but also to speak out for

To get a nine-minute version of Dungeons and Dragons, watch [Doodling in Math Class: Dragon Dungeons](#) by Vi Hart.

Donegeons and *Draggones* are pronounced as *Dungeons* and *Dragons*.

Musical scales are composers' building blocks.



others. “I promise never to interrupt your practise and hope that you get your scales soon!” Fire Horse shouted out her first sentence in the Donegeon.

Flightgone

Fire Horse was amazed that the fiery breath of Silencegone did not even make her hot. “I wonder what is next after Silencegone.” As Silencegone retreated to their cave, a Draggone covered in a mixture of scales and feathers flew in to rest in front of Fire Horse. Suddenly, Fire Horse saw herself flying, free of the immense pressures of living. “Fire Horse, I am Flightgone. Use the **ASCII table** displayed by my scales and feathers to decipher the message: *4C 65 74 20 67 6F 21*”

“I am going to assume that these two digit numbers are in base-16 because of the presence of *C* and *F*.” Fire Horse started looking up the table by flying around Flightgone. According to the table, the code translated to *Let go!* “I know what Flightgone means: I must let go of my pressures to stop having dreams of flight, no matter how wonderful flying feels. I know now that I will never learn how to cope with the pressures of living by having dreams of flight.”

Rushgone

Flightgone read Fire Horse’s mind and blew their fiery breath to incinerate all the pressures Fire Horse carried on her. As Fire Horse realized what just happened to her and was going to thank Flightgone, they had already flown back to their cave. In their place now sat Rushgone.

Right away, Fire Horse was rushing to make it to all the important events of her life, but was now hopelessly late for everything. No matter how much she hurried, another emergency seemed to pop up to detain her, from a final examination at the university, from a presentation in a lecture, from a flight she must catch, from a ceremony she must attend, and on and on. “What must I do to get out of this mad dash?” cried out Fire Horse to Rushgone.

Due to the Chinese censorship of social media, the #MeToo movement morphed into #RiceBunny because *rice* is pronounced ‘mi’, and *bunny*, ‘tu’.

ASCII stands for American Standard Code for Information Interchange. The **ASCII character table** contains 7-digit binary, or 3-digit octal, or 2-digit hexadecimal, and decimal code for each character. Flightgone gave the hexadecimal code of characters from the table. ◀



“You must read my light and dark scale patterns on my neck, Fire Horse, to free yourself.” Rushgone stated quietly, “They are coding two digits using the encoding method of **Universal Product Code**.”

With great effort, Fire Horse took a deliberate, slow, deep breath and recollected what she read before. After a few more deep, cleansing breaths, she began talking to herself. “I know that each digit is coded by 7 modules, or strips to be filled or left blank, similar to the 7-digit binary code in the ASCII table. Vaguely, I remember the digit 0 is coded 3 – 2 – 1 – 1 to mean that the first 3 strips are blank, then 2 strips filled, 1 strip blank, and 1 strip filled to end. Rushgone’s neck starts with 2 – 2 – 2 – 1 that encodes 1 and 1 – 4 – 1 – 1 for 3. Oh, my lucky number!” Fire Horse yelled out, “Thirteen!”

With that, Rushgone gently refined Fire Horse with their calming fiery breath which stopped her sense of urgency and her feeling of perpetually being late.

Paingone

Rushgone disappeared faster than Fire Horse could say thank-you. As the next Draggone emerged slowly from their cave, Fire Horse was struck with debilitating pain all over, the worst of which was on her left elbow.

“My name is Paingone,” hissed the Draggone. “To stop your pain, tap all over your body including your skull and joints regardless of how painful it is to lift your hands. For your left elbow, in particular, look at the pattern my eyes shine on it and tap on the red regions harder.”

“I have to overcome the pain and start tapping to get better,” Fire Horse mumbled to herself arduously, “What a strange pattern on my left elbow! How did Paingone shine a **QR code** on my elbow with their laser-piercing eyes? Lucky my elbow wasn’t burnt.” Fire Horse tapped determinedly all over her body and according to the QR code on her left elbow. The moment her tapping was done, Paingone breathed their

Universal Product Code or UPC, in short, is the unique barcode assigned to a product, the code that a cashier points the scanner at during check out.



The Art of Living Peacefully combines many methods of **self healing** from different cultures.

Quick Response code or QR code in short is a two-dimensional code with three identical squares at the corners and filled or blank squares in specific zones for information. When a smart phone camera detects the QR code, it converts the code to a link.

pain-relieving breath; all Fire Horse felt was a slow release of pain's grip on her body upon Paingone's departure.

Aquagone

Even before the next Draggone entered, Fire Horse felt icy cold water under her feet. The water rose fast to her neck, arresting her breath while, at the same time, forcing her to stand on her toes to behold a very long Draggone partly covered by the water. "Fire Horse," thundered the Draggone, "you will soon drown in the icy cold water unless you tell me what is at the bottom of the pool."



As the water covered her head, a sense of calm overtook her panic. She realized that she could see under water, and immediately dove down to the bottom of the pool. "Aquagone is the Draggone's name", spelled out with small mosaic tiles. "I also see a 4×4 square, with only 4 dark squares, one in each row and column. Hmm, how odd that the **displacements** of all distinct pairs of dark squares are different! Am I looking at a Costas array?" Fire Horse swam up to talk to Aquagone, "Aquagone, you have a Costas array of 4 dark squares at the bottom!"

This was the pattern on the left elbow of Fire Horse's arm.



Cold water swimming is an extreme sport with benefits and risks. The British author, **Katherine May**, poetically described the benefits to mental health from cold water swimming in *Wintering: The Power of Rest and Retreat in Difficult Times* [20].

A **Costas array** is a square two-dimensional code with 0s and 1s where every row and column contains exactly one 1, and no two line segments joining a pair of distinct 1s have the same displacement vector. This property is capitalized to control the frequencies of a sequence of sonar pulses in order to determine correctly both the distance and velocity of a target, even in the presence of noise.

Some **small Costas arrays** up to 4×4 are given. The one Aquagone gave to Fire Horse was a symmetric one about the red diagonal. The **displacements** between AC and AD are different even though the lengths or distances are the same.

A			
	B		
			C
		D	

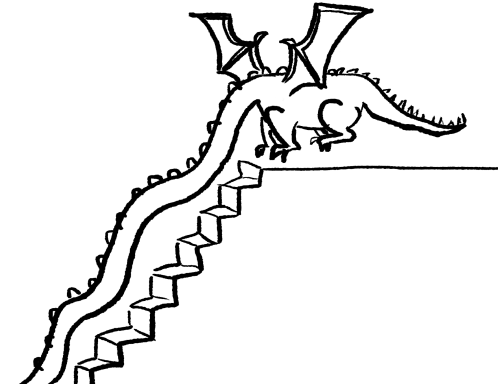
No sooner had Fire Horse announced her findings than Aquagone started swimming away. With that retreated the icy cold water leaving Fire Horse shivering. “Y-y-y-you f-f-forgot to blow your fiery breath on me.” Fire Horse called after Aquagone. With a swipe of their long tail, Aquagone turned around to give Fire Horse a warm drying breath. Not only was Fire Horse dry, but also all her worries seemed to have been washed away.

Tumblegone

Before Fire Horse could figure out what just happened to her, a big earthquake shook the Donegeon and a long staircase opened down where she was standing. “Oh no, I know this dream too well. I am running away from a monster down an infinite flight of stairs. So many times in the same nightmare, I even decided to get it over with by jumping down into the abyss, never mind stepping on every step just to wake myself up from the nightmare. Guess it’s not going to work here no matter how hard I jump down because this is only the sixth Draggone I am about to encounter,” Fire Horse thought. “It is not easy to run fast down the stairs while looking back up!” Fire Horse screamed.

A strange phenomenon began to unfold as Fire Horse raced down the stairs: the steps were coloured either olive green or purple; when Fire Horse stepped on a step, the step multiplied to four steps of the same colour. “If I keep going down, it is going to take me forever to get anywhere with these ever-growing stairs. Only those steps above me

An example of an 18×18 Costas array is shown. ◀



Suppose we want to send the message, *GCAT*, as our **source**, with the **source encoder** which associates the letters as follows:

$$G \rightarrow 00 \quad C \rightarrow 01 \quad A \rightarrow 10 \quad T \rightarrow 11.$$

The **channel encoder** then takes each digit and repeats it 4 times before sending the message through the channel. The **channel decoder** then breaks the message into blocks of 4, and takes the digit that repeats itself the most number of times in each block of 4 digits, then use the **source decoder** to go back to the letters before giving the output to the user.

are finite. Since I have to encounter the Draggone anyway, I am racing up to the Donegeon instead of down to the abyss.”

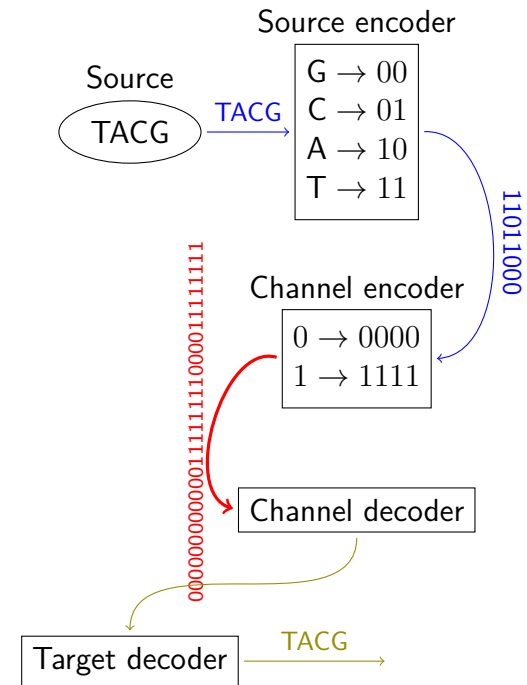
As Fire Horse got closer to the top, a frightful looking Draggone reached their long neck down the staircase to meet her. “Don’t think you can get out of climbing stairs that quickly. To get the fiery breath from me, Tumblegone, you must tell me what the lengthening staircase means.”

Fire Horse thought as fast as she could, “Could this possibly be a simple **error correcting code**? By repeating each colour four times, it would be easier to detect when a mistake occurs.” Tumblegone wasted no time in blowing their fiery breath. All quaking and racing on the staircase disappeared; instead, Fire Horse found herself firmly planted and newly charged to face whatever monsters were hiding in the closet. “Why did I never try running up the staircase in such nightmares before? I am going to try next time, but hopefully, no such nightmare will ever plague me again.”

Beatgone



A sudden stillness descended upon the Donegeon after Tumblegone’s withdrawal into their cave. Fire Horse finally got a chance to catch her



Error correcting codes are tools used in reliable transmission and storage of data. The simplest type is to repeat each symbol the same number of times, trading speed of transmission for accuracy. Today’s digital communication and storage depend on error correcting codes.

Reed-Solomon error correcting codes apply higher mathematics like group theory of finite fields to increase reliability. ◀

breath after encountering six Draggoness in a row. “All six Draggoness looked ghastly, but all helped me out of my nightmares the moment I decided to face them.”

“I don’t think you ever want to face me, Fire Horse.” An enormous Draggoness squeezed into the Donegeon and filled all available space. Very soon, Fire Horse was pushed against the walls of the Donegeon. “Fire Horse, the Donegeon master kept me, Beatgone, to the end as your last encounter. From all your nightmares, you never dealt with those in which you beat an oppressor with all your might and won. What could you do now to fight oppression?”

“You are oppressing me by squeezing me breathless against the wall.” Fire Horse shot back. “The courage to speak freely was what I lacked when I was bullied because I could get hit, get punished, get education and other similar privileges taken away from me. To avoid confrontation, I needed to invent many ways of communicating with others.”

Beatgone interrupted, “I know your ways. Sometimes you used **Caesar ciphers**, sometimes backward or reflected writing. As you learned more, you started using cryptography. It’s a wonder that you didn’t end up a spy or a detective.”

“You are still squeezing me to death. I have an overwhelming urge to beat you.” Fire Horse answered mockingly.

“While on the subject of beating, try to see the images I am sharing with you.” Beatgone began filling Fire Horse’s head with images of her oppressors, except much, much older. Some of them were frail and small, some could not stand, some were demented; all, extremely pitiful. Certainly, none looked threatening to Fire Horse. “If you want images of the oppressors’ bullied childhoods, I could share them with you, too.”

“I don’t need to see any more.” With that reply, Fire Horse forcefully pushed Beatgone away; and to her surprise, a fiery breath escaped from Beatgone which reduced Beatgone to the size of a cute little **dormouse**. Fire Horse was liberated by the last breath, patted tiny Beatgone good-bye, and exited the Donegeon triumphant with an exuberant shout for joy. “I am *totally done* with nightmares!”

Since a **Caesar cipher** shifts each letter of a message a fixed number positions down the alphabet, it is the easiest and most famous encryption system. An example of encoding

If she had anything confidential to say, she wrote it in cipher, that is, by so changing the order of the letters of the alphabet, that not a word could be made out.

becomes

Pm zol ohk hufaopun jvumpkluaphs av zhf,
zol dyval pa pu jpwoly, aoha pz, if zv johunpun
aol vykly vm aol slaalyz vm aol hswohila, aoha
uva h dvyk jvbsk il thkl vba.

slightly modified from **Cryptii**.

What is the number each letter is shifted according to the alphabet? ◀

Johanne Sebastian Bach coded the “Crab Canon” which was played both forwards and backwards. When written as two parts on a Möbius strip, it merged into the **same piece** of two voices. Marcus Du Sautoy explained in the 12-minute video how Bach coded creatively.

Public key cryptography is explained here as secure communication between Alice and Bob who tried to prevent the letter carrier from knowing the message. ◀

2.5 TRACKING REMAINDERS—ACCOUNTING FOR SEVEN LADIES WITH SEVEN DRAGGONES

The Blessings Club had seven Ladies each of whom hosted a club with its own mascot according to the table.

Lady	Activity	Day of Week	Mascot
A-má ⁿ	singing	Sunday	Silencegone
Lady Khepri	drawing	Monday	Flightgone
Lady Lucia	sculpting	Tuesday	Rushgone
Lady Bachue	archery	Wednesday	Paingone
Lady Chimalmat	pottery	Thursday	Aquagone
Lady Shala	gardening	Friday	Tumblegone
Lady Aua	ice fishing	Saturday	Beatgone

Fire Horse visited on a Sunday when she also joined the ladies in singing under the direction of A-máⁿ. As she departed, she promised, “I hope to visit again in 30 days.” Which activity could she join then?

With the help of **modular arithmetic**, Fire Horse found that she would be in Lady Lucia’s sculpting club because

$$30 \bmod 7 = 2,$$

two days forward from Sunday.

If she visited in 60 days, then

$$2 \times 30 \bmod 7 = 2 \times 2 \bmod 7 = 4$$

would mean that she would be in Lady Chimalmat’s pottery club on some Thursday, four days forward from Sunday.

Fire Horse loved gardening, but always lost the garden to the weeds as the growing season progressed. For this reason, Fire Horse really would like to attend Lady Shala’s gardening club. How many months



Because there are seven days in a week, if Fire Horse came back in seven days, she would join the singing club again; similarly, for 14, 21, or 28 days. What matters here is the remainder after $30 \div 7$. Using 28 which is the largest multiple of 7 less than 30, we find the remainder $30 - 28 = 2$. Two days forward from Sunday is Tuesday; therefore, Fire Horse would be in Lady Lucia’s sculpting club.

The arithmetic of remainders is **modular arithmetic**. In this example, $30 \div 7 = 4$ with remainder 2 is denoted by $30 \bmod 7 = 2$. The quotient 4 is immaterial. It is read as “30 modulo (or mod) 7 is equal to 2.” When 30 and 16 both give remainder 2 when divided by 7, we use the

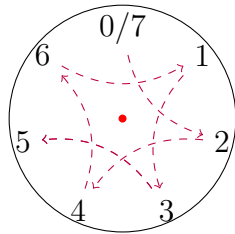
of 30 days would Fire Horse need to wait? Fire Horse expressed this problem as an algebraic equation, letting n be the number of months.

$$30 \times n \pmod{7} = 5$$

She needed 5 because according to the schedule, Lady Shala’s gardening club on Friday was 5 days forward from Sunday. Building a table of values for n from 1, 2, 3, ..., Fire Horse found that after six months, she would be able to join Lady Shala’s gardening club on Friday.

n	1	2	3	4	5	6	7
$30 \times n \pmod{7}$	2	4	6	1	3	5	0

Fire Horse also tried to find the answer on a 7-hour clock starting at 0, jumping 2 steps at a time until landing at 5. The answer she wanted was the number of jumps, 6, in this case.



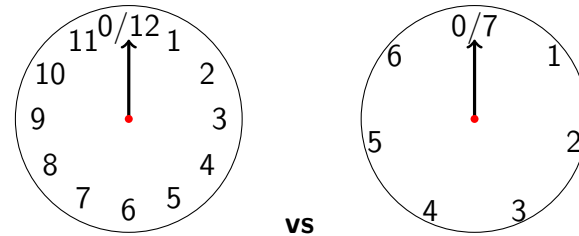
Finally, after six months, Fire Horse joined the ladies in the gardening club hosted by Lady Shala. They were trying to plant pink seed begonias in a rectangular plot. If they planted 6 in a row, they had 5 left over; if 7 in a row, they were short of 3. How many pots of seed begonias might have been prepared by Lady Shala?

Fire Horse supposed that they had n begonias in total and expressed the arrangements in two equations:

$$n \pmod{6} = 5, \quad n \equiv -3 \pmod{7}, \quad \text{or} \quad n \pmod{7} = 4,$$

notation $30 \equiv 16 \pmod{7}$, read as “30 is **congruent** to 16 modulo 7.” A quick way of visualizing modular arithmetic is to treat the **modulus** (7 in the example) as the total number of hours in a clock and perform **clock-arithmetic** on it.

A 12-hour clock versus a 7-hour clock:



Examples:

1. If Silencegone showed up in the Donegon at 3 am, what time would he show up in his next visit to the Donegon?
Assuming a 12-hour clock for am and pm, or 24 hours per day, since the schedule states that Silencegone visited every 50 hours, $50 \pmod{24} = 2$, so 2 hours forward from 3 am, or 5 am two days later.
2. If Fire Horse came back to visit 20 days after singing with the ladies, what activity would she join?
 $20 \pmod{7} = 6$, so Fire Horse would join Lady Aua’s ice fishing club, 6 days forward from Sunday.
3. If Flightgone showed up in the Donegon at 3 am, what time was he there the previous time?

because $-3 + 7 = 4$. She started a table of values for possible ns only for $n \bmod 6 = 5$ from the smallest possible answer, 5. By adding 6 to each answer iteratively, she generated all possible answers of n for $n \bmod 6 = 5$. Out of these numbers, Fire Horse picked out 11 and 53 which also satisfied $n \bmod 7 = 4$.

Furthermore, she noticed that $53 - 11 = 42 = \text{lcm}(6, 7)$, namely, the **least common multiple** of 6 and 7 is 42. This allowed her to list all possible number of seed begonias, $n = 11 + 42 \times k$, for some positive integer k . If Lady Shala cultivated more than 50 pots of begonias, she would have at least 53 pots to arrange using $k = 1$ in $n = 11 + 42 \times 1$.

$n \bmod 6 = 5$	5	11	17	23	29	35	41	47	53
$n \bmod 7 = 4$	5	4	3	2	1	0	6	5	4

In her visits, Fire Horse chatted with the ladies about their mascots. She found out that every Draggone visited the Donegeon according to a fixed schedule. Fire Horse remembered that they all showed up between 3 am and 4 am early Friday morning on March 13th, 2020. When would all seven Draggones be in the Donegeon within the same hour next time?

Draggone	Every x hours
Silencegone	50
Flightgone	60
Rushgone	70
Paingone	80
Aquagone	90
Tumblegone	100
Beatgone	120

To simplify the question, Fire Horse started with just Silencegone and Flightgone to compare the schedule each followed for showing up in the Donegeon after Fire Horse first met them.

Since Flightgone visited every 60 hours, $60 \bmod 24 = 12$, or two-and-a-half days before the 3 am visit, or 3 pm three days before. ◁

Arithmetic operations in Modulo- m systems:

1. **Equivalence:** $a \equiv b \pmod{m}$ if $a - b$ is a multiple of m . For example: $285 \equiv 200 \pmod{17}$ because $285 - 200 = 85$ and $85 \div 17 = 5$, or 85 is a multiple of 17.
2. **Addition and subtraction:** $(a \bmod m) \pm (b \bmod m) = (a \pm b \bmod m)$. For example: $(27 + 37 \bmod 7)$ is easier done knowing that $27 \bmod 7 = 6$ and $37 \bmod 7 = 2$, and $2 + 6 = 8 \equiv 1 \pmod{7}$. Compared with $64 \bmod 7 = 1$.
3. **Multiplication:** $(a \bmod m) \times (b \bmod m) = (a \times b \bmod m)$. For example: $(8^{100} \bmod 7) = (8 \bmod 7)^{100} = 1^{100} = 1$. ◁

Solving equations in modular arithmetic often involves knowing the **inverse** of a number. For example, in the equation

$$30 \times n \equiv 5 \pmod{7},$$

one can find the inverse of 2 (since $30 \bmod 7 = 2$) first, that is 4, because $4 \times 2 \equiv 1 \pmod{7}$. Using the property of multiplication, to obtain $5 \pmod{7}$, one simply multiplies the inverse, that is, 4, by 5 to get $4 \times 5 = 20 \equiv 6 \pmod{7}$. This agrees with the answer, $n = 6$, from the table. ◁

Silencegone	50	100	150	200	250	300	350
Flightgone	60	120	180	240	300	360	420

The matching 300 from each meant that after 300 hours, they would appear in the Donegeon during the same hour next time. Using this idea, Fire Horse found that it would take 25200 hours before all 7 Draggoes appeared in the same hour in the Donegeon. Converting 25200 hours to 1050 days, and using 365 days in a year (except for the leap year 2024 when February has 29 days), Fire Horse determined that the Draggoes would next appear in the same hour on January 27th, 2024.

Just before Fire Horse was ready to bid farewell to the Ladies and Draggoes, Rushgone flew over to show her how the 12 digits in a UPC (Universal Product Code) used modular arithmetic to authenticate the code. Rushgone let the digits from the left be $d_1d_2d_3 \dots d_{12}$, where d_3 means the third digit from the left. Then Rushgone shared the formula:

$$3d_1+d_2+3d_3+d_4+3d_5+d_6+3d_7+d_8+3d_9+d_{10}+3d_{11}+d_{12} \equiv 0 \pmod{10}$$

which multiplied all odd-placed digits by 3 and check that the sum was a multiple of 10. Since the first 11 digits coded manufacturer and product information, the last digit, d_{12} was a **check digit**.

“Thank you, Rushgone, for teaching me something new.” Fire Horse waved good-bye as Rushgone flew away.

2.6 EXERCISES

Counting rods

2.1 The turtles A-máⁿ took out to race wondered off to her cabbage patch and started devouring the cabbages. Fire Horse found $\overline{\mathbf{T}}$ cabbages partially eaten by turtles. How many cabbages were partially eaten?

[IM]

The list of multiples of 50 and 60 first match at 300, so the **least common multiple** of 50 and 60, denoted by $\text{lcm}(50, 60)$ equals 300. A **least common multiple calculator** finds the $\text{lcm}(50, 60, 70, 80, 90, 100, 120) = 25200$. ◁

Check digit algorithm from IBM **program segment** provides some computer coding help.

2.2 Fire Horse imagined visiting the Blessings Club in the year 2567 and wondered for what the ladies would be blessing their descendants. Express the year with counting rods.

2.3 One year A-máⁿ and her daughters harvested 358 pounds of rice from their farm. Express the amount of rice with counting rods. [IM]

Egyptian numerals

2.4 Lady Khepri loved to paint cats. On a large wall, she painted 𐀀𐀀𐀀𐀀𐀀𐀀 cats. How many cats did she paint?

2.5 Lady Khepri loved to feed fish. In a big lake she frequented, there were 𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀 tiny fish. How many tiny fish were there in the lake?

2.6 A big water serpent in the lake ate 𐀀𐀀𐀀𐀀𐀀𐀀𐀀𐀀 tiny fish. How many tiny fish did the water serpent eat? [IM]

2.7 A water serpent in the lake ate 𐀀𐀀𐀀𐀀𐀀𐀀𐀀 tiny fish from 𐀀𐀀𐀀𐀀𐀀𐀀𐀀 tiny fish originally in Lady Khepri's big lake. How many tiny fish were left? Carry out the arithmetic in both Egyptian numerals and in Hindu-Arabic numerals. Express the answer in both numerals also.

2.8 Lady Khepri counted 𐀀𐀀𐀀𐀀𐀀 frogs in a small section of a river. After a crane came to eat 𐀀𐀀𐀀𐀀 frogs, how many were left? Express your answer in Egyptian numerals. [IM]

Roman numerals

2.9 *Ben-Hur*, the movie, was made in **MCMLIX**, and earned almost **CXLVII** million dollars in the box office from its initial release. Express both Roman numerals in Hindu-Arabic numerals.

2.10 *Gladiator*, the movie, started filming in January of **MCMXCIX**. Which year did Ridley Scott start filming *Gladiator*? [IM]

2.11 *Centurion*, the movie, is a **MMX** British film written and directed by Neil Marshall. When was the movie released?

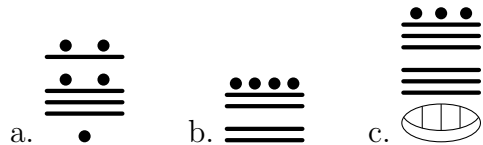
2.12 *Agora* is a **MMIX** film by Alejandro Amenábar about a female mathematician, Hypatia from the 4th-century Roman Egypt. When was the film released? [IM]


2.13 What is the year in the base of Dante's statue in Florence, Italy? [IM]


2.14 What are the years in the last line of the plaque from the Jewish Ghetto in Venice, Italy? [IM]

Mayan numerals

2.15 Convert the following to Hindu-Arabic numerals.



2.16 If Lady Chimalmat brought  cobs of Mayan corn to share with her friends in the Blessings Club, how many cobs did she bring?

2.17 Lady Chimalmat put  beans in a salsa bean salad for the Blessings Club potluck. How many beans were in the salad?

Base of Dante's statue in Florence, Italy.



Photographed by Jonathan Shapiro, Cal Poly State University.


Plaque from Jewish Ghetto, Venice, Italy.




Photographed by Jonathan Shapiro, Cal Poly State University.

William Shakespeare's play, *The Merchant of Venice*, was misused by Nazi Germany as part of its anti-Semitic campaign. For a debate on whether Shakespeare's work is still relevant today, in particular, whether *The Merchant of Venice* is anti-Semitic, please watch *Done/Undone* by *Bard on the Beach*, Vancouver, B.C.




2.18 Lady Chimalmat blessed her  descendants in Lakamha, or today's Palenque in Chiapas. How many descendants did Lady Chimalmat bless in Lakamha? [IM]




2.19 Lady Chimalmat blessed  students studying the **Temple of the Murals** in Bonampak. How many students did Lady Chimalmat bless?




2.20 Lady Chimalmat's descendants harvested bananas from  banana plants. How many banana plants were there? [IM]



2.21 Lady Chimalmat's descendants grew  squash plants this year. How many squash plants did they grow? [IM]



2.22 Lady Chimalmat's descendants grew  stalks of corn this year. How many stalks of corn did they grow?



2.23 The Maya calendar has  days, or approximately  solar years. [IM]



2.24 Perform the following addition in Mayan numerals. Answer in both Mayan and Hindu-Arabic numerals.

a.

b.

c.

2.25 Perform the following subtraction in Mayan numerals. Answer in both Mayan and Hindu-Arabic numerals.

a.

b.

2.26 Perform the following subtraction in Mayan numerals. Answer in both Mayan and Hindu-Arabic numerals.

a.

b.

[IM]

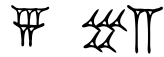
Babylonian numerals

2.27 On a hot day, the Hanging Gardens of Babylon used



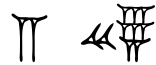
litres of water. How much water was used?

2.28 One year during the quince harvest, Lady Shala collected



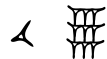
quinces. How many quinces did Lady Shala harvest? [IM]

2.29 Lady Shala's favourite snack to serve to her guests was almonds. One day for the gardening club, she prepared



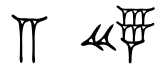
green almonds for her club members. How many green almonds did she put out? [IM]

2.30 Lady Shala was famous for her cultivation of grape vines. One year, she gathered



bunches from her vines. How many bunches of grapes did she harvest?

2.31 Lady Shala, one day, gathered



bunches of grapes. Unfortunately, the rats ate a third of the grapes that night. How many bunches of grapes were left? [IM]

2.32 Lady Shala had her sister and brother help her harvest pears one year. If her sister harvested



pears, and her brother,

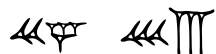


pears, and all together that day, they had



pears in total, how many pears were harvested by Lady Shala? Express your answer in Babylonian numerals also.

2.33 One year, Lady Shala had a bumper crop harvest of dates from her date palms. She shared



fresh dates with her friends and family, and dried



dates. How many dates did she harvest that year? Express your answer in Babylonian numerals also. [IM]

2.34 Perform the following addition in Babylonian numerals. Answer in both Babylonian and Hindu-Arabic numerals.

a. $\triangleleft \text{III} \quad \lll \nabla + \text{III} \quad \lll \nabla$

b. $\lll \text{III} \quad \nabla + \lll \text{II} \quad \triangleleft \text{III}$

c. $\overline{\text{I}} \overline{\text{II}} \overline{\text{III}} + \overline{\text{II}} \overline{\text{III}} \overline{\text{III}}$

2.35 Perform the following subtraction in Babylonian numerals. Answer in both Babylonian and Hindu-Arabic numerals.

a. $\overline{\text{III}} \overline{\text{III}} - \overline{\text{II}} \overline{\text{III}}$

b. $\overline{\text{III}} \overline{\text{III}} \overline{\text{III}} - \overline{\text{II}} \overline{\text{III}}$

c. $\overline{\text{II}} \overline{\text{III}} \overline{\text{III}} - \overline{\text{II}} \overline{\text{III}} \overline{\text{III}}$

2.36 The **Babylonian clay tablet** shows an approximation of $\sqrt{2} \approx 1 + 24/60 + 51/3600 + 10/216000$. Use a calculator to compute both $\sqrt{2}$ and the base-60 expansion to approximate $\sqrt{2}$.

Kaktovik numerals

2.37 Convert the following to Hindu-Arabic numerals.

a. $\overline{\text{V}}$ b. $\overline{\text{V}} \overline{\text{W}}$ c. $\overline{\text{V}} \overline{\text{W}}$ d. $\overline{\text{V}} \overline{\text{V}}$

2.38 Lady Aua visited the school library in Barter Island and found $\overline{\text{V}} \overline{\text{W}}$ children's books about polar bears. How many children's books were on polar bears? [IM]

2.39 Lady Aua counted $\overline{\text{V}} \overline{\text{W}}$ children's books about polar bears and $\backslash \overline{\text{W}}$ children's books about black bears, how many books were there about these two types of bears? Express your answer in Kaktovik numerals also.

2.40 According to a 2019 report from the **World Wildlife Fund**, or WWF in short, the population of polar bears was estimated to be between 2500 and 3000 in Chukchi Sea. Express the range in Kaktovik numerals.

2.41 According to the 2019 report from the WWF, declining polar bear populations were mostly in Canada. For example, Southern Hudson Bay had between 500 and 1000 polar bears. Express the range in Kaktovik numerals. [IM]

According to the same report from the WWF, around Barter Island in the Southern Beaufort Sea region, polar bear population was also declining, estimated to be between 500 and 1000.

2.42 Perform the following operations in Kaktovik numerals and express your answers in both Hindu-Arabic and Kaktovik numerals.

a. $\overline{\overline{\overline{v}}} + \overline{\overline{\overline{w}}}$ b. $\overline{\overline{vw}} + \overline{\overline{\overline{w}}}$ c. $\overline{\overline{w}} - v - \overline{\overline{\overline{v}}}$
 d. $\overline{\overline{\overline{v}}} - \overline{\overline{\overline{w}}}$

Different bases

2.43 For each given number, in the same base, write two numbers before and two numbers after the given number.

a. 310_5 b. 254_6 c. 32_4 d. 1101_3 e. $A39F_{16}$ f. 1011_2
 g. 677_8

2.44 Convert each number to base-10 (also called, a **decimal** number).

a. 372_9 b. 253_7 c. 425_6 d. 1012102_3 e. $AB9_{16}$
 f. 1010010101_2

2.45 Convert each base-10 (or **decimal**) number to the base specified.

- a. 34 to base 2
- b. 77 to base 2
- c. 123 to base 2
- d. 394 to base 7
- e. 572 to base 6

f. 2592 to base 8

g. 297 to base 16

h. 2938 to base 16

i. 9848 to base 16

j. 12 192 to base 16

2.46 Perform each addition in the given base.

a. $314_5 + 233_5$ b. $354_6 + 445_6$ c. $323_4 + 222_4$

d. $1121_3 + 2121_3$ e. $A39F_{16} + 9387_{16}$ f. $110\ 011_2 + 101\ 101_2$

g. $676_8 + 354_8$

2.47 Perform each subtraction in the given base.

a. $421_5 - 232_5$ b. $534_7 - 246_7$ c. $423_5 - 134_5$

d. $110\ 101_2 - 10\ 011_2$ e. $A30E_{16} - 938F_{16}$ f. $112\ 211_3 - 101\ 122_3$

g. $751_8 - 364_8$

Binary, octal, hexadecimal conversions

2.48 Convert each binary first to an octal then to a hexadecimal number.

a. $101\ 111\ 110\ 011_2$ b. $110\ 101\ 111\ 001_2$ c. $111\ 100\ 010\ 101_2$

d. $100\ 011\ 101\ 111_2$

2.49 Convert each octal number first to a binary then to a hexadecimal number.

a. 1345_8 b. 2634_8 c. 5471_8 d. 7531_8

2.50 Convert each hexadecimal number first to a binary then to an octal number.




- a. $3C5_{16}$ b. $E3D_{16}$ c. $B47_{16}$ d. $8AB_{16}$

[IM]




2.51 Project topic suggestion: Study the game **2048** to focus on the last digit cycles of multiples and exponents.

Mixed systems

2.52 Order the following in ascending order (that is, small to big).

- a. **MMMCDXCVIII** b.  c. $DA0_{16}$
 d. 110110100011_2 e.  

2.53 Order the following in descending order (that is, big to small).

- a. **MCXLIX** b.  c. 2171_8 d.  e. 

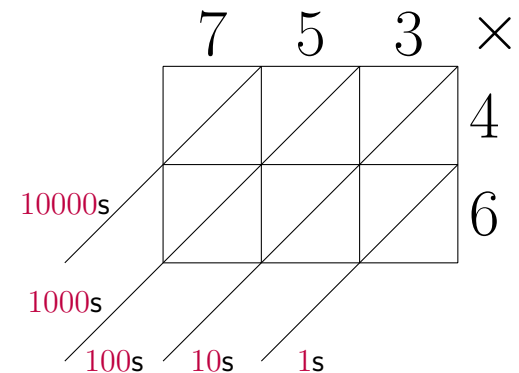
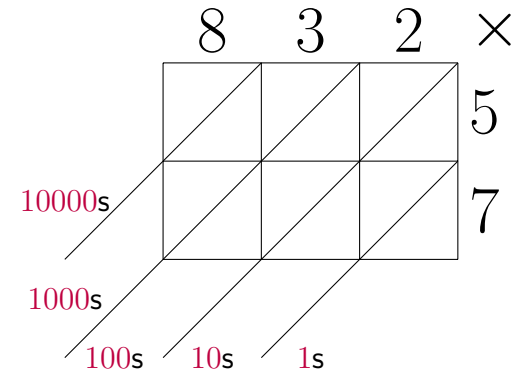
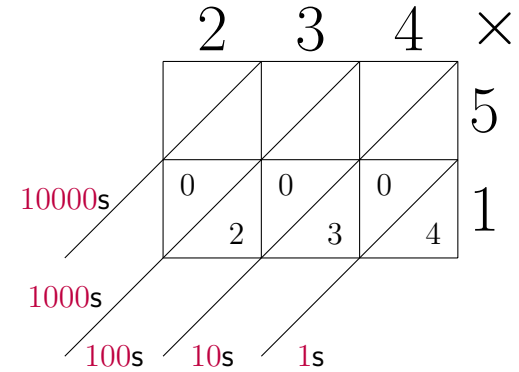
[IM]

Galley method

2.54 Use the Galley Method to multiply 234×51 (in base 10). See the partially filled galley in the sidebar.

2.55 Use the Galley Method to multiply 832×57 (in base 10). See the galley in the sidebar. [IM]

2.56 Perform the base-10 multiplication of 753×46 using the Galley Method. See the galley in the sidebar. [IM]

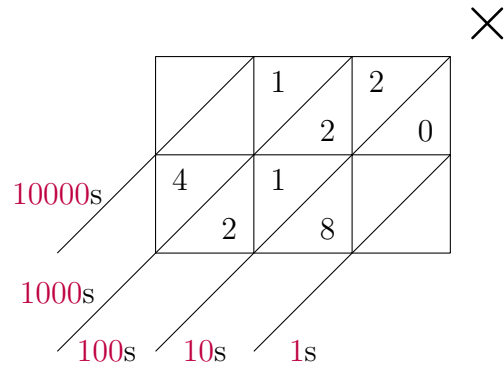


2.57 Perform the base-5 multiplication of $412_5 \times 34_5$ using the Galley Method. See the galley in the sidebar.

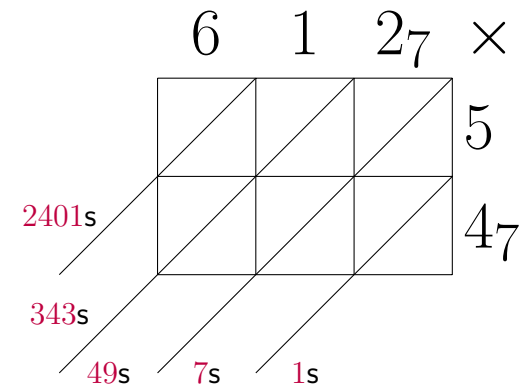
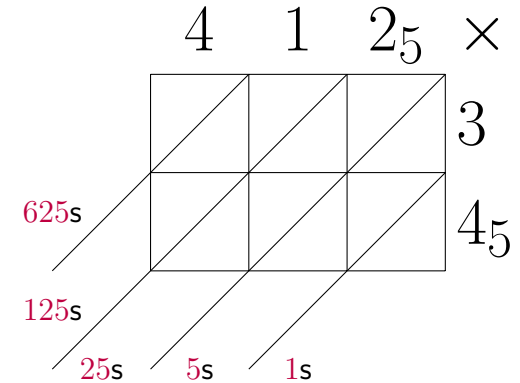
2.58 Perform the base-7 multiplication of $612_7 \times 54_7$ using the Galley Method. See the galley in the sidebar.

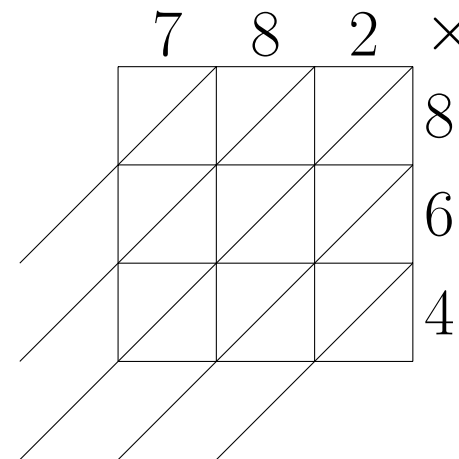
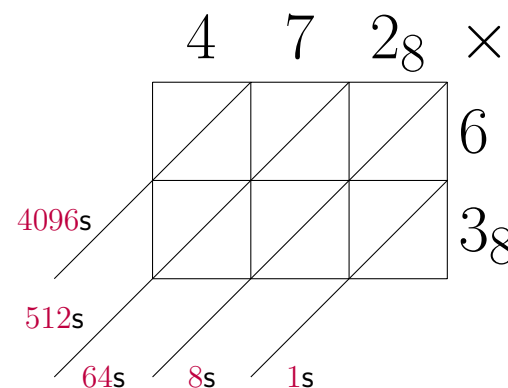
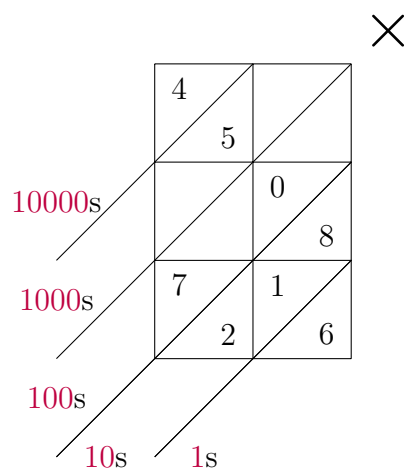
2.59 Perform the base-8 multiplication of $472_8 \times 63_8$ using the Galley Method. See the galley in the sidebar on the next page.

2.60 Determine the numbers around the galley to complete the multiplication.



2.61 Determine the numbers around the galley to complete the multiplication.





[IM]

2.62 Use the Galley Method to multiply 782×864 (in base 10). See the galley in the sidebar [IM]

2.63 Use the Galley Method to multiply the binary number 101×110 . See the galley in the sidebar on the next page

2.64 Use the Galley Method to multiply the hexadecimal number $E87A \times D2$. See the galley in the sidebar on the following page

2.65 Multiply the Kaktovik numbers $\overline{w} \setminus \overline{r}$ and $\overline{r} \setminus \overline{w}$ using the Galley Method. See the galley in the sidebar on page 81 and the multiplication table on page 83.

2.66 Multiply the Kaktovik numbers $\vee \overline{w} \overline{w}$ and $\overline{w} \overline{w}$ using the Galley Method. See the galley in the sidebar on page 81 and the multiplication table on page 83. [IM]

Modular arithmetic

2.67 True or false:

- a. $25 \equiv 47 \pmod{8}$ b. $15 \equiv 38 \pmod{7}$ c. $43 \equiv 79 \pmod{12}$
 d. $32 \equiv 96 \pmod{6}$ e. $93 \equiv 18 \pmod{15}$ f. $82 \equiv 26 \pmod{4}$

2.68 True or false:

- a. $35 \equiv 91 \pmod{7}$ b. $105 \equiv 32 \pmod{9}$ c. $132 \equiv 29 \pmod{13}$
 d. $101 \equiv 16 \pmod{5}$ e. $323 \equiv 108 \pmod{16}$
 f. $202 \equiv 67 \pmod{8}$

[IM]

2.69 Compute the following:

- a. $2 + 17 \pmod{9}$ b. $15 - 8 \pmod{3}$ c. $4 \times 19 \pmod{7}$
 d. $22 + 46 \times 3 \pmod{16}$ e. $93 - 38 \times 2 \pmod{25}$
 f. $11 \times 26 - 7 \pmod{8}$

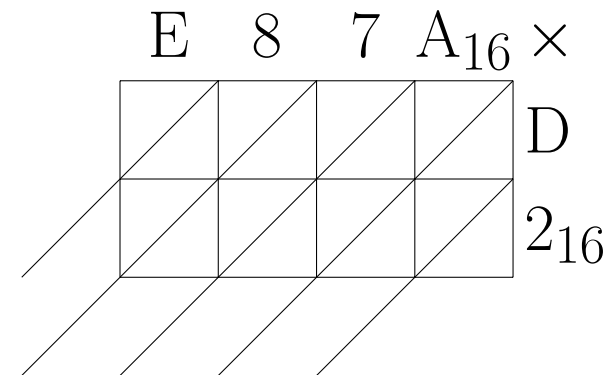
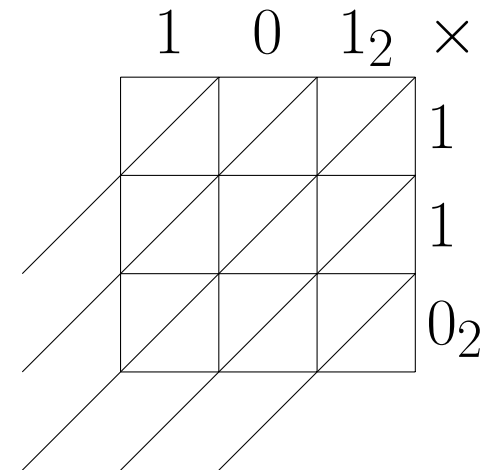
2.70 Compute the following:

- a. $72 + 37 \pmod{4}$ b. $65 - 12 \pmod{8}$ c. $3 \times 29 \pmod{6}$
 d. $52 - 26 \times 13 \pmod{11}$ e. $103 + 41 \times 3 \pmod{23}$
 f. $121 \times 2 + 7 \pmod{9}$

[IM]

2.71 Solve for the unknown in each of the following:

- a. $7x \equiv 3 \pmod{4}$
 b. $6 - a \equiv 5 \pmod{8}$
 c. $3 \times (2 + n) \equiv 4 \pmod{6}$
 d. $52 - 2y \equiv 3 \pmod{21}$



e. $10b \equiv 9 \pmod{13}$

f. $12u + 2 \equiv 7 \pmod{9}$

2.72 Solve the following systems of congruences:

a. Find the smallest positive integer x such that both congruences are true: $x \equiv 7 \pmod{9}$ and $x \equiv 2 \pmod{5}$.

b. Find the second smallest positive integer y such that both congruences are true: $y \equiv 1 \pmod{3}$ and $y \equiv 4 \pmod{7}$.

c. Find the positive solution nearest 100 such that all three congruences are true: $z \equiv 2 \pmod{6}$, $z \equiv 4 \pmod{10}$, and $z \equiv 0 \pmod{8}$.

d. Find the positive solution nearest 200 such that all three congruences are true: $u \equiv 2 \pmod{7}$, $u \equiv 3 \pmod{11}$, and $u \equiv 4 \pmod{9}$.

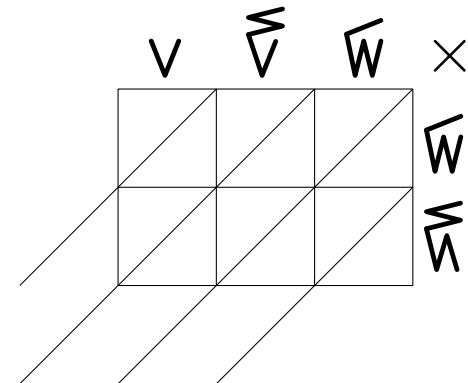
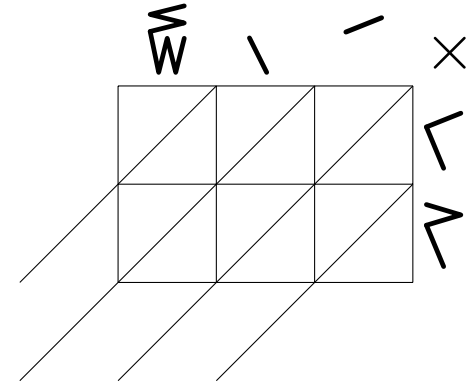
2.73 Once, Fire Horse joined the sculpting club led by Lady Lucia on Tuesday. What day of the week would it be 59 days later?

2.74 Tumblegone was in the gardening club some Friday. What day of the week would it be 90 days earlier? [IM]

2.75 One year, Fire Horse celebrated her birthday on Tuesday. What day of the week was her birthday the following year (a leap year, that is, 366 days)?

2.76 One year, Lady Khepri celebrated her birthday on Thursday. What day of the week was her birthday the previous year (365 days)? [IM]

2.77 Paingone appeared in the Donegon at 2:35 am on a Wednesday. What time and what day would it be 33 hours and 27 minutes later?



2.78 Aquagone appeared in the Donegeon at 8:27 pm on a Sunday. What time and what day would it be 52 hours and 48 minutes later?

[IM]

2.79 Fire Horse was born in the year of the horse according to the **Chinese Zodiac**. One of her cousins was 11 years older. Which animal would be her cousin's according to the Chinese Zodiac?

[IM]

2.80 The following UPC was found on a package of glazed puff pastries from Milano: 062858103203. Use the check digit algorithm given by Rushgone to verify the UPC.

2.81 The following UPC was found on a package of breakfast cereal: 055577312551. Use the check digit algorithm given by Rushgone to verify the UPC.

[IM]

2.82 Project topic suggestion: Study **Japanese multiplication method**.

2.83 Project topic suggestion: Study **Perfect numbers**.

2.84 Project topic suggestion: Study **Automorphic numbers**.

×	v	h	w	l	r	d	n	m	y	g	t	f	u	e	a	s	o	i	o
v	w	r	n	y	d	m	f	u	o	v	w	r	n	y	d	m	f	u	o
h	r	m	d	e	n	l	w	l	v	h	u	o	v	h	u	o	v	h	u
w	n	d	f	o	w	l	d	v	w	v	h	u	o	v	h	u	o	v	h
l	v	e	o	l	y	h	u	o	v	h	u	o	v	h	u	o	v	h	u
r	d	n	w	y	r	m	f	u	o	v	w	r	n	y	d	m	f	u	o
d	m	l	w	l	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
n	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
m	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
y	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
g	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
t	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v
u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
v	w	r	n	y	d	m	f	u	o	v	w	r	n	y	d	m	f	u	o
h	r	m	d	e	n	l	w	l	v	h	u	o	v	h	u	o	v	h	u
w	n	d	f	o	w	l	d	v	w	v	h	u	o	v	h	u	o	v	h
l	v	e	o	l	y	h	u	o	v	h	u	o	v	h	u	o	v	h	u
r	d	n	w	y	r	m	f	u	o	v	w	r	n	y	d	m	f	u	o
d	m	l	w	l	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
n	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
m	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
y	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
g	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
t	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v
u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
v	w	r	n	y	d	m	f	u	o	v	w	r	n	y	d	m	f	u	o
h	r	m	d	e	n	l	w	l	v	h	u	o	v	h	u	o	v	h	u
w	n	d	f	o	w	l	d	v	w	v	h	u	o	v	h	u	o	v	h
l	v	e	o	l	y	h	u	o	v	h	u	o	v	h	u	o	v	h	u
r	d	n	w	y	r	m	f	u	o	v	w	r	n	y	d	m	f	u	o
d	m	l	w	l	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
n	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
m	f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
y	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
g	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u
t	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o
f	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v
u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h
o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u	o	v	h	u

Table 2.3: Kaktovik multiplication table

Measuring and Tracking—A discourse among Pythagoras, Euclid, and Archimedes

3.1 COMPARISON OF ANGLES—ARRIVAL OF THE STARS

Geometric elements—Introducing Pythagoras, Euclid, and Archimedes

Pythagoras of Samos (c. 570–495 BC), **Euclid of Alexandria** (c. 325–270 BC), and **Archimedes of Syracuse** (c. 287–212 BC) spent countless hours philosophizing, composing music, creating new mathematics, inventions, and the sciences in general—all in all, leading a meaningful and productive existence in the Outerworld of **Hades** and **Cerberus**, his three-headed guard dog. However, talking to the same colleagues and students for over 2000 years became so monotonous that when an invitation unexpectedly arrived from Capilano University for the three ancient mathematicians to spend a sabbatical from the Outerworld, Pythagoras, Euclid, and Archimedes immediately jumped at the opportunity, and began planning for their space-time travel.

The moment all three were prepared to their satisfaction, they asked **Scotty** to beam them to Capilano University to begin their sabbatical.

Vi Hart showed how to use a music box to mix space and time in **Folding Space-Time**.

Euclid in his *Thirteen Books of the Elements* [7] pioneered rigorous mathematical writing via definitions, axioms, postulates, and theorems. In that spirit, we say

1. a **point** is “that which has no part,”
2. a **line** is “breadthless length,”
3. a **plane** has “length and breadth” only.

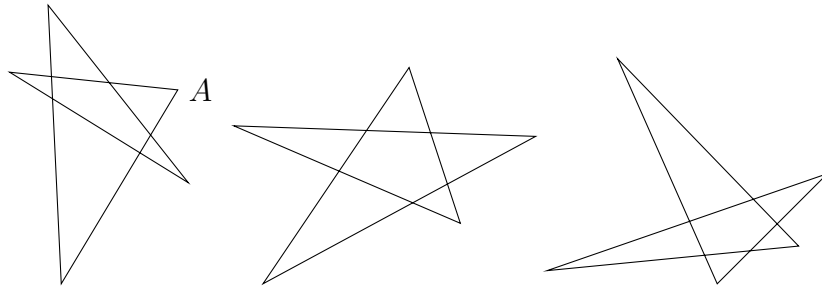
Start by marking two points, P and Q , on a piece of paper. Then, use your straight edge to draw a line through these two points, thus partitioning the line into three parts:



After landing, while getting over time-space-travel lag, they decided to etch stars of their own designs on the pavement of the parking lot where they landed. When Euclid saw how different their stars looked, he launched into a series of constructions to share a **theorem** on five-point stars.

“Do we really have to hear you lecture again?” Pythagoras yawned for the fourth time after arrival. “Wasn’t it enough to compile all your lectures into *thirteen* books, Euclid? Being as ancient as I am, I would prefer having a good nap to help realign all my particles into proper proportions. Once refreshed, I want to visit the music department first, just to replace the ringing in my ears with something more harmonious.”

Archimedes was, however, buoyant about what Euclid wanted to share, “I loved your thirteen books from the Library in Alexandria. Please tell me about the theorem of stars. But—remember how I died by the sword of a Roman soldier in the *Siege of Syracuse*? Wherever you want to draw, I will stand guard for you—just in case.” Archimedes slowly scanned the parking lot and the surrounding woods for any sign of danger.



Euclid’s discourse on angles

Euclid needed no more encouragement. True to his teaching style, Euclid commenced with a question:

PUZZLE 19 When you draw your own five-point star (hopefully, very different from the three above) and mark the

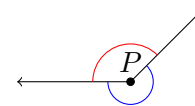
two half lines and a **line segment** \overline{PQ} in between.



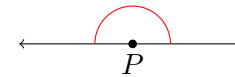
If we include point P with a **half line**, we get a **ray**.



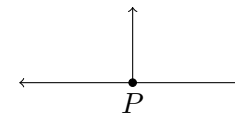
Two rays sharing the same point P make an **angle**. We usually mean the smaller one (marked in red) of the two angles.



When the two rays form a straight line, we called the angle a **straight angle**.



Half of a straight angle is a **right angle**. The straight line standing on the horizontal line is a **perpendicular** to the horizontal line.



We use the Babylonian measurement of angles where a

angles of the five points, what is the angle sum?

“Can you explain this quickly?” Archimedes asked anxiously because lots of people suddenly started leaving a big building.



“No, I can’t. To make sure I am clear in my explanations, I must start from the *very* beginning,” replied Euclid, painstakingly emphasizing every syllable.

1. Euclid began carefully drawing two straight lines l and m intersecting at point P . He then marked three angles around P , and started.

Since $\angle 1 + \angle 3 = 180^\circ$ due to the straight line m , and similarly, $\angle 2 + \angle 3 = 180^\circ$ on line l , by subtracting both equations by $\angle 3$, it follows that $\angle 1 \cong \angle 2$.

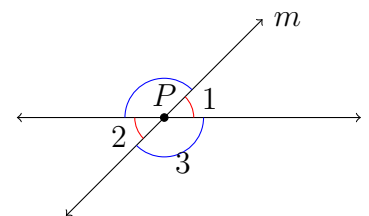
full turn is 360° , half a full turn is 180° , i.e., the **straight angle**, and a quarter turn is 90° , i.e., the **right angle**.

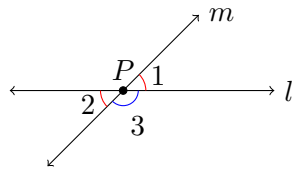


To make special angles like 60° , 90° , and 180° by folding paper, please watch Vi Hart’s [Angle-a-trons](#).

Angles drawn differently may have the same measure. If $\angle 1$ and $\angle 2$ have the same measure, we use the notation $\angle 1 \cong \angle 2$, and say “Angle one is **congruent** to angle two.”

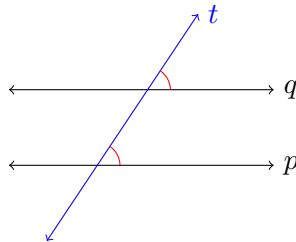
Congruent (non-adjacent) angles formed by two intersecting lines are **vertical angles** or **vertically opposite angles**. Around point P , angles $\angle 1$ and $\angle 2$ are **vertical angles**; the blue angles form another pair of **vertical angles**.





“Now wait a minute,” Archimedes interrupted, “don’t you mean that *the measure of $\angle 1$ plus the measure of $\angle 3$ equals 180° ?” “Indeed,” replied Euclid, “but that’s so many words to say so little, so the symbol $\angle 1$ shall mean the measure of $\angle 1$ when the context demands it. Computer programmers call this **type coercion**.”*

- Next, Euclid drew two parallel lines p and q , (slowly to ensure that they looked really parallel) and a line t intersecting both of them.



“Hey, look, Archimedes! By sliding the lower red angle up along line t (also called the **transversal**), you see that it coincides with the upper red angle exactly. Isn’t it a great way to see that the red angles are congruent?” Euclid looked up to see if Archimedes was watching.

“Oh yes. I remember my father explaining it that way to me once.” Archimedes glanced at the drawing and quickly returned his focus to the crowd leaving a big building, coming down a set of steps.

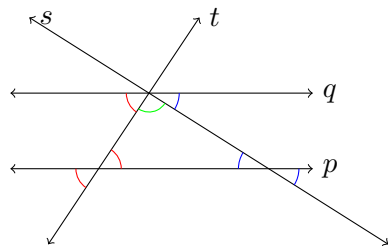
Parallel lines are in the same plane, infinitely long in both directions, and never intersect.

The **transversal** to parallel lines p and q is the line intersecting both parallel lines. In the diagram, the blue line t is the transversal.

Getting the same pattern by sliding is a **slide or glide symmetry** or a **translational symmetry**. The red angles (called **corresponding angles**) formed by parallel lines p and q and the blue transversal t are congruent. \triangleleft

3. Euclid added another **transversal**, line s , to the previous figure so that line s intersected lines t and q at the same point.

“Can you see the three coloured angles (red, green, and blue) in the triangle formed by the transversals, lines s and t and the bottom line p ? When you use vertical angles to flip both blue and red angles in the triangle out to the bottom of line p , then use two slides to bring both angles up along their transversals to line q , you can see that the three coloured angles form a straight angle, or the red, green, and blue angles in the triangle add up to 180° .” Euclid looked very proud of this flip-and-slide maneuver.



“Have you been trying to show me that the angle sum in a triangle is 180° ?” Archimedes sounded rather impatient now. “You should have just drawn a random triangle on paper, torn the three angles, and put the pointy ends together. They would make a straight angle. Done.”

“Do you see any paper here?” Euclid retorted. “You do what you can with the technology available to you.” Euclid waved the stick he was holding with one hand and pointed to the dirt with the other.

4. “Now, we mark the lower right hand corner of the triangle, point A . The two angles sharing sides with the interior angle of the triangle at A are **exterior** angles (one of which is marked in purple). Notice that an interior angle and its corresponding exterior angle

The **interior angles** of a triangle on a plane add to 180° . This is not true for a triangle drawn on a sphere. Watch Vi Hart’s $4\frac{1}{2}$ -minute video, [Triangle Party](#), for discussions on acute or obtuse triangles at 1:35, “Bermuda” triangle whose angle sum is greater than 180° at 2:40, and Euclidean triangle (the normal two-dimensional type) whose angle sum is 180° at 2:45, as described by Archimedes in the text. ◀

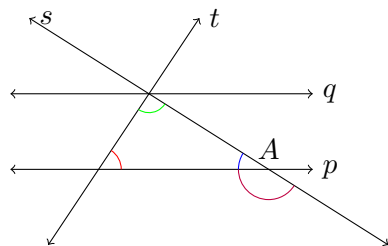
In mathematics, a series of logical arguments to convince some audience that a statement is true is a **proof**. The statement of importance is a **theorem**. Sometimes, many smaller facts need to be established before proving the theorem. We call such a smaller fact a **lemma**. For a more thorough and academic exposition, consult Leslie Lamport’s [How to Write a Proof](#).

Formally, one can organize Euclid’s five-step explanation to Archimedes of the angle sum of a five-point star with four lemmas and one theorem.

- a) LEMMA 1 *Vertically opposite angles are congruent.*
- b) LEMMA 2 *Corresponding angles are congruent.*
- c) LEMMA 3 *The sum of interior angles in a triangle is 180° .*
- d) LEMMA 4 *An exterior angle of a triangle is the sum of the two non-adjacent interior angles.*
- e) THEOREM 1 *The sum of five pointy angles of a five-point star is 180° .*

◀

form a straight angle (or the blue angle and the purple angle add to 180°). Therefore, an exterior angle (the purple one) has the same measure as the sum of two non-neighbouring (**non-adjacent**) interior angles (the red and the green angles here) in a triangle.” Euclid thought that was a short enough explanation for Archimedes, and checked if Archimedes was following.



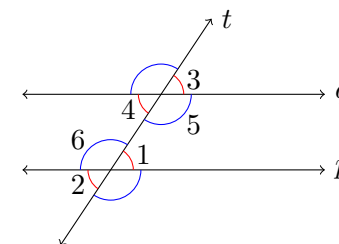
“Oh, you were trying to tell me that an exterior angle of a triangle is congruent to the sum of the two non-adjacent interior angles! You could use the paper angles for that, too.” Archimedes chimed, “but, you should really finish soon before all those people come to this spot.”

“I have prepared the way for the punchline—honestly, almost done.” Euclid reassured Archimedes.

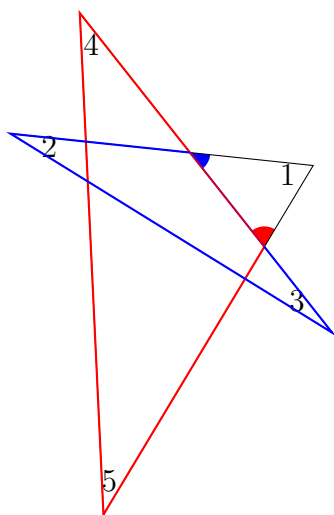
5. “Now we can look at any of the five-point stars we drew earlier (I picked the one labelled *A*), then choose one of the five smaller triangles containing one of the sharp points of the star on which we focus. We know that the interior angles of a triangle add up to 180° . With the specific triangle (say, the one containing $\angle 1$) in the star, notice that the blue angle is the exterior angle of the blue triangle, so the blue angle $\cong \angle 2 + \angle 3$, i.e., the sum of non-adjacent interior angles. Similarly, the red angle $\cong \angle 4 + \angle 5$ in the corresponding red triangle. So using the angle sum of a triangle, $180^\circ = \angle 1 +$ the blue angle $+ \text{the red angle} \cong \angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5$. Here you are!” Euclid beamed at Archimedes.

Angles formed by two parallel lines and a transversal have many relationships. Therefore, names were given to relative positions of such angles.

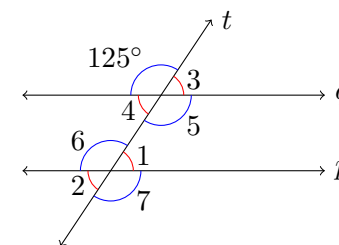
- Corresponding angles** are pairs of angles, like $(\angle 1, \angle 3)$ or $(\angle 2, \angle 4)$ in the figure on the current page, related by a slide symmetry.
- Alternate interior angles** are pairs of angles inside the parallel lines on different sides of the transversal, like $(\angle 1, \angle 4)$ or $(\angle 5, \angle 6)$.
- Alternate exterior angles** are pairs of angles outside the parallel lines on different sides of the transversal, like $(\angle 2, \angle 3)$.
- Interior angles on the same side of the transversal** are pairs like $(\angle 1, \angle 5)$ or $(\angle 6, \angle 4)$. These pairs also form straight angles, namely, $\angle 1 + \angle 5 = 180^\circ$. We call such angles **supplementary**.
- Analogous to **supplementary** pairs of angles, when two angles add up to 90° , we call them **complementary**.



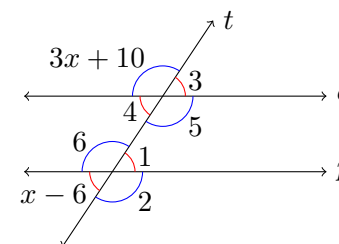
An angle puzzle: Find all the angles $\angle 1$ to $\angle 7$ and provide a reason for each angle found.



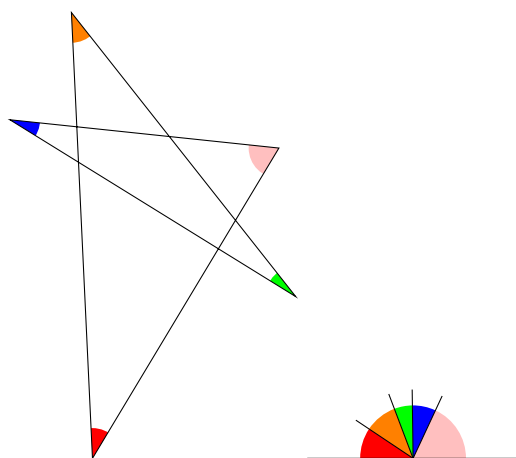
“But, Euclid, if we had paper, and could tear all five pointy angles from the star, then putting the five angles’ pointy ends together would form a straight angle. I agree, though, that your way is more convincing because the cutting and putting together could make some people suspicious of this maneuver working only for the particular five-point star chosen.” Archimedes quickly said, and began shaking Pythagoras, trying desperately to wake him. Meanwhile, the crowd was only a couple of cars from the three mathematicians of antiquity.



An angle puzzle using algebra: Solve for x .



An angle puzzle for paper cutting: Draw a seven-point star similar to how you would draw a five-point star. Cut all seven pointy angles and put all seven pointy ends together, leaving no gap between angles. What do you find? Can you guess in general for a star drawn this way with n points for odd n 's? For example, $n = 7$ or 9 .



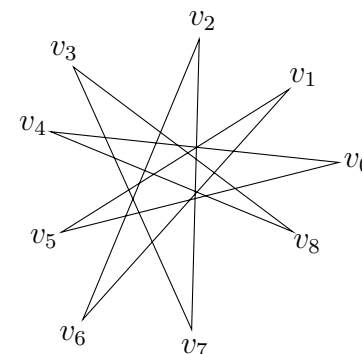
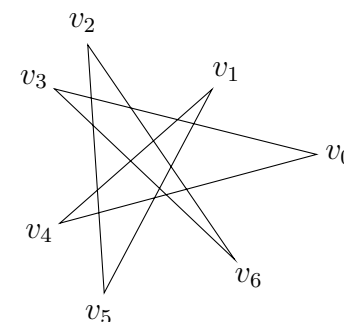
3.2 POLYGONS AND SIMILARITY—WELCOMING PARTY

Pythagoras rubbed his eyes and forced himself to open them. “Wo! Who are all these people?” Archimedes and Euclid huddled close to Pythagoras, wondering what to do with such a crowd around them.

“Welcome to Capilano University!” A most pleasant smile greeted the three star mathematicians. “We are so honoured to have all three of you among us. Please do not be alarmed by our faculty and students, all of whom are dying to meet you and learn from you.” Upon hearing the word, *dying*, Archimedes jumped a step back to hide behind Pythagoras and Euclid. “Oh, Sorry, Archimedes. In our welcome basket, you will find refreshments and a map of the campus. Please come to the STEM office for a place to rest when you have regained your space-time legs.”

Pythagoras’s polygonal angle sums

Pythagoras thanked the welcoming party; all three waved good-bye to the big crowd so that they could be alone again. “Wow, what a nice star you have here! I can see that the five pointy angles are marked. Did you talk about the angle sum of the inner pentagon? Since Euclid already established that the angle sum of a triangle is 180° , the angle

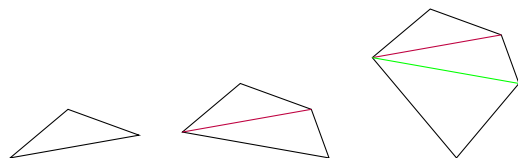


◁

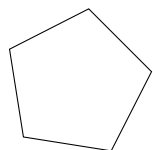
A figure on a plane is **closed** if it can be drawn without lifting the drawing instrument and if the initial and terminal points coincide. Such a figure is **simple** if the drawing never crosses itself except at the initial and terminal points. The five-point star is **closed** but not **simple**.

A **convex** polygon is a many-sided **simple, closed** figure where each interior angle is less than 180° . Another way of describing a **convex** polygon is to look at lines joining any two points inside the polygon: if all such lines lie entirely

sum of a **convex** quadrilateral would be 2 triangles put together, or 360° . Similarly, 3 triangles' angle sum ($3 \times 180 = 540$) for a **convex** pentagon."



Pythagoras was on a roll, "in fact, when one grows the **convex polygon** from a triangle, replacing one side of the triangle by two sides, this increases the polygon by one more side while increasing the angle sum by 180° . This gives the general formula that $(n - 2) \times 180$ is the angle sum of a general n -sided convex polygon. I am all for beautiful proportions. When all the side lengths are congruent, then the interior angles are also congruent. In the case of a **regular** pentagon, one would compute one interior angle to be $(5 - 2) \times 180/5 = 108$, drawn like this."

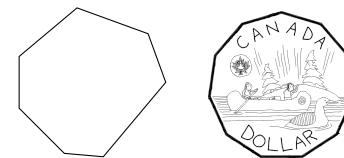


Pythagorean Theorem and beyond

Euclid could no longer refrain from speaking, "Pythagoras, from your background in a **Geomori** family on Samos, please tell us about how the Geomori laid out their land: 9 plots, 16 plots, then 25 plots around a right triangle."

inside the polygon, it is **convex**; otherwise, **concave**. The outline of a five-point star is a **concave** polygon, whereas the triangle, quadrilateral, and pentagon shown are all **convex**.

A convex heptagon (7-sided) and the regular hendecagon (11-sided) are shown here.

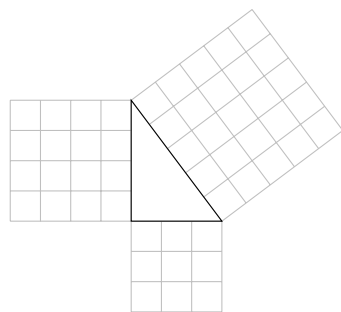


When all sides and angles of a **convex** polygon are congruent, then the polygon is **regular**. An **equilateral triangle** is a **regular polygon** with the fewest number of sides. ◀

The longest side of a **right triangle** is the **hypotenuse**. The two shorter sides are the **legs**.

If a triangle has two congruent sides, then it is an **isosceles** triangle. An **isosceles** right triangle has side lengths a , a , $a\sqrt{2}$. ◀

Two polygons are **similar** if their corresponding angles are congruent. More precisely, $\triangle ABC \sim \triangle DEF$ if $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. A property of similar polygons is that the ratio of corresponding side lengths is a constant. In the case of similar triangles, $\overline{AB}/\overline{DE} = \overline{AC}/\overline{DF} = \overline{BC}/\overline{EF}$. ◀



Pythagoras was more than happy to oblige, “I still think ‘all things known have number.’ The 3–4–5 triangle has such a useful proportion. We get the same general shape whether the lengths are doubled to 6–8–10 or tripled to 9–12–15. The study of **similar** triangles and polygons in general was really helpful in building all those impressive monuments.”

Archimedes chuckled to himself, “I hope Pythagoras doesn’t get bogged down in proportions again, like the longest side length of an **isosceles** right triangle, because the square root of 2 ($\sqrt{2}$) cannot be expressed as a ratio of two positive integers.”

Euclid reminded Pythagoras, “the relationship of the side lengths of right triangles was generalized as $a^2 + b^2 = c^2$ where c is the length of the longest side. There are so many beautiful ways of proving this relationship, like drawing a square of $(a + b)$ by $(a + b)$ in two different ways.”

For a detailed account of the 3–4–5 triangle, please consult Part I Chapter 6 of *The Music of Pythagoras: How an ancient brotherhood cracked the code of the universe and lit the path from antiquity to outer space* by Kitty Ferguson. [8]

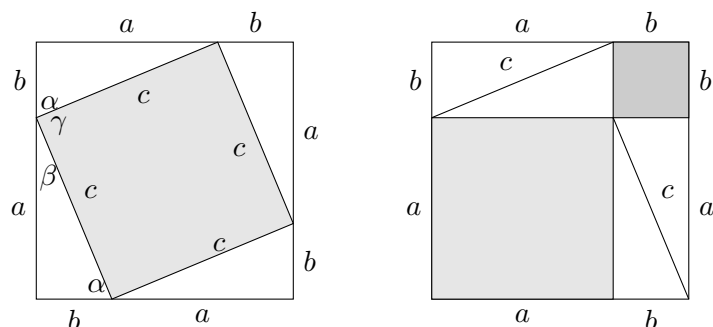


Pythagorean Theorem states that the lengths of right triangles satisfy the equation:

$$a^2 + b^2 = c^2,$$

where a and b are the lengths of the legs and c is the length of the hypotenuse.

Euclid’s formula from Book **X** Proposition **XXIX** states that if $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, where $m > n > 0$, and m, n are both integers, then a, b , and c satisfy $a^2 + b^2 = c^2$. Please consult [Euclid’s proof](#) for details.



Archimedes looked at the pictures from Euclid, and remembered, “I know a paper-folding way of showing that $a^2 + b^2 = c^2$, too.” To that comment, Euclid answered, “OK, Archimedes. We will go to the STEM office to get some paper for you to fold. Since you know my *thirteen books* so well, you must remember how to generate all primitive **Pythagorean triples**, too, including the proof.”

Indeed, Archimedes remembered the triples fondly and began reciting to a tune, “3–4–5, 5–12–13, 8–15–17, 7–24–25, ...”

3.3 AREA AND VOLUME—ARCHIMEDES’S PROUDEST MOMENT

“Can you stick out your tongue and touch your nose?” Archimedes challenged Euclid and Pythagoras. “Ha-ha-ha, I know that’s your favourite joke.” Pythagoras stuck out his tongue and touched his nose with his finger to humour Archimedes. “But, I would rather know about the picture carved on your tombstone, Archimedes,” implored Pythagoras.

“Oh, you mean a sphere in a circumscribing cylinder? That was my favourite, a proportion of 2 : 3 for both surface areas and volumes.” Archimedes continued, “I can see why you would like that—such simple proportions and all. Let me start from the beginning like Euclid would prefer.” Archimedes made sure Euclid heard the last sentence.

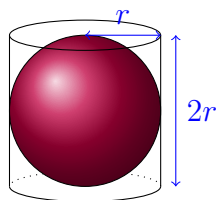
For a visualization of Pythagorean triples on the $x - y$ -plane, please watch a 15-minute video, [All possible Pythagorean triples, visualized](#) from 3 Blue 1 Brown. Right before the end of 2 minutes into the video, the two-square proof of Pythagorean Theorem is given.

A paper-folding proof of the **Pythagorean Theorem** is demonstrated in the [origami proof of the Pythagorean Theorem](#) by Vi Hart. ◀

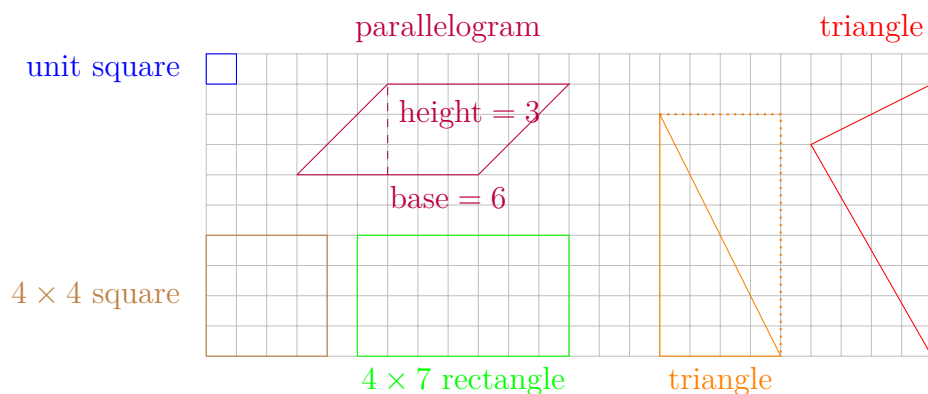
Visualizing a square can be used for relaxation and the reduction of anxiety. Use the square below to practise [Square breathing](#) or **Box breathing**.



Areas



“When we start with a unit length, like we saw in your picture of the 3–4–5 triangle, then a 3×3 square would have an area of 9 square units, meaning, that a 1×1 square is our **square unit**.” Archimedes started drawing a large grid to show areas of different shapes.



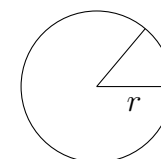
“With the help of the grid, it is clear how the area formulas of these shapes were derived; for example, a rectangle’s area is its width times length and a triangle’s area is just half of its base times height. Once we have the area of a parallelogram which is its base times height, all we need is the **circumference** of a circle of radius r . This is where we involve the curious number $\pi \approx 3.14159$ which baffled us for so long because it was worse than $\sqrt{2}$ to understand. Now we know that the circumference is $2\pi r$. As usual, I like to cut a shape into small pieces called **sectors** for the circle. Then, let’s shade every other **sector** grey.

Area formulas of the shapes in Archimedes’s grid:

1. A square of side length n has an area of $n \times n$.
2. An n by m rectangle has an area of $n \times m$.
3. A parallelogram of base b and height h has an area of $b \times h$.
4. A triangle of base b and height h has an area of $b \times h/2$.
5. A circle of radius r has an area of πr^2 .
6. A cylinder of radius r height h has a surface area of $2\pi r^2 + 2\pi r h$.
7. A sphere of radius r has a surface area of $4\pi r^2$.

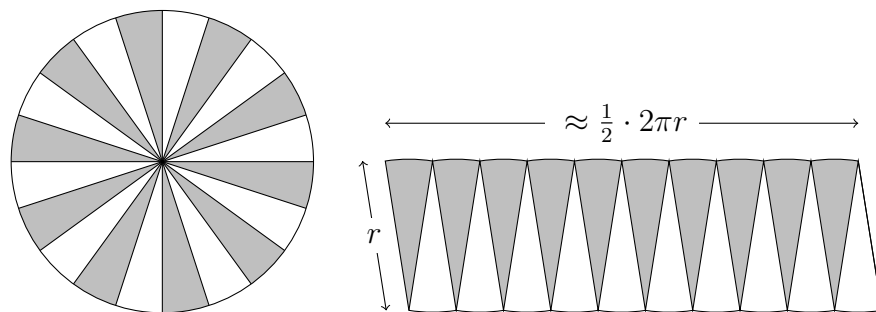
The area formulas of both the trapezoid and the rhombus can be derived by decomposing the shapes into triangles and/or rectangles. Such derivations can be found in the exercise section. ◀

A **circle** consists of the set of points on a plane equidistant to a given point. The distance from the centre to any point on the circle is the **radius**. The length around the circle is its **circumference**. A connected portion of the circumference is an **arc**.



A **sector** is a closed shape formed by two radii and an

As we cut out all the sectors and interlace them, shaded ones pointing down, unshaded ones pointing up, we would see an almost rectangle (more like a parallelogram if sectors are not small) of length πr and width r , so a total area of πr^2 for the circle.”



Euclid remarked, “I like your arguments of the infinitesimals, fine tuning lower bounds and upper bounds for any tricky area or volume you tried to find.” Euclid would know best because he had indicated such an idea in his books, just didn’t seriously apply the method.

“When I cut open the surface of the cylinder, I see the curved surface (a rectangle) and two circles, so an area of $2\pi r \times 2r + 2\pi r^2 = 6\pi r^2$. I showed that the surface area of the sphere of the same radius r is $4\pi r^2$, so the sphere surface area is $2/3$ of the circumscribing cylinder.”

Volumes

“I am impressed by the area arguments for the circle, cylinder, and sphere, Archimedes.” Pythagoras nodded enthusiastically, “but my curiosity is solely focused on the volume of the sphere. How did you find it?”

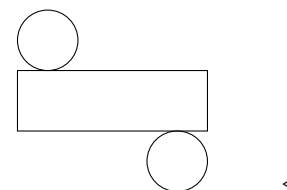
“Here, Euclid would agree that I need to start with the unit **cube**, a $1 \times 1 \times 1$ box which will be our **cubic unit**. From here, it is possible to count the number of unit cubes in a $3 \times 3 \times 3$ cube, i.e., if the top level contains $3 \times 3 = 9$ cubes, then having three levels altogether yields

arc.

The formula for the **area of a circle** is explained by mathematicsonline’s YouTube channel. ◀

For the proof of the sphere’s surface area formula, please watch the 17-minute **video** from 3 **Blue** 1 **Brown** which used properties of similar triangles, angle sum on a line, and the Pythagorean theorem to show that the curved surface area of the enclosing cylinder to a sphere of the same radius is equal to the sphere’s surface area.

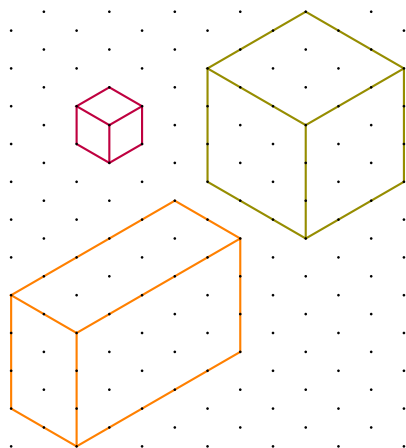
The surface area of a cylinder decomposed into two circles and a rectangle:



An $l \times w \times h$ **rectangular prism** has a volume of lwh cubic units. In general, any two-dimensional shape of area a on the plane extended h tall (vertical distance to the base plane) in the third dimension gives a solid with a volume of ah , or its base area multiplied by the height. In particular, the volume of a cylinder is $\pi r^2 h$.

This way of looking at volumes is Archimedes’s trademark: the examination of cross sectional areas. Imagine a tall deck of cards. By shifting the deck slightly to one side, its volume remains unchanged because the cross sectional areas are still the same. ◀

$3 \times (3 \times 3) = 27$ unit cubes. This yields a volume of 27 cubic units for a $3 \times 3 \times 3$ cube.”



“I like your two-dimensional representations of the cubes. The hexagonal grid helps a lot in representing three-dimensional objects on a plane.” Euclid tried to sound encouraging.

Archimedes continued, “when the top face is a 2×5 rectangle, like in the orange box, one sees $2 \times 5 = 10$ unit cubes on the top level, and with three levels in total, the volume of the **rectangular prism** is $3 \times (2 \times 5) = 30$ unit cubes. Similarly for a cylinder, the volume is the area of the circle multiplied by the height.”

Archimedes's proudest Eureka!

Pythagoras interrupted Archimedes, saying “This is not enough to get the volume of the sphere. Your idea of cutting prisms into thin cross sections works well for prisms and pyramids, but not spheres. I think you are keeping something crucial from us.”

In response to this, Archimedes smiled with a twinkle in his eye, “You are right, Pythagoras. The crux of the matter is a double cone of the same height as the cylinder or the sphere, that is, all are $2r$

Figure 3.7 on page 140 provides a background grid for your own 3d drawings.

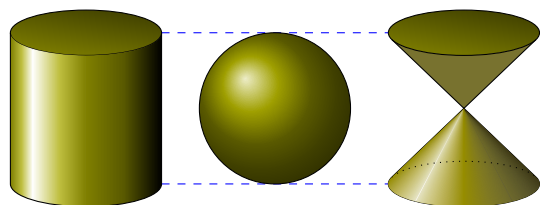


Volume formulas of some geometric solids:

1. A cube of side length n has a volume of n^3 .
2. A rectangular prism of base $l \times w$ and height h has a volume of $l \times w \times h$.
3. A general prism of base area a and height h has a volume of $a \times h$.
4. A cylinder of radius r and height h has a volume of $\pi r^2 \times h$.
5. A pyramid is $\frac{1}{3}$ the volume of its enclosing prism.
6. A sphere of radius r has a volume of $\frac{4}{3}\pi r^3$.

tall. Since you know that the volume of a pyramid is $1/3$ the volume of the rectangular prism enclosing it, a cone is also $1/3$ the volume of its enclosing cylinder. When I put a cylinder, a sphere, and a double cone on the same level and studied the cross sectional area from the top down, the sphere and the double cone compensated each other to make up the same area as the cylinder's circle." Pythagoras inhaled sharply keeping all the air in, lifted his eyebrows, and looked astounded.

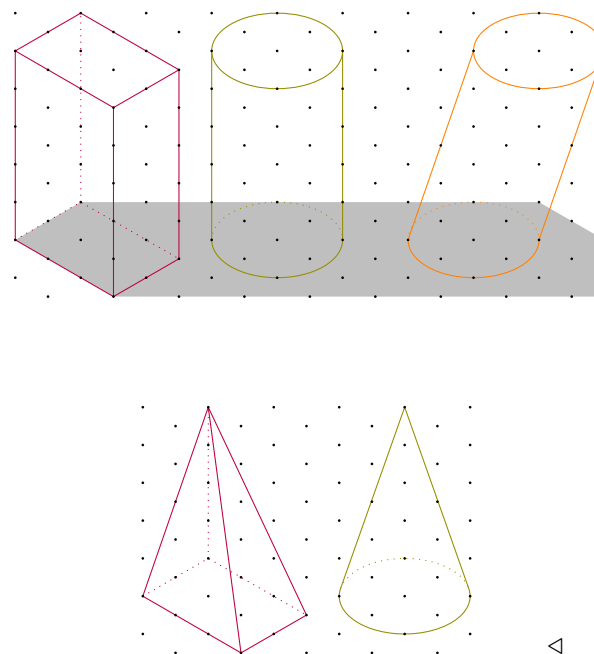
Archimedes continued, "This guess led me to dunking the two objects into my famous bathtub of water, comparing the rise in the water level to that of the cylinder by itself. It turned out that the combined volume of the sphere and the double cone was the same as the cylinder. That was another *Eureka* moment for me!" Archimedes spoke faster and faster because he feared that the continuous inhaling of Pythagoras without any exhaling might lead to a lung explosion.



"How ingenious!" Pythagoras finally exhaled, totally amazed, "Now since the volume of the cylinder is $\pi r^2 \times 2r = 2\pi r^3$, the sphere's volume must be two-thirds of the cylinder's volume because one-third is taken by the double cone. This gives us the formula for the sphere's volume, $\frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3$. That *is* elegant!" Pythagoras concluded the derivation himself.

"Now you understand why I wanted the diagram carved on my tombstone," continued Archimedes, "for the ratio of $2/3$ is magical, both for the surface areas and the volumes."

"I think we should start heading toward the STEM office. According to the campus map, the music department is in the same building, so perhaps, on the way, we can stop by the music department to uplift our



souls with some music and help Pythagoras get rid of the ringing in his ears?” Euclid started leading them in the direction of Fir building.

3.4 ORIGAMI SYMMETRY—ARCHIMEDES AT PLAY

Entering Fir building from the upper level, Euclid found STEM division office without any trouble. The moment they stepped into the reception area, all the scientists emerged from their offices to greet them, jumping straight into discussions about all the discoveries Archimedes made in his bathtub, Pythagoras’s students, enemies, and beanophobia, and of course, Euclid’s elements. Since the three guests were famished from their travel, many also offered food from the mail room. While still munching, Euclid remembered his promise to Archimedes. “Could we please have some paper for Archimedes?” Euclid asked the scientists surrounding them. One of them quickly ran into the mail room and came back with an assortment of coloured paper, all rectangular.

“I need to make the paper square first. With a diagonal fold bringing the shorter edge to coincide with the longer edge, I can use the paper cutter to remove the unmatched excess at the bottom.” Archimedes began folding and walking towards the paper cutter.

Reflection from nature

Holding up his first batch of square paper all folded down the diagonal, Archimedes began folding at an incredible speed, producing one wonder after another: first a **whitetail doe**, then an **orchid blossom**, finally, finishing with a **cardinal**. When Archimedes looked up to see his audience shaking their heads in disbelief, he smiled shyly, “I have been practising some of the amazing creations by **Robert Lang**. Just look at the symmetry of the crease patterns.” While marvelling, he unfolded all three works of art to study the crease pattern from each. “The **whitetail doe** has an **axis of symmetry** vertically down the middle of the square corresponding to the bilateral symmetry of the animal. It is harder

Archimedes’s discovery of the sphere’s volume formula is in the video, **The volume of a sphere** from **Numberphile-YouTube**. The video also shows why a pyramid is $\frac{1}{3}$ the volume of its enclosing prism. <



A **TED talk and other videos** by Robert J. Lang show us how origami combined with mathematics is applied to space exploration, medicine, and airbag folding.

Artists through the ages have used symmetry to create beauty. We study four types of symmetry in geometry: a **reflection**, a **translation**, a **glide reflection**, and a **rotation**, all of which result from a **rigid motion**, that is, an action that takes a geometric object in the plane and move it to another position in the plane without altering its shape or size.

To recognize **reflection symmetry**, one can fold the paper vertically down the middle and check that the crease

to see why the **orchid blossom** has two axes of symmetry: the vertical and the horizontal midlines since the blossom only shows a bilateral symmetry. The **cardinal's** bilateral symmetry comes from a diagonal axis of symmetry.”

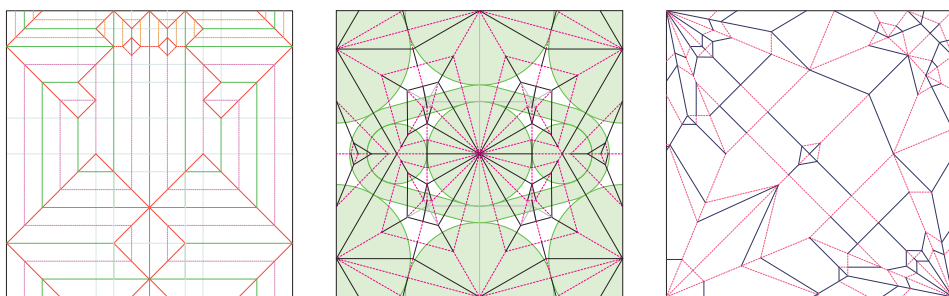


Figure 3.1: **Whitetail doe**, **Orchid blossom**, and **Cardinal**

Rotation and translation on pots and flowers

“Archimedes, you don’t just do origami with square paper.” Euclid prompted him.

“Oh yes, when circular paper or regular polygonal paper is used, another explosion of possibilities ensues. Just look at how Robert Lang makes flowers, pots, and **tessellations** from them.” Archimedes couldn’t wait to show them more. “Circular paper naturally produces **rotational symmetry** like the **pot's** crease pattern showing 24 divisions of mountain and valley folds giving rise to 24 rotational symmetries. Robert Lang also created a **pot** out of a regular heptagon giving rise to 7 rotational symmetries.” Archimedes was eager to show more origami crease patterns. “When we thought of tiling as *ancient Greeks*, we had stones or tiles to play with. Later on, I saw that M. C. Escher turned **tessellation** into an intricate mathematical design. But imagine tessellation in origami—crease patterns tiling a piece of paper resulting in an abundance of **translational symmetry** while creating a **rose!**”

pattern on the left of the **axis of symmetry** coincide with the crease pattern on the right. Similarly, when a mirror is placed perpendicular to the paper on the **axis of symmetry**, then the mirror’s image matches the hidden side behind the mirror.

A **rotation** of a figure about a point, P , on the plane can be visualized with a pin and tracing paper: First put a piece of tracing paper on top of the figure to copy both the figure and P ; second, put the pin through P on the tracing paper and the original plane where the figure is. Without moving the plane of the figure, rotate the tracing paper around the pin at P the specified angle of rotation. More formally to rotate a point, A , of the figure, form a vector from the point of rotation, P , to A , then rotate the vector \overrightarrow{PA} around P the specified angle. Exercises in rotation at the end of this chapter may be done with point-wise rotations when crucial points in any given figure are marked first.

A **translation**, also called a **slide** or a **glide**, moves all points of a figure according to the same direction and displacement. **Corresponding angles** in a parallel-line-and-transversal arrangement are shown to be congruent using a **translation**. ◀

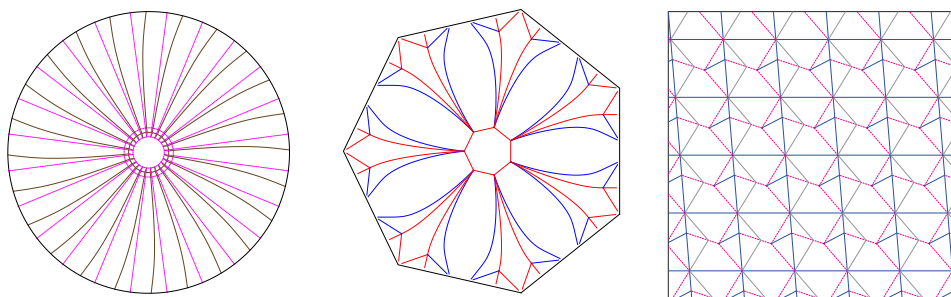


Figure 3.2: Otomipot, Tensionpot, and Hollow rose

3.5 GPS IN TWO DIMENSIONS—ANCIENTS WITH A MODERN TOY

Archimedes could have gone on forever with the wonders of origami, but Euclid noticed that the scientists were discretely glancing at the clock because of their upcoming meetings or classes. Very gently, Euclid interrupted Archimedes, “Thank you for all the food, paper, and great conversation. We would like to explore the campus and the surrounding area.”

To prevent them from getting lost, the Mathematics Department presented them with a GPS (Global Positioning System) to navigate between Capilano University’s North Vancouver campus, their residence near Dollarton Highway, and grocery shopping at Superstore.

Immediately upon receiving the new toy, Euclid went straight for the manual and began studying how to operate a GPS while Pythagoras was trying to gather another group of followers among the scientists to start a new school of Pythagoreans. Instead of the manual, Archimedes, the inventor, turned on the GPS and wondered aloud:

PUZZLE 20 What is the least number of satellite stations needed to locate our position?

For those interested in making origami, it is helpful to have clear **steps**.

Origami techniques helped design an **aluminum structure**.

Tomohiro Tachi of the University of Tokyo has a **laboratory** devoted to new designs, some of which were shown in **Bunny fold and more**.

The Danish-Icelandic artist, Olafur Eliasson, uses **light and reflections** to engage viewers to participate in his art.

The Israeli-American designer, Neri Oxman, creates **bio-architecture or material ecology** by guiding nature in geometric constructions.

A **sphere** is the surface of a perfectly round ball. Mathematically, it is the set of points in three dimensions equidistant to a fixed point, called the centre. Figure **3.3 on the next page** shows a 2-dimensional representation of spheres.

Note: Only the location of the GPS is considered in the discussion to simplify the model. In reality, time of transmission, time of arrival, atmospheric interference with signal paths, and stationary or orbiting satellites all need to be considered. For more, please visit **multilateration**.

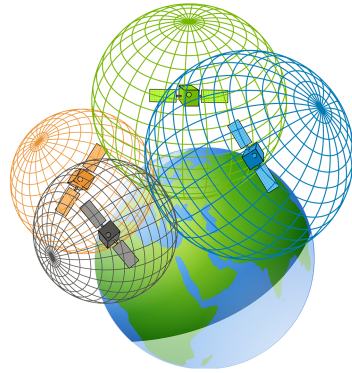


Figure 3.3: GPS pinned by 4 satellites from GISGeography

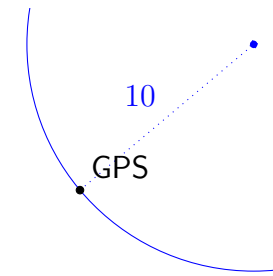
Euclid looked up from the GPS manual to answer Archimedes, “Our GPS calculates the difference between its time now and the delay in receiving the same time reported by a satellite to find the distance to the satellite. This distance can be visualized as the radius of a **sphere** centred at the satellite with our GPS on the sphere’s surface. The second satellite with its own radius, thus its own sphere intersects the first sphere at a **circle**. The third satellite’s sphere intersects the circle at just two points. Finally, we must require at least four satellites to pin point our GPS, cutting down from two points to just one.”

The three ancient mathematicians couldn’t wait to test out the new toy, once they understood a little how GPS might work with satellites. They quickly found the shortest way to the staircase, walked down to the music department following wonderful melodies to the band room, sat down for a quick fix for Pythagoras’s ears.

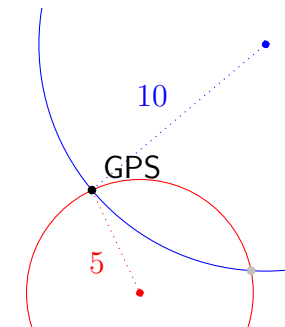
After uplifting their souls with music, they felt totally recharged to embark on a walk to Superstore. Euclid wanted to find a shortcut through the woods by Sportsplex, so Archimedes followed the GPS passing by Cedar, the Library, Arbutus, Birch, and finally, Sportsplex. Pythagoras saw a group of young people on a field running after one

Since it is not easy to draw spheres on paper, we simplify the three-dimensional case to two dimensions. Imagine that the GPS measures the distance to one satellite to be 10 units, a second satellite to be 5 units, and a third, 7 units. The **diagram** shows a plane finding its distances to three stations. Use an appropriate scale factor to draw three circles on paper to locate the GPS.

First circle



Second circle



ball, and whenever possible, kicking it. “Hey, look! I can gather another group of followers. Should we approach them?”

Euclid looked up from his manual and shook his head, “I can never understand why they have to fight for just one ball. Why don’t they get a ball each?” To answer Pythagoras, “No, don’t let your hubris lead to another desperate escape from your enemies. We are not here to form another exclusive school of Pythagoreans. I want to get egg tarts from Superstore. Let us catch up to Archimedes.”

“Hurry up, Euclid and Pythagoras!” Archimedes called after them. “This walk to Superstore is not a **Marathon**. This reminds me of a poster I read outside the STEM office. Do you know what a **Hackathon** is? A long run from *Hack*? Deducing from the origin of the word, *Marathon*, which was a long run from Marathon to Athens, I don’t know a place called *Hack*; perhaps *hackathon* was just a misspelling. Replacing the first *a* by an *e* would make more sense—at least, to me.” Archimedes chuckled to himself having fun with words again.

With that, Pythagoras, Euclid, and Archimedes began their first adventure outside campus.

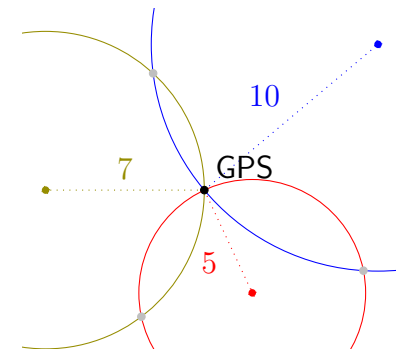
3.6 EXERCISES

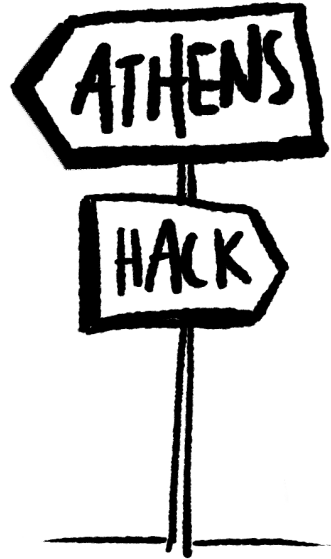
Angles

3.1 Euclid drew a diagram with two parallel lines and a transversal in the horizontal direction. Given the 75° angle as one of the interior angles, answer the following questions.

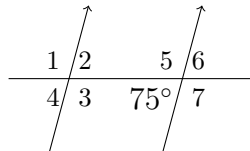
- Name the angle corresponding to the 75° angle.
- Name the angle vertically opposite to the 75° angle.
- Name the angle forming an alternate interior angle pair with the 75° angle.

Third circle





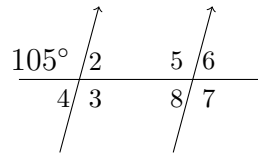
- d. Name the angle forming an *interior angle on the same side of the transversal* pair with the 75° angle.
- e. Name a pair of alternate exterior angles congruent to the 75° angle.



3.2 Euclid drew a diagram with two parallel lines and a transversal in the horizontal direction. Given the 105° angle, answer the following questions.

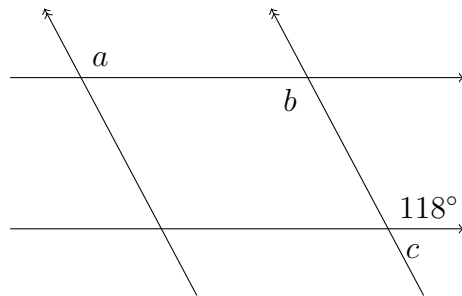
- a. Name the angle corresponding to the 105° angle.

- b. Name the angle vertically opposite to the 105° angle.
- c. Name the angle forming an alternate exterior angle pair with the 105° angle.
- d. Name a pair of alternate interior angles congruent to the 105° angle.

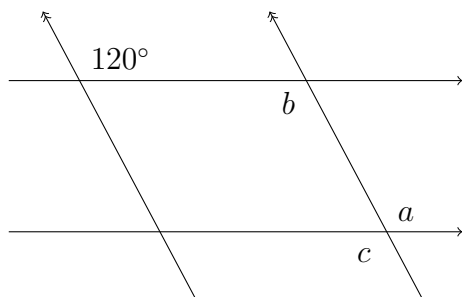


[IM]

3.3 Euclid drew two pairs of parallel lines and marked angles $\angle a$, $\angle b$, and $\angle c$ with 118° above $\angle c$. Find all three marked angles, each with reasons.

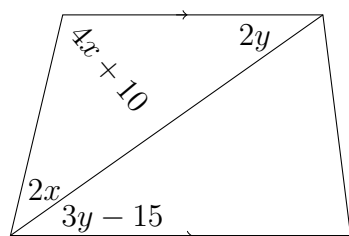


3.4 Euclid drew two pairs of parallel lines and marked angles $\angle a$, $\angle b$, and $\angle c$ with a distinguished angle 120° . Find all three marked angles, each with reasons.

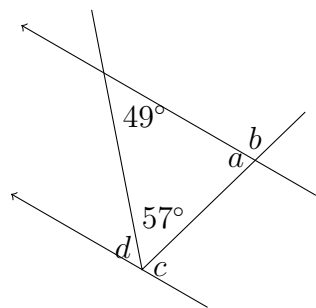


[IM]

3.5 Euclid drew a trapezoid with horizontal parallel bases. Solve for x and y .

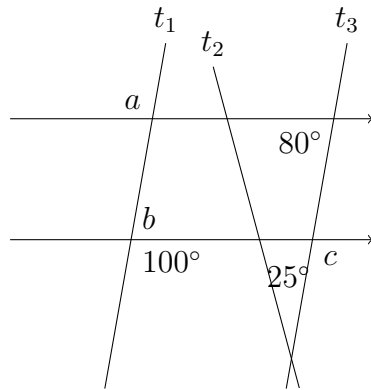


3.6 Euclid drew two parallel lines and two transversals intersecting at the angle 57° . Find a , b , c , and d with reasons.

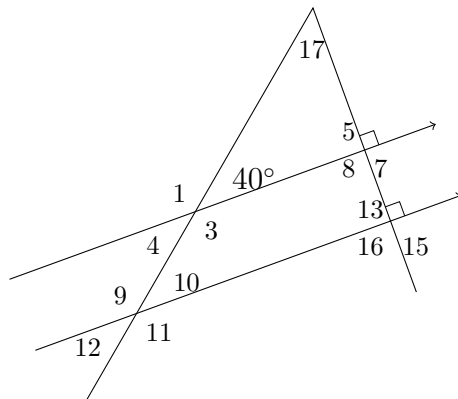


[IM]

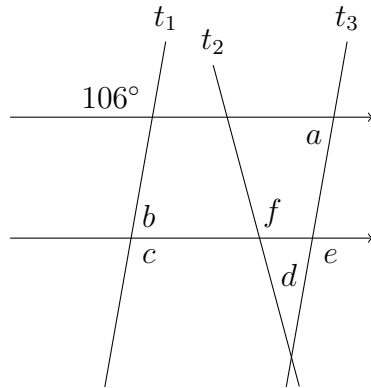
3.7 Euclid drew two horizontal parallel lines and three transversals, t_1 , t_2 , and t_3 with three given angles: 25° at the bottom of the little triangle, 100° , and 80° . Find angles $\angle a$, $\angle b$, and $\angle c$. Can one deduce that t_1 is parallel to t_3 ?



3.8 Euclid drew two parallel lines and two transversals intersecting at $\angle 17$ where one of the transversals is perpendicular to the parallel lines. Find angles $\angle 12$, $\angle 13$, and $\angle 17$.



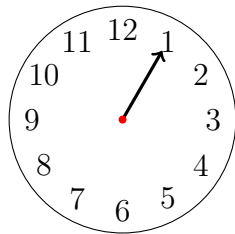
3.9 Euclid drew two pairs of parallel lines, the horizontal pair, and the pair t_1 and t_3 . Suppose $\angle f \cong 115^\circ$, find the other five angles: $\angle a$, $\angle b$, $\angle c$, $\angle d$, and $\angle e$.



[IM]

3.10 In a twelve-hour clock, how many degrees does the hour hand sweep in one hour?

[IM]



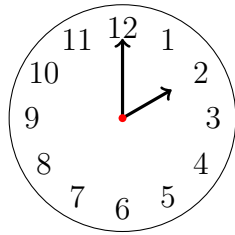
3.11 In a twelve-hour clock, how many degrees does the hour hand sweep in one minute?

3.12 In a twelve-hour clock, how many degrees does the minute hand sweep in one hour?

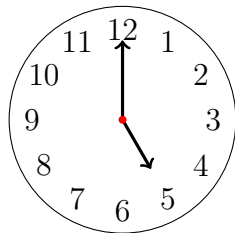
[IM]

3.13 In a twelve-hour clock, how many degrees does the minute hand sweep in one minute?

3.14 Find the acute angle formed by the hour hand and the minute hand at two o'clock on a twelve-hour analogue clock.

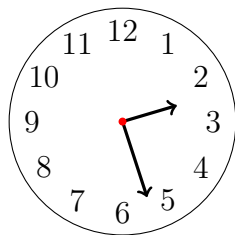


3.15 Find the smaller angle formed by the hour hand and the minute hand at five o'clock on a twelve-hour analogue clock.

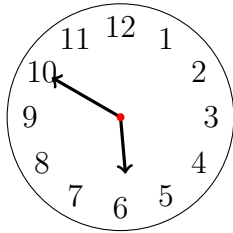


[IM]

3.16 Find the acute angle formed by the hour hand and the minute hand at twenty-seven past two on a twelve-hour analogue clock.



3.17 Find the smaller angle formed by the hour and minute hands at ten to six on a twelve-hour analogue clock.

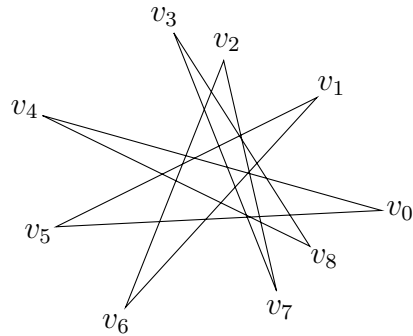


[IM]

Stars

3.18 In Figure 3.4 on the following page, Euclid drew a seven-point star for Archimedes and asked him to find the angle sum of the seven pointy angles. What is the angle sum?

3.19 Euclid drew a nine-point star for Archimedes and asked him to find the angle sum of the nine pointy angles. What is the angle sum?



[IM]

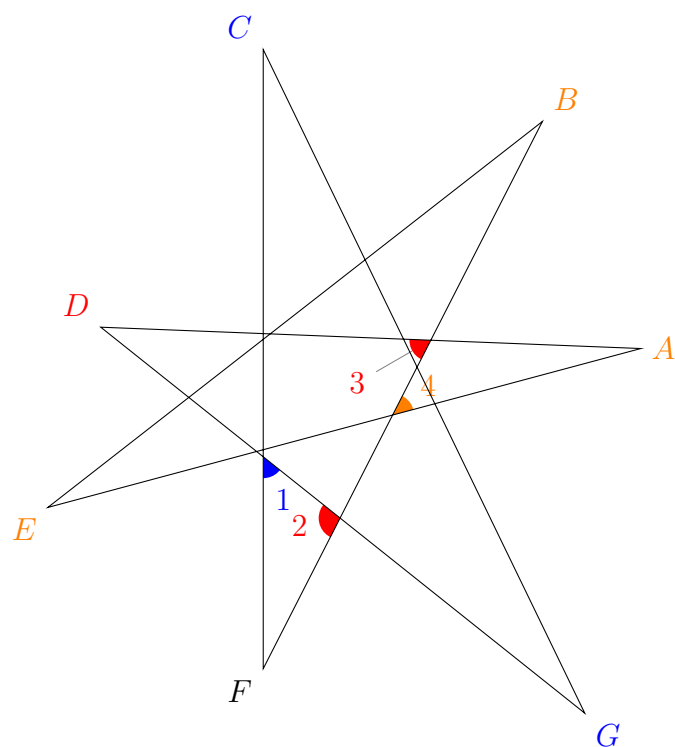
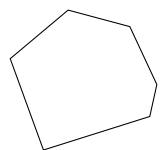


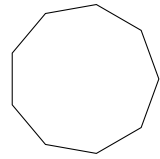
Figure 3.4: Euclid's seven-point star

Polygons

3.20 Pythagoras drew a convex hexagon. Find its interior angle sum.



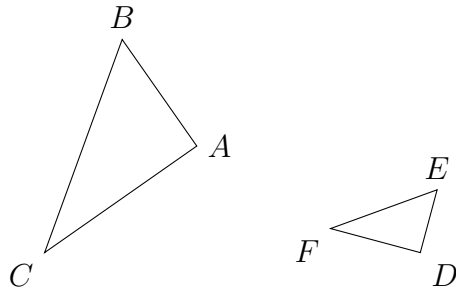
3.21 Pythagoras drew a regular convex nonagon. Find the measure of one interior angle.



The **altitude** relative to a side in a triangle is the height or distance from the side to its opposite vertex.

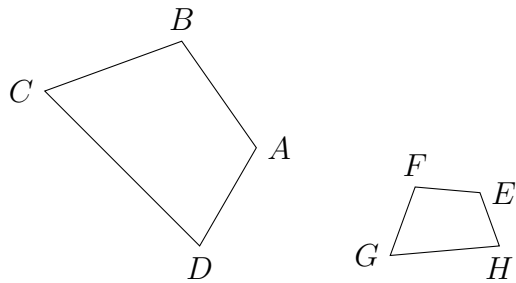
[IM]

3.22 Pythagoras drew a pair of triangles, $\triangle ABC$ and $\triangle DEF$. Are they similar?



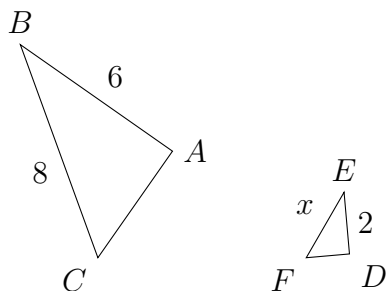
3.23 Project topic suggestion: Study **Altitude reciprocals** of triangles.

3.24 Pythagoras drew two quadrilaterals: quadrilateral $ABCD$ and quadrilateral $EFGH$. Are they similar?

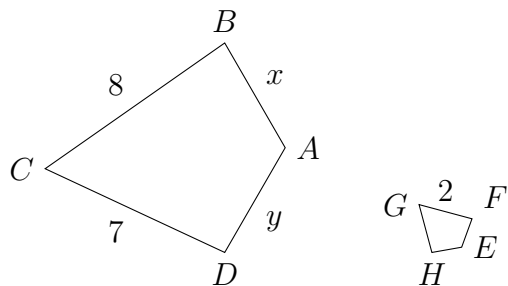


[IM]

3.25 Pythagoras drew two similar triangles: $\triangle ABC \sim \triangle DEF$. Solve for x .

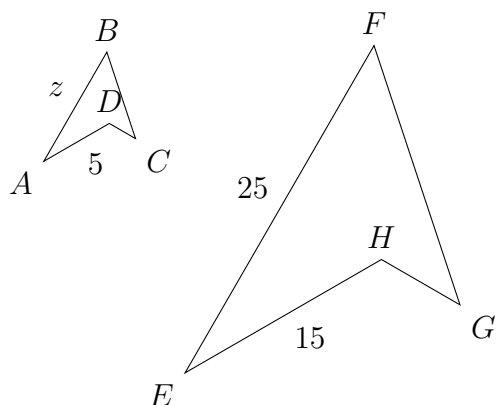


3.26 Pythagoras drew two similar quadrilaterals: quadrilateral $ABCD$ is similar to quadrilateral $EFGH$. Find x and y given that $\overline{FE} \cong 0.5$, and $\overline{HE} \cong 0.6$.



[IM]

3.27 Pythagoras drew two similar quadrilaterals: quadrilateral $ABCD$ is similar to quadrilateral $EFGH$. Find z given that $\overline{AD} \cong 5$, $\overline{FE} \cong 25$, and $\overline{HE} \cong 15$.



3.28 Pythagoras had many congruent equilateral triangles at his disposal. What solids could he form with congruent equilateral triangles?

3.29 Pythagoras had many congruent squares at his disposal. What solid(s) could he form with congruent squares? [IM]

3.30 Pythagoras had many congruent regular pentagons at his disposal. What solid(s) could he form with congruent regular pentagons? [IM]

3.31 Project topic suggestion: Study the **Platonic Solids** and their dual pairs.

3.32 Project topic suggestion: Study **Tile patterns of regular polygons**.

Pythagorean Theorem

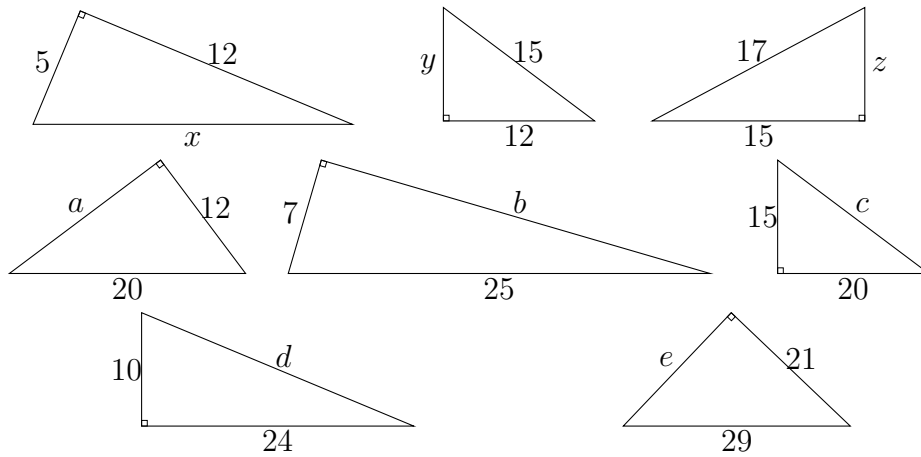
3.33 Pythagoras drew eight right triangles with a missing side length each, some missing a leg while others missing a hypotenuse. Find the missing side length for each.

The **Platonic solids**, also called **regular solids**, are constructed using congruent convex regular polygons. For example, 4 equilaterals form a **tetrahedron**. Euclid proved that there are only five Platonic solids.

Similar to the construction of **Platonic solids**, when one constructs a highly symmetric solid using more than one type of congruent regular polygons, one obtains an **Archimedean Solid**. Truncated Platonic solids are **Archimedean solids**.

[IM]

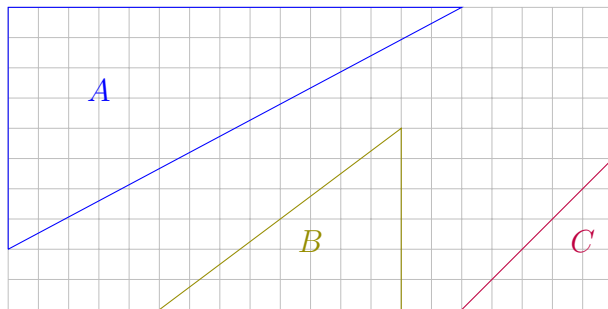
The side opposite the right angle is the **hypotenuse**, i.e. the longest side in a right triangle.



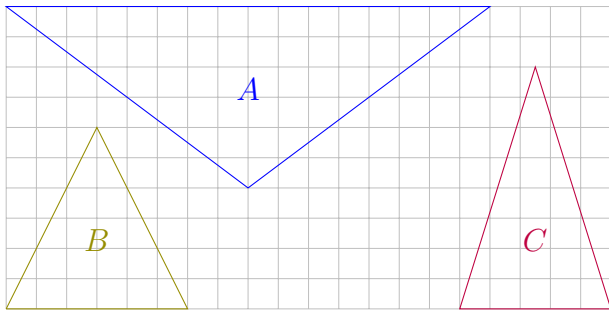
The **perimeter** of a polygon is the sum of all its side lengths. For example, if a triangle has three sides of lengths a , b , and c , then its perimeter is $a + b + c$.

3.34 Pythagoras looked at all eight right triangles he drew and wondered which ones were similar?

3.35 Pythagoras drew three right triangles: $\triangle A$, $\triangle B$, and $\triangle C$, on a square grid. Find the length of the hypotenuse of each triangle.

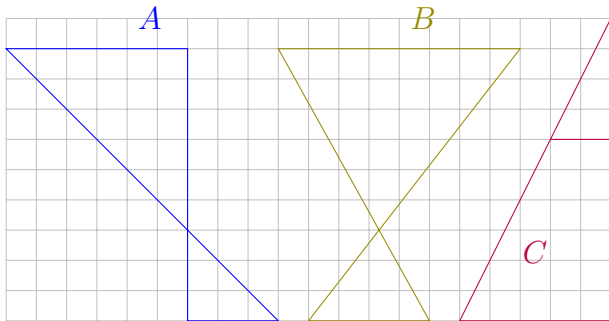


3.36 Pythagoras drew three isosceles triangles: $\triangle A$, $\triangle B$, and $\triangle C$, on a square grid. Find the length of the **perimeter** of each triangle.

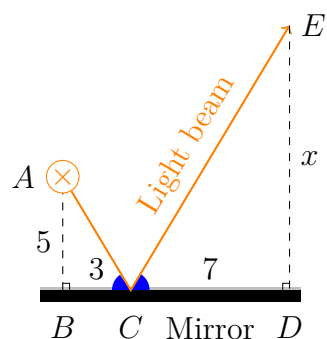


[IM]

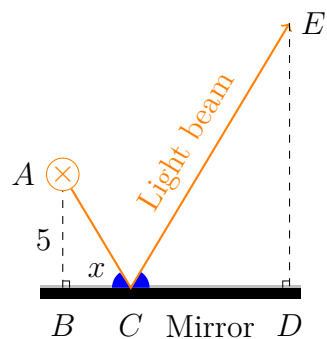
3.37 Pythagoras drew three pairs of intersecting triangles. Prove that each pair has similar triangles and find the perimeter of the smaller triangle in each.



3.38 Pythagoras pointed a laser from A towards C on a mirror on \overline{BD} ; the laser reflected to a wall containing both points D and E . The angles formed by the laser with the mirror are congruent. Find x , the distance DE where the laser would land.

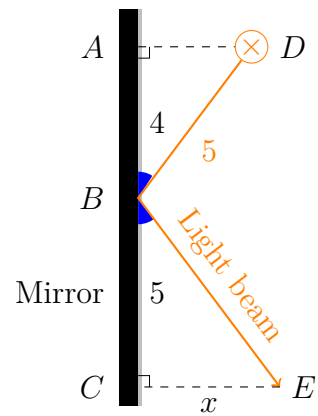


3.39 Pythagoras pointed a flash light from A towards a point, say C , on a mirror of length 10 from B to D ; the light reflected to a wall containing both points D and E . The angles formed by the light beam with the mirror are congruent. Find x , the distance of \overline{BC} that the light needed to land to achieve the shortest total distance from A to E on the wall on \overline{DE} .



[IM]

3.40 Pythagoras pointed a light from D towards B , a point on a mirror; the light reflected to a wall containing both points C and E . The angles formed by the light beam with the mirror are congruent. Find the distance \overline{CE} .



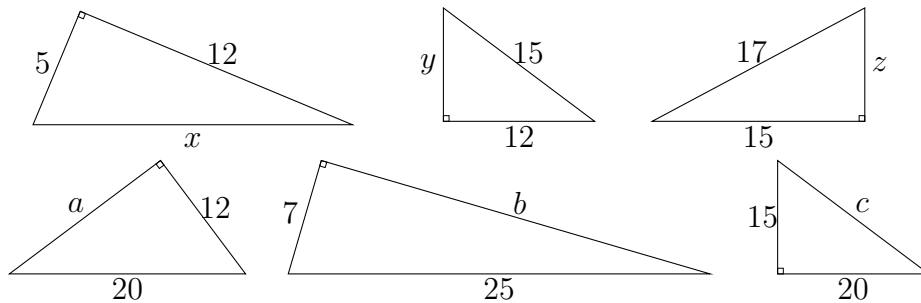
[IM]

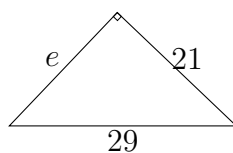
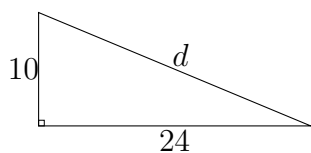
3.41 Pythagoras drew a right triangle similar to a 3–4–5 triangle. Suppose its perimeter was 60 units long, find all three side lengths.

3.42 Pythagoras drew a right triangle similar to a 5–12–13 triangle. Suppose its perimeter was 60 units long, find all three side lengths. [IM]

Areas

3.43 Pythagoras drew many right triangles missing one of its side lengths. Find the area of each.

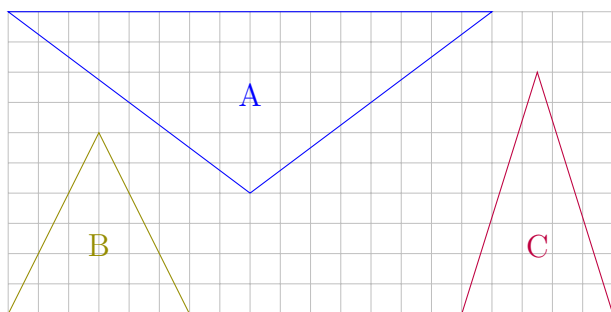




[IM]

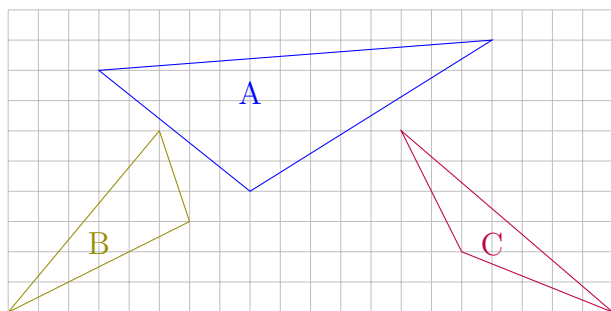
Perimeters, areas, and volumes cover one-dimensional, two-dimensional, and three-dimensional measurements respectively. When different units need to be unified in a particular dimension, one can consult a [unit converter](#). For one-dimensional units, the most common conversion is $100 \text{ cm} = 1 \text{ m} = 1000 \text{ mm} = 0.001 \text{ km}$. The [Physical Measurement Laboratory](#) has a table of metric prefixes.

3.44 Archimedes drew the three isosceles triangles below, $\triangle A$, $\triangle B$, and $\triangle C$, on a square grid. Find the area of each triangle.



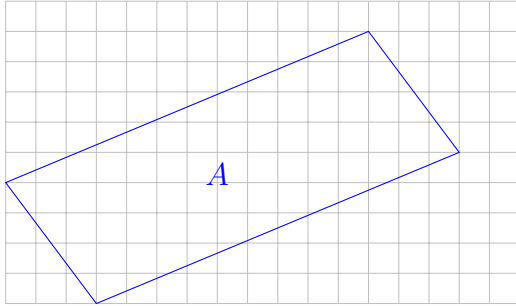
[IM]

3.45 Archimedes drew the three triangles below. Use an enclosing rectangle to find the area of each triangle.

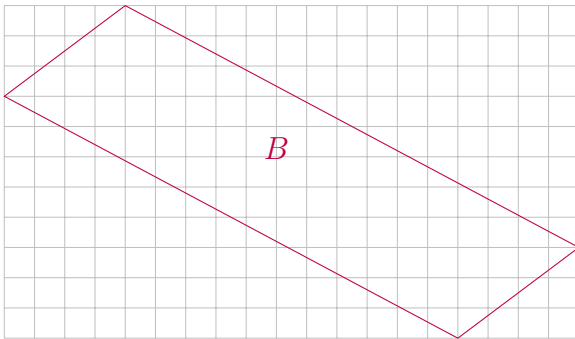


3.46 Project topic suggestion: Study [Triangles inscribed in rectangles](#).

3.47 Archimedes drew quadrilateral *A* below on a grid. Find both the perimeter and the area of the quadrilateral.

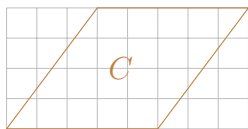


3.48 Archimedes drew quadrilateral *B* on a grid. Determine what type of quadrilateral Archimedes drew, then find both the perimeter and the area of the quadrilateral.

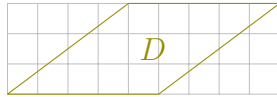


[IM]

3.49 Archimedes drew quadrilateral *C* below. Determine the type of quadrilateral he drew and find its area and perimeter.



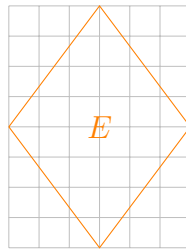
3.50 Archimedes drew quadrilateral D below. Determine the type of quadrilateral he drew and find its area and perimeter.



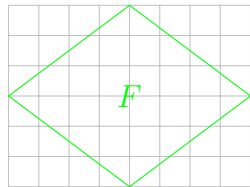
[IM]

The classification of quadrilaterals can be done according to the number of parallel pairs first followed by either congruent angles or congruent sides: trapezoid, parallelogram, rectangle (congruent angles) or rhombus (congruent sides), square.

3.51 Archimedes drew quadrilateral E below. Determine the type of quadrilateral he drew and find its area and perimeter.

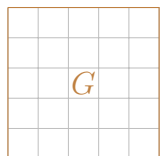


3.52 Archimedes drew quadrilateral F below. Determine the type of quadrilateral he drew and find its area and perimeter.



[IM]

3.53 Archimedes drew quadrilateral G on the next page. Determine the type of quadrilateral he drew and find its area and perimeter.



To see how to construct a circumscribing circle containing a given triangle, please watch [Kelly Castaneda's](#) construction.

[IM]

3.54 Archimedes looked at the five quadrilaterals C , D , E , F , and G above and compared their perimeters and areas. What is your conclusion after the comparison?

3.55 Archimedes studied rectangles whose areas were all 12 square units. If his rectangles all had integer side lengths, how many such rectangles did he study? Which one had the longest perimeter?

3.56 Archimedes studied rectangles whose areas were all 18 square units. If his rectangles all had integer side lengths, how many such rectangles did he study? Which one had the shortest perimeter? [IM]

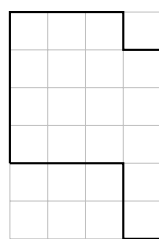
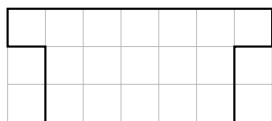
3.57 Archimedes found some rectangles of integer side lengths where the area and the perimeter matched numerically. (These are called **equable** rectangles.) How many can you find? [IM]

3.58 Archimedes found some triangles of integer side lengths where the area and the perimeter matched numerically. (These are called **equable** triangles.) How many can you find?

3.59 Archimedes found some **equable** parallelograms. How many can you find? [IM]

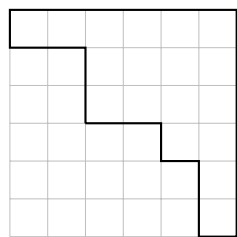
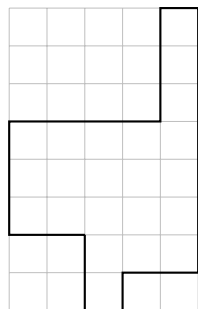
3.60 Archimedes generated all **equable** regular polygons. How did he construct them? [IM]

3.61 Archimedes drew polygon A on the next page on a grid. Find its enclosed area and perimeter.

Polygon *A*Polygon *B*

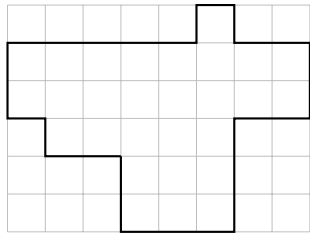
3.62 Archimedes drew polygon *B* above on a grid. Find its enclosed area and perimeter. [IM]

3.63 Archimedes drew polygon *C* below on a grid. Find its enclosed area and perimeter.

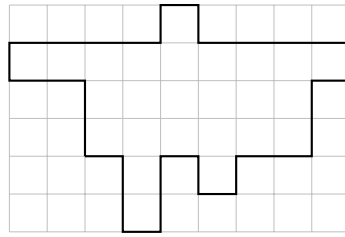
Polygon *C*Polygon *D*

3.64 Archimedes drew polygon *D* above on a grid. Find its enclosed area and perimeter. [IM]

3.65 Archimedes drew polygon *E* on the next page on a grid. Find its enclosed area and perimeter. Compare the perimeters of polygons *A*, *B*, *C*, and *D* above to their corresponding enclosing rectangles' perimeters. Can you find a shortcut to the perimeter of polygon *E*?



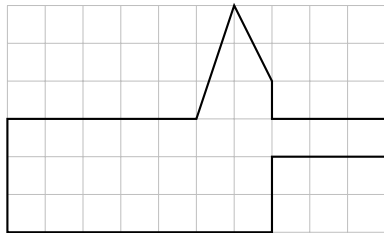
Polygon E



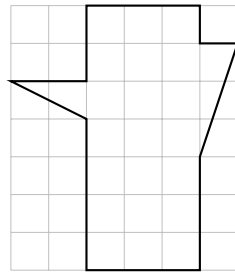
Polygon F

3.66 Archimedes drew polygon F above on a grid. Find its enclosed area and perimeter. Does the shortcut found in Exercise 3.65 on the preceding page for the perimeter work for polygon F ? [IM]

3.67 Archimedes drew polygon G below on a grid. Find its enclosed area and perimeter.



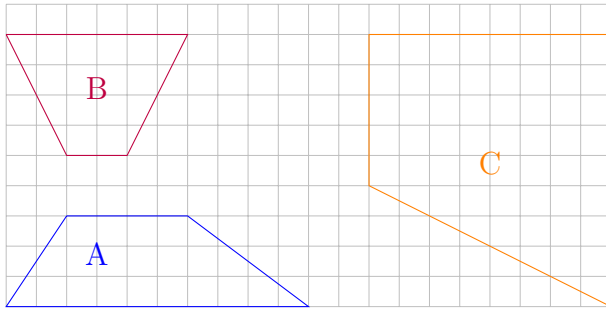
Polygon G



Polygon H

3.68 Archimedes drew polygon H above on a grid. Find its enclosed area and perimeter. [IM]

3.69 Archimedes drew three trapezoids. For each, find its area and perimeter.



3.70 Archimedes drew an $l \times 5$ rectangle of area 95 square units. Find the rectangle's perimeter.

3.71 Archimedes drew a $17 \times w$ rectangle of area 153 square units. Find the rectangle's perimeter. [IM]

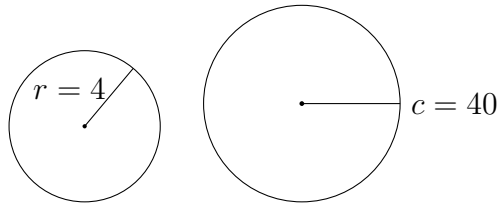
3.72 Archimedes drew an $l \times 5$ rectangle of perimeter 42 units. Find the rectangle's area.

3.73 Archimedes drew a $23 \times w$ rectangle of perimeter 74 units. Find the rectangle's area. [IM]

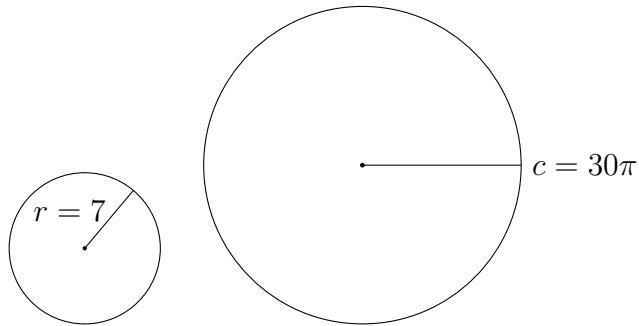
3.74 Archimedes drew a rhombus of diagonals 6 and 8 units long. Find its area and perimeter.

Circles

3.75 Archimedes drew two circles where the radius was $r = 4$ in one and the circumference, $c = 40$ in the other. Find the area of each.



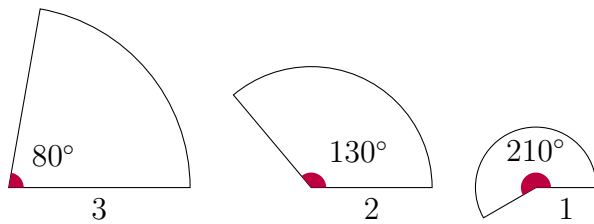
3.76 Archimedes drew two circles where the radius was $r = 7$ in one and the circumference, $c = 30\pi$ in the other. Find the area of each.



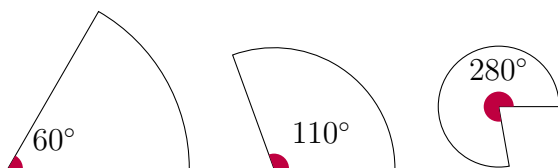
[IM]

3.77 Archimedes obtained a rope of 24 units and wanted to use the rope to enclose three different shapes, one at a time, using all the rope: a circle of circumference $c = 24$, a square of perimeter 24, or an equilateral triangle with perimeter 24. Which shape had the largest area?

3.78 Archimedes drew three sectors of sector angles 80° , 130° , and 210° . Suppose the radii, respectively, were 3, 2, and 1. Find the sector areas.



3.79 Archimedes drew three sectors of sector angles 60° , 110° , and 280° . Suppose the perimeters, respectively, were 30, 26, and 20. Find the sector areas.

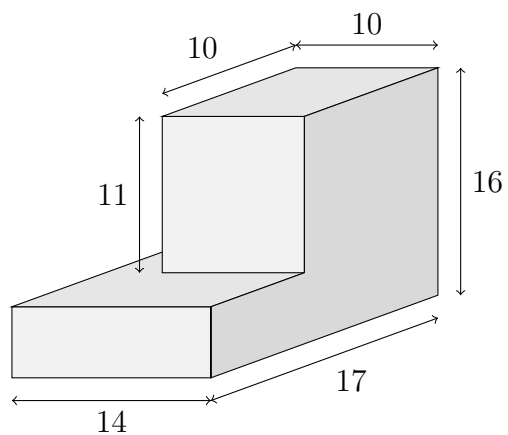


3.80 Archimedes took a sector with sector angle 110° and radius $r = 5$ and glued the straight edges together. What shape would he see?

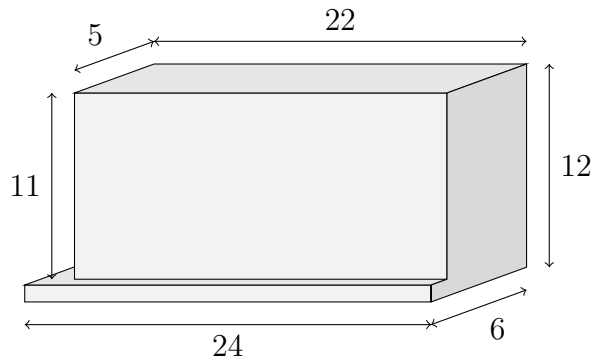
3.81 Project topic suggestion: Study [Circles of Apollonius](#) of a triangle.

Volumes

3.82 Archimedes placed one rectangular prism on a flatter rectangular prism to form solid **A** below. Find both the surface area and the volume.

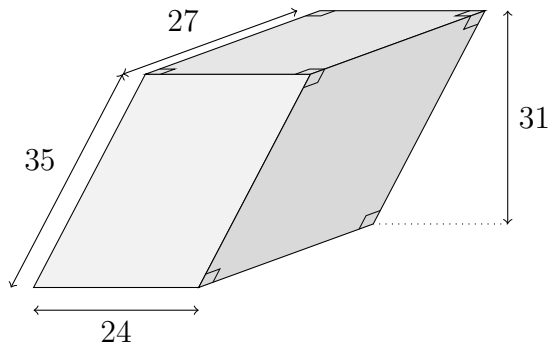


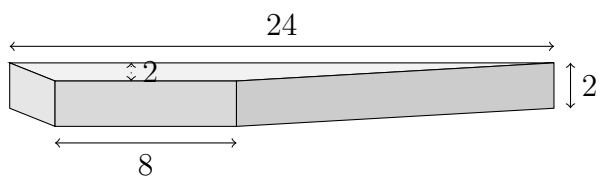
Solid A

Solid B

3.83 Archimedes placed one rectangular prism on a flatter rectangular prism to form solid B above. Find both the surface area and the volume. [IM]

3.84 Archimedes constructed a parallelepiped, i.e. a prism with a parallelogram (actually a rectangle) as its base, seen in solid C below with faces consisting of only parallelograms. Find both the surface area and the volume.

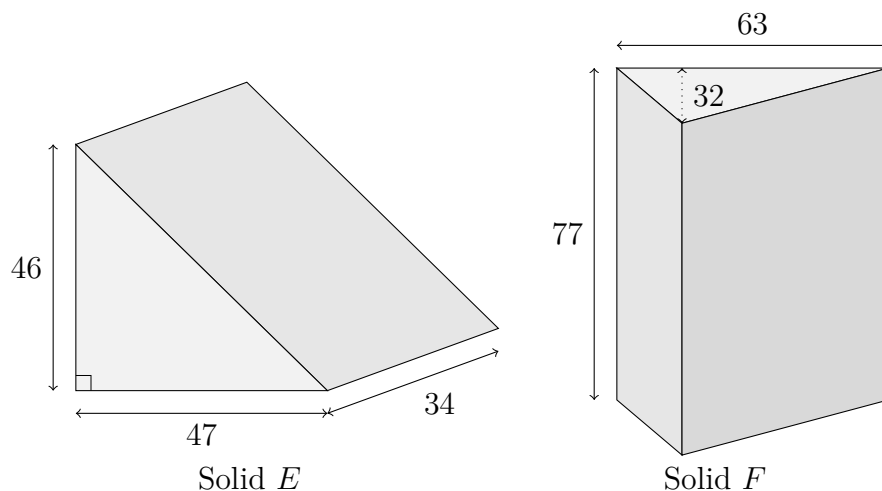
Parallelepiped C



Trapezoidal prism D

3.85 Archimedes constructed a trapezoidal prism to form solid D above. Find the volume. [IM]

3.86 Archimedes constructed a triangular prism for solid E below. Find its surface area and the volume.



Solid E

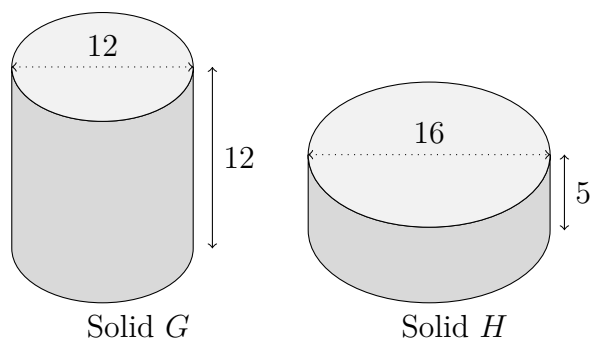
Solid F

3.87 Archimedes constructed a triangular prism to form solid F above. Find the volume. [IM]

3.88 Archimedes constructed a cylinder shown in solid G on the following page. Find the surface area and volume.

Otomipot folded from the first crease pattern in Figure 3.2 on page 101 by Robert Lang.





Tensionpot folded from the second crease pattern in Figure 3.2 on page 101 by Robert Lang.



3.89 Archimedes constructed a cylinder shown in solid H above. Find the surface area and volume. [IM]

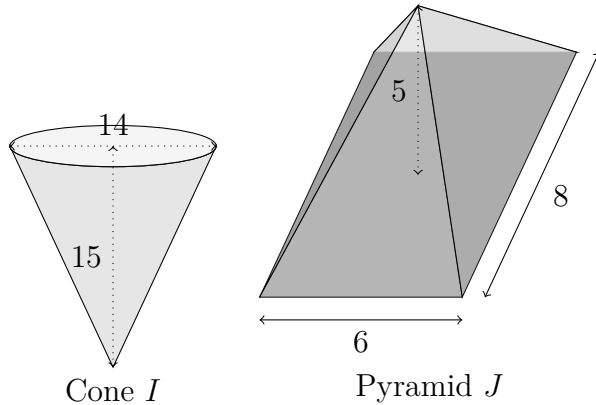
3.90 The STEM faculty presented a vase of cylindrical height 20 cm and base diameter 14 cm as a gift to Archimedes.



Find the volume of the cylindrical vase.

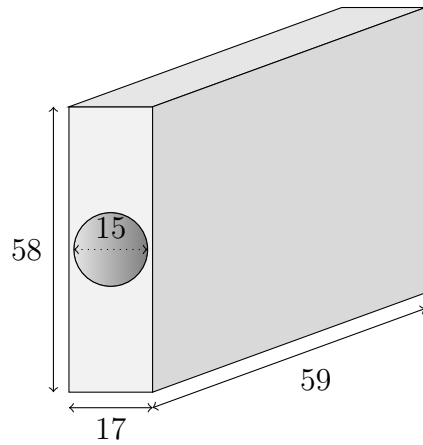
[IM] A **right cone** or a **right pyramid** has its pointy end directly above the centre of the base.

3.91 Archimedes constructed a right cone (solid *I* below) and a right rectangular pyramid (solid *J* below). Find the volume and surface area for each.

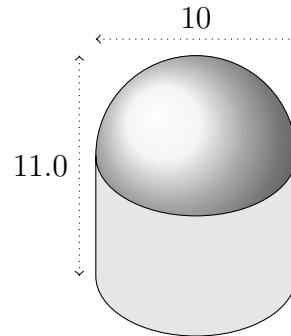


[IM]

3.92 Euclid drilled a cylindrical hole through a rectangular prism as shown in solid *K* on the next page. Find its surface area and volume.



Solid *K*



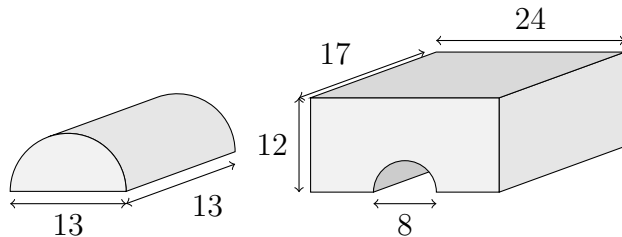
Solid *L*

Hollow rose folded from the third crease pattern in Figure 3.2 on page 101 by Robert Lang.

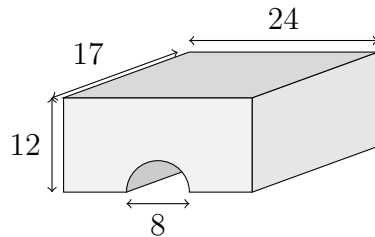


3.93 Euclid placed a hemisphere on top of a cylinder as shown in solid *L* above. Find its surface area and volume.

3.94 Euclid halved a cylinder as shown in solid *M* below. Find its surface area and volume.



Solid *M*



Solid *N*

3.95 Euclid drilled a half cylindrical hole through a rectangular prism as shown in solid *N* above. Find its surface area and volume.

3.96 Euclid put two half cylinders on a square prism to make a heart prism as in Figure 3.5. Find its surface area and volume.

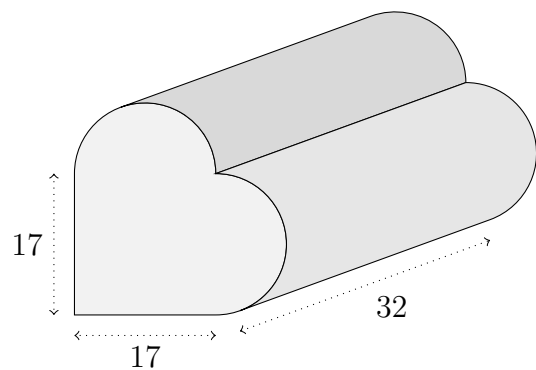


Figure 3.5: A heart shaped prism

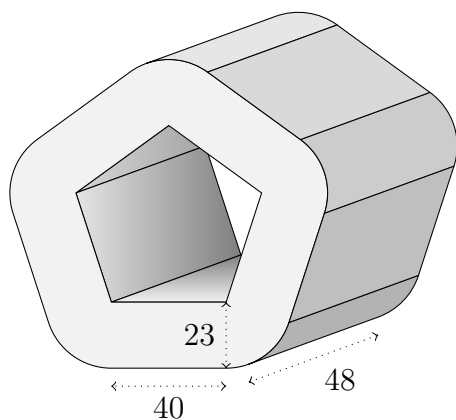


Figure 3.6: Five cylinder parts reassembled


Whitetail doe folded from the first crease pattern in Figure 3.1 on page 100 by Robert Lang.

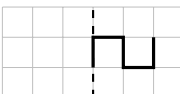


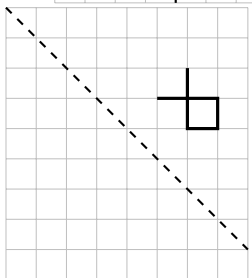
3.97 Euclid cut a cylinder lengthwise in five equal parts. Then he inserted identical boxes between the cylinder parts as seen in Figure 3.6. Find the surface area and volume of the figure.

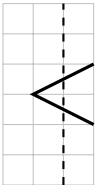
Symmetries

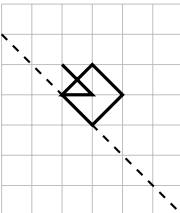
3.98 Reflect each figure along the dashed line.

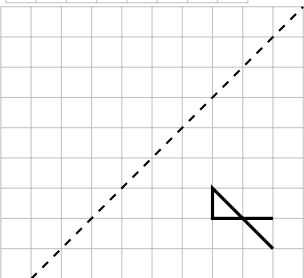
a. 

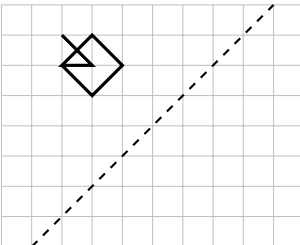
b. 

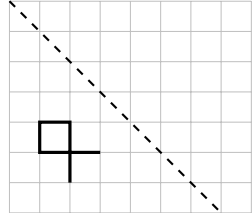
c. 

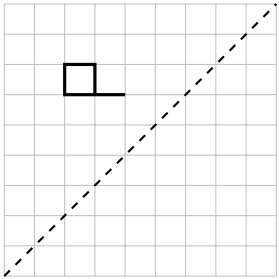
d. 

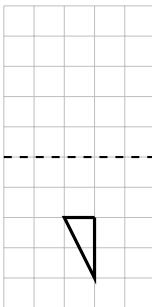
e. 

f. 

g. 

h. 

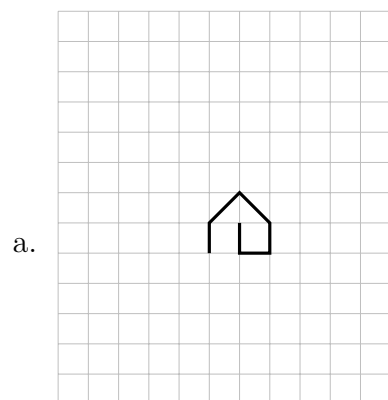
i. 

j. 

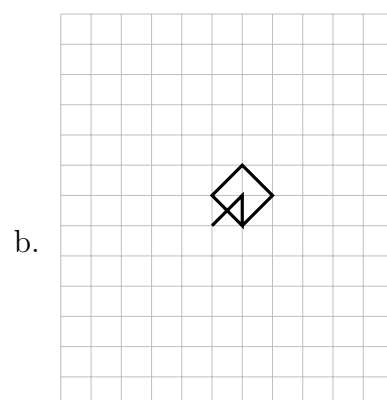
3.99 Translate each figure.

Orchid blossom folded from the second crease pattern in Figure 3.1 on page 100 by Robert Lang.

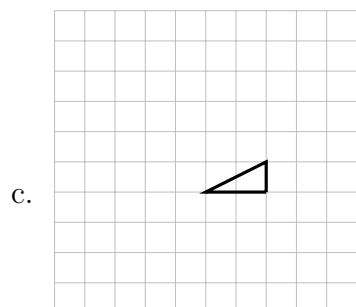




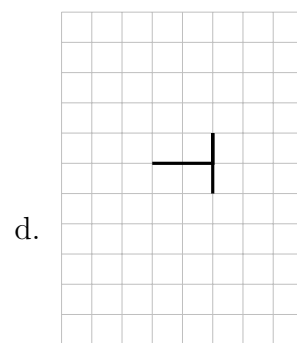
Three units right and
four units up.



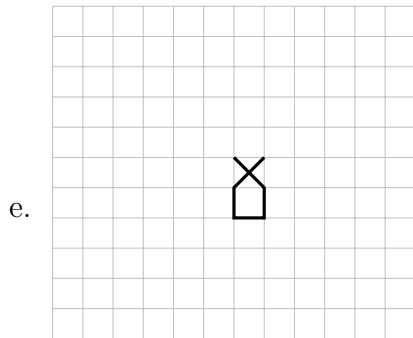
Four units left and one
unit down.



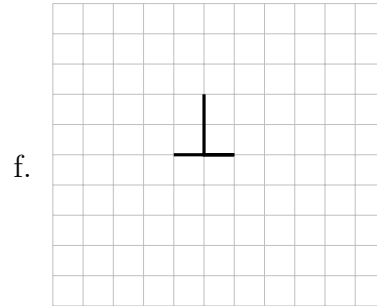
Three units right and
two units down.



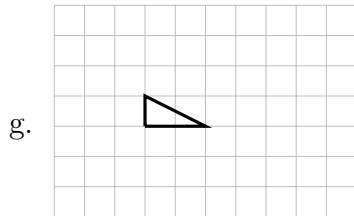
Three units left
and one unit
down.



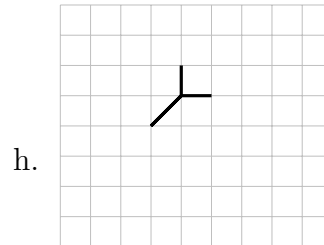
Three units right and four units up.



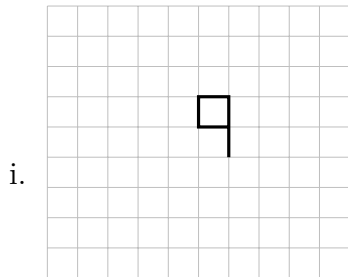
Three units left.



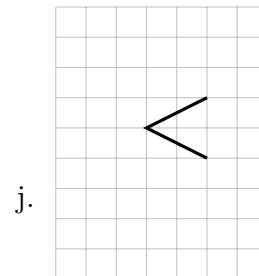
Three units left.



One unit right and two units up.



One unit right and three units down.

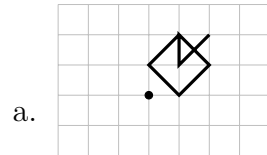


Two units right and one unit up.

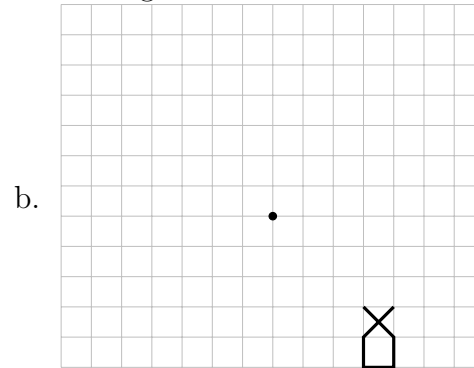
Cardinal folded from the third crease pattern in Figure 3.1 on page 100 by Robert Lang.



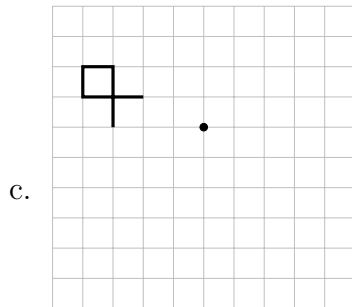
3.100 Rotate each figure around the point. Hint: Apply point-wise rotation by first naming turning points (A, B, C , etc.), also labelling the point of rotation P , second, rotating each vector ($\overrightarrow{PA}, \overrightarrow{PB}, \overrightarrow{PC}$) according to the direction and number of degrees.



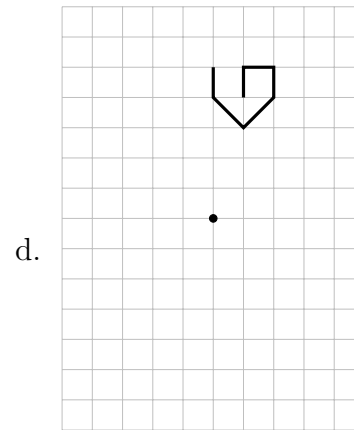
270° counter-clockwise.



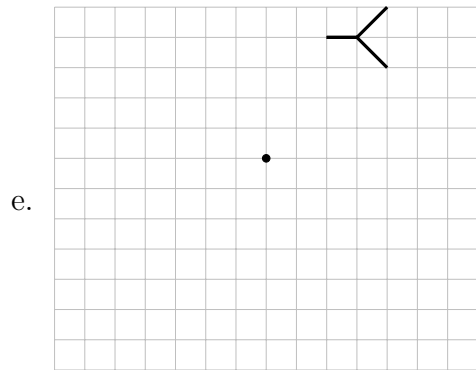
180° counter-clockwise around.



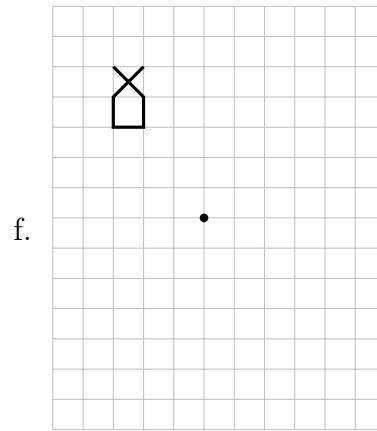
270° counter-clockwise.



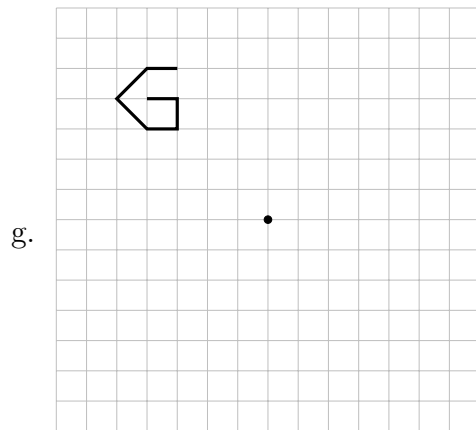
180° counter-clockwise.



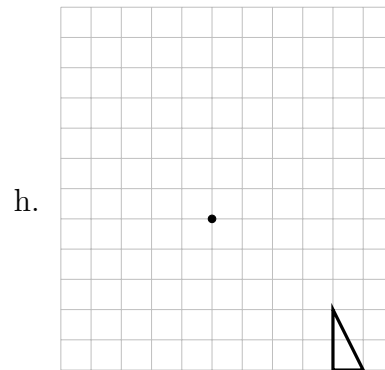
180° counter-clockwise.



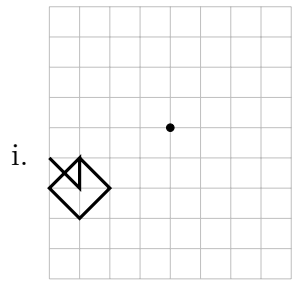
90° counter-clockwise.



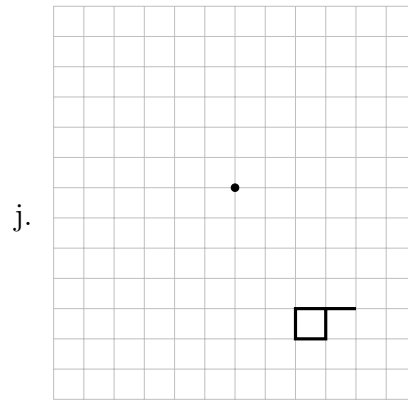
270° counter-clockwise.



180° clockwise.



270° clockwise.



270° clockwise.

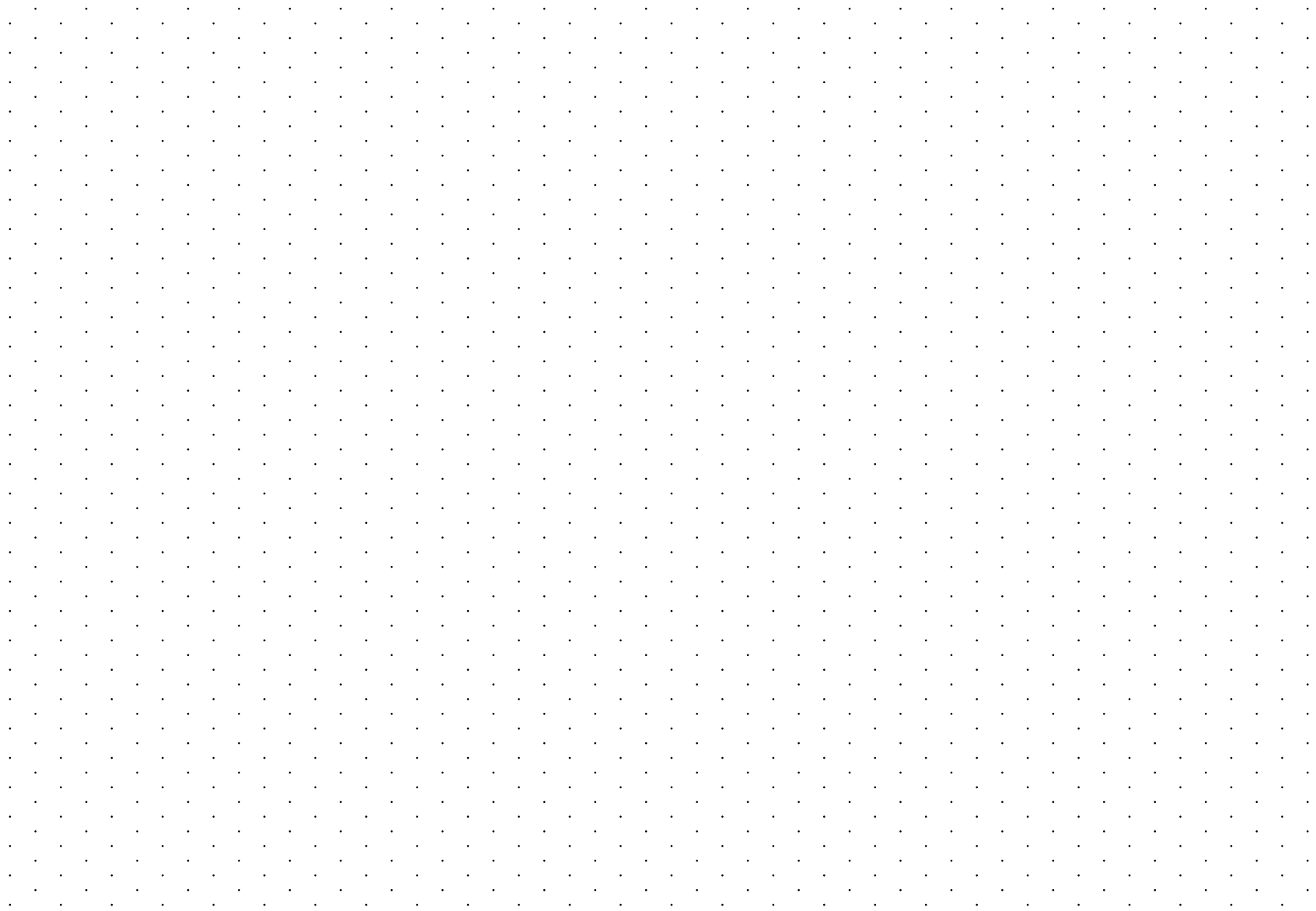


Figure 3.7: A grid for your 3d drawings

Saving, spending, borrowing, and giving—Brian, David, and Janette on a work trip

4.1 SAVING

Fraction, percent, decimal—Introducing Brian, David, and Janette

Capilano University students, **Brian**, **David**, and **Janette** from the Nomads Club wanted to spend a summer with the co-founders of the **Buy nothing—get everything plan** [19] in a small island community off Puget Sound in the Pacific North West of the United States.

Ever since childhood, **Brian**, the oldest child of ten, knew that he had to work and save for his post secondary education. Because feeding ten children on a pastor's income was nearly impossible, Brian's parents made their children work to earn pocket money. Also, to teach their children the importance of giving and saving, Brian's parents would take a tenth ($1/10$), or 10% of each child's earnings for charity donation, another 10% for savings according to the advice of the **Wealthy Barber** [5], leaving the rest for each child to manage. With 80% (or $4/5$) of his earnings at his disposal, Brian learned to budget for short

The fraction $\frac{1}{2}$ conveys three sequential operations:

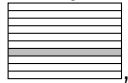
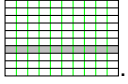
1. Consider an entity as a whole.
2. Equally divide the entity into 2 parts. (The 2 comes from the **denominator**, the number below the middle horizontal line.)
3. Take 1 part out of the 2 equal parts. (The 1 comes from the **numerator**, the top number that shows how many parts to take.)

The fraction $1/10$, read as *a tenth*, means *take 1 part out of 10 equal parts from a whole*. If the *same whole* is divided into 100 parts instead of 10, then an equivalent fraction to $1/10$ would be $10/100$, or 10% because **percent**, denoted by % means *divided into 100 equal parts*.

term purchases and long term goals.

David was the younger child from a family with two children. Both working full time, David's parents capitalized on the Canadian Government's savings incentive, the **Registered Education Savings Plan** (RESP) which provided a 20% ($= \frac{1}{5}$) matching grant to what they contributed to a maximum of \$2500 for each child per year from birth to age 18. Whatever David needed or wanted, David's parents gladly provided, always after careful comparisons of prices and qualities. Therefore, David did not need to take odd jobs before going to university. Furthermore, David was not explicitly taught how to save when he received allowances and monetary gifts. Instead, David just loved seeing things grow—counting an ever-growing number of coins and bills or watching his savings account show larger and larger numbers gave him immense joy. Regarding spending, all David knew was to pay for quality after careful price comparison whenever he needed or wanted to buy something. Strangely enough, David seldom wanted anything.

Janette was the middle child of three, the only daughter. Having a penny-pinching mother, Janette learned to hunt for bargains at a young age. Her photographic memory came in handy whenever she encountered a bargain, namely, the lowest price ever seen according to her memory. The trouble was that she could be easily tempted into buying too much like 10 pairs of Toms slip-ons at 75% off and 5 pairs of Hunters rubber boots at 80% off. Divorced long ago and bringing up three children on her own, Janette's mom could barely save for her own retirement never mind her children's post secondary education. Fortunately, Janette's maternal grandparents helped their grandchildren by paying for extra-curricular activities and matching their scholarships. Before going to university, Janette taught piano lessons to beginners to earn pocket money. As a diligent student, Janette received stellar grades to earn enough scholarship to attend Capilano University—also thanks to her grandparents' matching of her scholarship.

To see that $\frac{1}{10} = \frac{10}{100}$, one can take the fraction picture of $\frac{1}{10}$, , and subdivide each part into 10 equal parts, . Converting from $\frac{1}{10}$ to $\frac{10}{100}$ is a **fraction expansion**.

The fraction $\frac{80}{100}$ or 80% is equivalent to $\frac{4}{5}$ by **fraction reduction**: bundle 20 parts into 1 to see that the numerator, 80 yields 4 parts whereas the denominator, 100 yields 5 parts. This fraction, $\frac{4}{5}$ cannot be reduced further, and is thus called **reduced to the lowest terms**. ◀

Because $\frac{1}{5}$ of \$2500 is \$500, if a child's parents put aside to the maximum, i.e., \$2500 every year for 18 years, when the child turns 18, they will be entitled to a total grant of $18 \times 500 = 9000$ dollars from the Canada Education Savings Grant (CESG). The Liberal Party under **Jean Chrétien** as the prime minister established the RESP in 1998; however, the CESG seemed to help the rich who could save to support their children's education rather than the poor. To correct this oversight, from 2005, additional CESG could be awarded to families according to their net income.

Place values of a base-10 system into the negative powers of 10 are the $10^{-1} = 1/10$ (tenth's place), $10^{-2} = 1/100$ (hundredth's place), $10^{-3} = 1/1000$ (thousandth's place), and so on. Because *percent* means *divided by 100*, $35\% = \frac{35}{100} = 0.35$ where 5 is at the *hundredth's place*. Many of us learned also that dividing by a hundred could be achieved by moving the decimal point two places to

Percent/Decimal equation—Nomads Club

Brian, David, and Janette joined the Nomads Club because of their shared love for **low cost travels**. Unfortunately, the COVID pandemic made it unsafe and expensive to travel outside the North American Continent; therefore, the club members creatively channelled the energy of their travel bugs by playing images of **Google Earth** or documentaries with **David Attenborough** on a big screen in the student lounge, or by planning no-fly trips in Canada or the States.

Because all three were also keen to participate in **Greta Thunberg's** environmental movement, they decided to spend a summer learning how to wean a community slowly from addictive overconsumption. "I know we can bus to Puget Sound and pollute less, but I have a car anyway for school and work. Would you two like to carpool with me? You can pay me the bus fare of \$40 each way." Brian suggested.

"That is such a generous offer. A car is so expensive to run now with the **global energy crisis**. Gasoline prices going from \$1.36 in January of 2020 to \$1.73 a year later is an increase of more than 27%! We have not seen **inflation** this high in Canada before." Janette truly appreciated Brian's offer but just had to make the other two worry about inflation as much as she did.

David thoughtfully added, "Also, Brian's car can be another resource for the community once we get there. I am sure our symbolic contribution of bus fare equivalent to Brian can't even begin to cover the costs of insurance and maintenance for the car. Thank you for offering, Brian."

"I never think of it that way, only that I need the car to go to school and to work. However, when I budget for a year, the car costs me at least \$5000, almost as much as my tuition fee. After the first \$13000 being tax free, I need to pay a little more than **20%** in taxes, plus all the deductions from my employers, how much do I have to make on paper before I could pay for my car?" Brian sounded more anxious the more he thought about the cost of having his car.

the left, like $35.0\% = 0.350$. Note that 0 trailing a number with decimal places does not add any value except **significant digits** because $\frac{35}{100} = \frac{350}{1000}$ where the fraction on the right can be reduced to the left by dividing both the numerator and the denominator by 10. \triangleleft

If a pair of Floral Toms slip-ons costs \$34.99, how much does one save after a 75% discount?

A discount of 75% from \$34.99 is $75\% \times 34.99 = 0.75 \times 34.99 = 26.2425$, or \$26.24 saved.

If a pair of Floral Toms slip-ons costs \$34.99 originally, after a 75% discount, how much does one pay before taxes?

Since taking 75% off 100% leaves 25% = $(100 - 75)\%$ or $1/4$, $25\% \times 34.99 = 0.25 \times 34.99 = 8.7475$, or $34.99 \div 4 = 8.7475$. Therefore one expects to pay \$8.75 before taxes.

With a \$8.75 purchase before taxes, how much is the total after a 5% Goods and Services Tax (GST) and a 7% provincial sales tax?

Adding the total percentages of taxes, $5 + 7 = 12$ percent, the taxes on \$8.75 will be $8.75 \times 0.12 = 1.05$ dollars. The bill will be $8.75 + 1.05 = 9.80$ dollars.

Using the distributive law, by combining 100% with 12%, one can find the total in one step.

$$8.75 \times (1 + 0.12) = 8.75 \times 1.12 = 9.80 \quad \triangleleft$$

Brian's 20% estimate for taxes consists of 5% provincial

Janette was eager to help Brian sort this out. “Should we estimate that the deductions from your pay were around 5% for the sake of simplicity? Suppose you want \$5000 for the car after a 25% deduction from your pay, i.e. 20% for taxes and 5% for others, then \$5000 is 75% of your earnings. Let your earnings be y dollars.

$$0.75 \times y = 5000, \quad \text{so} \quad y = 5000 \div 0.75 \approx 6666.67$$

Brian, you would need to make at least \$6666.67 to afford the car for a year. Say you are paid \$20 per hour, $6666.67 \div 20 \approx 334$ hours. For a full time student who can manage perhaps 20 hours of work every week, it would take you $334 \div 20 = 16.7$, or almost 17 weeks, that is, more than a semester!”

“My goodness! Janette, you are actually under-estimating the hours because I am paid less than \$20 per hour. If only I had put those hours into my studies instead of working, my grades would have been totally different! So ironic to realize that I may not need to work so much with a different life style. On top of that, since our **Upass** is included in the tuition and student fees, I can use public transit to save money and pollute the air less.” Brian was lost to his new plans for the next academic year.

“Brian, Brian!” David shook Brian’s shoulder to get his attention. “Do you know that you don’t have to sacrifice your love for the outdoors when you give up your car? Look at all the **car-sharing** and ride-sharing possibilities available to us now.”

Brian suddenly had a big smile on his face, “what if I sell my car now? If we just bus to Puget Sound with our bikes as luggage, we would still be mobile enough to be of use. Because of the shortage of rare metals, used cars are selling at a premium. Who knows? I might even get more for my car than what I paid for it two years ago.”

Janette’s wheels were turning again, “While in the midst of inflation, holding onto material goods would be more valuable than cash. I fear that a more careful analysis is needed to see if selling your car is the right

and 15% federal taxes for his income bracket.

Ideas on **no-flight holidays** are shared by the Guardian. Once flying is easy again, students should **avoid being kidnapped** and consider **35 countries to avoid** as prepared by the Australian government.

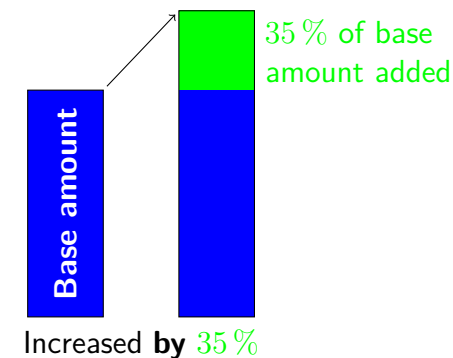
Percent increase is a **relative comparison**. The base one compares to is the first number, here \$1.36 which is increased to \$1.73.

The **absolute increase** is $1.73 - 1.36 = 0.37$.

The **relative increase** is $0.37 \div 1.36 \approx 0.272$.

The **percent increase** is $0.37 \div 1.36 \times 100\% \approx 27.2\%$.

To summarize, one says that \$1.36 is increased **by** 27.2%, or that \$1.36 is increased **to** 127.2% resulting in a final price of \$1.73. <



thing to do. Another factor we have not considered is the convenience of having a car and how much it is worth to you, Brian. Suppose commuting to school by car takes 15 minutes while taking Translink takes more than 45 minutes, would a 30-minute increase in travel time one way bother you?"

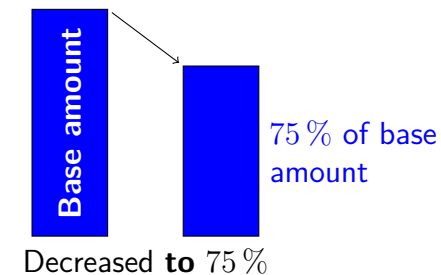
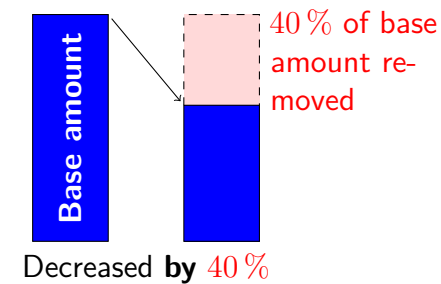
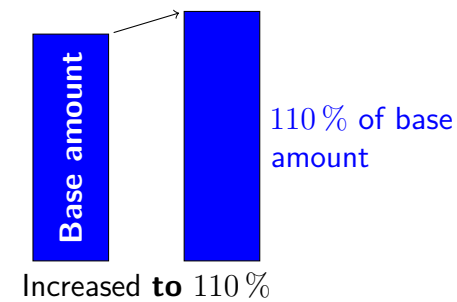
Brian thought about this, "You know how easily I waste more than 30 minutes on social media or games. In absolute terms, an hour more for commuting back and forth from school should not bother me, but in relative terms, it is a 200% increase!"

4.2 SPENDING

Simple interest—Cost of living

David took out a box of home made cookies from his backpack to share. "What I am curious about is how we plan to live over the summer. I know that Janette is living at home while attending school, Brian is living with his grandparents to save rent, and I live in the university dormitory. None of us need to pay for storage while we spend the summer in Puget Sound. That is fortunate! But how much have you saved to be able to spend a summer volunteering away from home?"

Janette was eager to jump in first. "You know that I only have my pocket money from teaching piano lessons and a small savings." Janette took out a box of cut fruit from her backpack to offer the other two. "Since the exchange rate is still pretty bad from Canadian dollars to American, even an average of US\$500 a month for room and board would make the work trip impossible for me to afford. So I searched for a family who needed child care and house help in exchange for room and board. I found a family with a triplet of boys, 3 years old, extremely active from early morning until late at night when both parents work. My hours will begin early, at 6:30 am, to help dress and feed the boys before they leave for pre-school. Then when I get home at 5 pm, I need to prepare dinner for the family, clean up after dinner and play with



Inflation rates from 1960 on showed the highest rate of 12.47% in 1982, but the energy crisis caused the price of gasoline to increase by more than twice that rate. ($27 > 2 \times 12.47 = 24.94$)

The Consumer Price Index or CPI measures the average

the boys until 8 pm, Monday to Friday.” Brian was surprised to hear how many hours Janette planned to work. “Can you manage the usual 8 hours of work plus 5 hours with your homestay family?”

Janette nonchalantly replied, “We all know that our eight hours of work with the Buy-Nothing co-founders would be lots of fun. The five hours of work looking after three boys, cooking, and cleaning won’t be too hard. I am sure I have enough energy for the tasks. Besides, the family will pay me \$15 per hour, 5 hours a day for 20 days per month. Look at this calculation,

$$20 \times 15 \times 5 - 500 = 1500 - 500 = 1000,$$

I will make \$1000US a month, at least \$3000 for the summer. Not bad for a volunteer work trip in a beautiful area getting inspired while making money. All I need is my bus fare and the cost of tuning up my bike.”

David shook his head at Janette. “I don’t know how you could be so comfortable living in someone else’s home. I would feel too self-conscious to know where to put my hands and feet. Fortunately, I love camping in the summer. With a really good tent and an air mattress, I plan to set up my tent on a camp ground while working for the camp ground owner part-time. I only need to clean the camp grounds for an hour early in the morning and to be a cashier for an hour in the evening. Besides bus fare and getting my bike ready, I just need to cover food costs which I budgeted \$15 a day, or $90 \times 15 = \$1350$ US for the summer.”

Brian opened his eyes super wide, “Where do you come up with that much in US dollars?”

David looked a little embarrassed answering Brian, “My parents said that their mutual funds did very well last year making double digit returns. They would pay for my food at home anyway if I was to stay with them in the summer. I know, it is about \$1700 they offer to pay for me.”

Brian’s brain began to churn. “Wait a minute! My savings account in the bank is only making 0.05%. Using simple interest calculation,

change in prices based on a collection of goods and services. The CPI is one of the tools for economists to study rates of inflation over time.

A consequence of industrial nations’ dependence on fossil fuel is the **plastic waste nightmare**. According to the **United Nations Environmental report**, about 300 million tonnes of plastic waste are produced every year, almost the weight of the entire human population. Of the 300 million tones, 9% has been recycled, 12% incinerated, leaving the rest 79% to landfills, dumps, or the natural environment.

Suppose the Insurance Corporation of British Columbia (ICBC) charges around \$2000 per year for a young adult to insure a car, and the maintenance of a car costs \$1000 per year, then having a car for four months ($= \frac{1}{3}$ of a year) in the summer would cost $3000 \div 3 = 1000$ dollars without including gasoline cost and **depreciation** of the car.

According to the local car-sharing company, **Modo**, the annual cost of car ownership hovers around \$10 000. For a round trip, Modo charges \$4 per hour including gas, insurance, maintenance, and parking privileges.

Suppose Brian takes a day-trip every weekend and needs a car for 10 hours. Modo Co-operative will charge Brian $4 \times 10 = 40$ dollars each week. Using 50 weeks a year, $50 \times \$40 = \2000 . Brian’s car insurance is enough to cover the use of a car from Modo. <

The Harvard University psychologist, **Daniel Gilbert** [15],

if I have to make \$1700 in interest one year, how much must I have in the savings account? Wait, wait, Janette! Don't do the calculation for me. I want to do it myself. Let y be the amount in the savings account earning 0.05 % annually to get \$1700 in interest after one year.

$$0.05\% \times y = 0.0005 \times y = 1700, \quad \text{so} \quad y = 1700 \div 0.0005 = 3\,400\,000.$$

Wow, I don't think I would ever have 3.4 million dollars in a savings account."

Janette was also shocked by Brian's calculation. "Of course, poor students like Brian and me plan and save for trips over a few years. But, even if Brian saved for five years using simple interest, at the rate of 0.05 %, it would still need \$680 000 in the savings account. How do you afford this trip, Brian?"

"The way I afford anything—by working for it. I don't have much saved because I have to pay for my education and entertainment. My grandparents help me so much by providing room and board while I go to university, hoping that I could save for the future. Unfortunately, my hobbies are expensive, photography, rock climbing, surfing, swing dancing, volley ball, just to name a few. For a while, I even took out a loan for a motorbike. My grandmother got so mad at me that she issued me an ultimatum: another purchase by borrowing, move out! For this summer's work trip, I budgeted \$500 per month for food. My father's friend will lend me a simple camper which I plan to hitch to my car. The camper is where I will live, borrowing showers from members of the community. Janette's work plan is very smart, making money instead of spending money from our volunteer work trip. I am going to look for such opportunities, like home care for seniors because I have such trainings already and I love spending time with seniors. My grandma would be so proud of me if I could bring home more money than I spend at the end of the summer."

warned of the brain's tendency to perform relative comparisons rather than absolute. According to Gilbert's book, *Stumbling on Happiness*, subjects participating in experiments repeatedly showed how attitudes towards money were vastly different depending on the situation. To compensate for the brain's relative logic, the *Wealthy Barber's* author, David Chilton [5] recommended shopping with coupons, going to specific stores for better deals, and saving whenever possible, regardless whether it was saving \$50 from \$100 (a 50 % discount) or from \$100 000 (a 0.05 % discount) because one's bank account only noted an increase of \$50, not where the savings happened.

According to the exchange rate on January 14th, 2022, $\text{US}\$500 = \text{CA}\626.83 .

In 1981, banks offered 20 % annual returns on guaranteed investment certificates, GIC in short. If a GIC was purchased at \$1000, after one year at 20 % using **simple interest**, the GIC would earn $20\% \times 1000 = 0.20 \times 1000 = 200$ in interest, resulting in a new balance of $1000 + 200 = 1200$ dollars.

A one-step calculation to find the total is $1.20 \times 1000 = 1200$ using the reverse of the distributive law for $1 + 0.20 = 1.20$.

The **simple interest formula** states $I = Prt$ where I is the interest earned on a principal of P at an annual rate of r over t years. ◀

If one wants to find the value A of an account that pays simple interest on a principal of P at an annual rate of r over t years, then $A = P + Prt = P(1 + rt)$, using the

Compound interest—David’s TFSA

After listening to Janette and Brian’s discussions on returns from mutual funds versus savings accounts, David suddenly remembered one of his family’s dinner table conversations about spending the earnings from savings. “One time, I asked my parents how they could afford any big-ticket item they needed without ever budgeting or planning. Their answers surprised me. First, they always spent way less than they earned because of their immunity to advertisements, trends, and fashions. For example, they never owned cell phones until they had to get one for a trip to Hawaii a few years ago; they still drive their cars from 2007/2008, well maintained though; their clothes last forever because they buy good quality and maintain the same sizes through exercise and healthy eating. Second, they enjoyed cooking with first ingredients and almost never took us to restaurants or ordered take-outs or deliveries because whatever food we wanted, they learned to cook. Cooking and eating together is an important family ritual for us daily. Finally, they saved with good advice from their financial planner, like maximizing the annual contribution to the **Tax-free Savings Account** or TFSA, maximizing the RESP contributions for their children, maximizing the **Registered Retirement Savings Plan** or RRSP contribution, and weighing risks wisely in mutual funds or other investments.”

Janette and Brian were dumbfounded. “Ten children make it difficult to make ends meet, never mind saving for a rainy day.” Brian sighed.

“How about a single mother bringing up three children? When not at work, my mother sewed and cooked, also from first ingredients. Still, it was not easy for her to save. David’s parents are clearly frugal, yet one must have enough left from the pay check to capitalize on the options of savings available to them. I see why the **wealth gap** is widening.”

David tried to brighten the conversation, “Let’s see how we can mimic my parents’ financial plans for ourselves. Since we turned 18, as Canadians, we could take advantage of the TFSA. Their advisor explained that whatever growth made in the TFSA is also tax free, so

distributive law. The value A is also called the **future value** or the **future amount** in accounting.

If David’s parents had \$200 000 in mutual funds earning 12 % one year, then the increase in value of their mutual funds would be $0.12 \times 200\,000 = 24\,000$. Here $P = 200\,000$, $r = 12\%$, and $t = 1$.

To find the **future value** A after one year, one adds P and $I = Prt$, i.e., $A = 200\,000 + 24\,000 = 224\,000$, or in one step, $A = 200\,000(1 + 0.12 \times 1) = 200\,000 \times 1.12 = 224\,000$. \triangleleft

If mutual funds earned 12 % the same year that savings account in banks earned 0.05 % (annually), how much better was it to own mutual funds?

Mutual funds earned $12 \div 0.05 = 240$ times savings account, so 239 times better, or 23 900 % more.

If the rate of inflation is greater than 0.05 %, then money in savings account is **depreciating** at the difference of the rates, that is losing its purchasing power.

Danske Bank (a major bank in Denmark) charged -0.6% for overnight lending rate in 2021. As a result, many Danish banks charged a negative rate for savings accounts. A negative rate means that the owner of the account pays the bank to keep the money. In this case, one should revert to the traditional method of keeping cash under the mattress as long as rats, roaches, floods, or fire could not get to it.

one could balance the entire investment portfolio by putting riskier, and generally higher yielding mutual funds in the TFSA.”

Janette was curious since she did not have a TFSA account. “If we don’t make much money as students, how does being tax-free benefit us?”

Brian answered quickly, “Up to a point, I don’t need to pay taxes. But, I have been making much more than that for a couple of years. If I had known about the TFSA, and had saved monthly, I would not have been tempted into getting a motorcycle.”

David was encouraged to share more, “Suppose we save \$6000 this year in our TFSA with no withdraws for 10 years. My parents’ advisor averaged a 6 % return annually. We could use the simple interest formula repeatedly on an ever-growing amount to get to a big number after ten years: see Table 4.1 where \$6741.6 is \$6360 ($= 6000 \times 1.06$) multiplied by 1.06, so equivalent to \$6000 multiplied by 1.06^2 .

Time	Amount
Start	\$6000
End of year 1	$6000 \times 1.06 = 6360$
End of year 2	$6000 \times 1.06^2 = 6741.6$
End of year 3	$6000 \times 1.06^3 = 7146.096$
End of year 4	$6000 \times 1.06^4 = 7574.86$
End of year 5	$6000 \times 1.06^5 = 8029.35$
End of year 6	$6000 \times 1.06^6 = 8511.11$
End of year 7	$6000 \times 1.06^7 = 9021.78$
End of year 8	$6000 \times 1.06^8 = 9563.09$
End of year 9	$6000 \times 1.06^9 = 10\,136.87$
End of year 10	$6000 \times 1.06^{10} = 10\,745.09$

Table 4.1: Earning 6 % per year for ten years

Notice how the first year’s gain of \$360 turned into a gain of nearly

Jamie Oliver, the famous no-fuss cook from Britain, is an activist against food ignorance as seen in this TED talk on [Teach every child about food](#).

For Canadian cooks whose recipes are easy to follow using simple ingredients, one can consult [Greta Podleski’s *Yum and yummer* \[25\]](#) or Chef Michael Smith’s [recipes](#).

The list of items from [Household budget: Food at home](#) by Sterling Price of Value Penguin shows that a significant portion of food cost goes into non-alcoholic beverages, canned and prepared foods, frozen prepared food, and processed food. Eliminating or drastically reducing non-essential foods not only results in large savings, but also enhances healthy habits.

A three-part documentary series explaining the [money gap](#) from Deutsche Welle, a German media company based in Bonn showed how ordinary German citizens earned almost no interest on their savings account while the rich made 10 % or more annually on low risk investments.

Money saved in the TFSA is after tax money. However, the growth is tax free when withdrawn in the future. This policy is depriving future generations tax revenues.

On the other hand, saving money in an RRSP provides a tax credit for the current year, and will be taxed when one withdraws during retirement, hopefully at a lower tax bracket.

The **compound interest formula** for the future value A with a principal P paying an annual rate of r compounding

\$600 in the last year. Whatever we can save early, our savings just work for us quietly in the background.”

Janette suddenly realized how David’s parents could afford anything without purchase-motivated savings. “I see—your parents just used the growth of the RESP in addition to the government’s grant to pay for your education. In fact, for all the luxury expenses like travel, the growth of their TFSA could comfortably cover. The amazing part of this type of financial planning is the self-sustaining model. As long as the principal is not reduced (and in David’s parents’ case, actually growing steadily), all the financial instruments help build their wealth.”

Looking puzzled, Brian still could not work out one small detail. “When I compared the interest rates of different banks on long term savings accounts to what David’s parents earned from different types of savings, I don’t understand why their financial advisor could average at least a 6% growth annually. Why can’t ordinary people like Janette’s mom and my parents get that type of rate for their savings?”

David felt awkward answering Brian, “One minor detail I forgot to mention. To have the advice of this type of financial advisor, one usually needs a referral from one’s bank after the total savings reach a certain amount, usually large. Before we could get to that level, all we could do is to spend less and save more. That is part of the reason I want to work with the co-founders of the Buy-nothing plan. Can you imagine, Brian, if you keep working at your jobs while saving all you earn? Now, that is a faster way than any financial plan can match.”





After thinking about David’s table of numbers, Janette would like to learn how to go backwards. “I can see that David just kept multiplying by 1.06, the factor from the **future value** formula to get the new balance for another year. But, how could we find out how much money to put aside now in order to have \$30 000 for a car in five years? Let’s assume that an annual rate of 6% is available compounded **monthly**, so we have $6\%/12 = 0.5\%$ growth each month. Five years would mean $5 \times 12 = 60$ months, or **compounding periods**. Let P be the amount

m times per year over t years is $A = P \left(1 + \frac{r}{m}\right)^{tm}$.

If we use 0.05% for \$6000 in a savings account, compounded annually for ten years, the increase will only be a little over \$30.

Time	Amount
Start	\$6000
End of year 1	$6000 \times 1.0005 = 6003$
End of year 2	$6000 \times 1.0005^2 = 6006$
End of year 3	$6000 \times 1.0005^3 = 6009$
End of year 4	$6000 \times 1.0005^4 = 6012.01$
End of year 5	$6000 \times 1.0005^5 = 6015.02$
End of year 6	$6000 \times 1.0005^6 = 6018.02$
End of year 7	$6000 \times 1.0005^7 = 6021.03$
End of year 8	$6000 \times 1.0005^8 = 6024.04$
End of year 9	$6000 \times 1.0005^9 = 6027.05$
End of year 10	$6000 \times 1.0005^{10} = 6030.07$

Repeated arithmetic operations are best handled in a spreadsheet.

1. Click on the icon , or on a computer, click on , then  Apps to find .
2. Type 6000 in cell A1 and push ‘enter’ or ‘return’.
3. In cell A2, type “=A1*1.0005” to get the future amount after one year at 0.05%.
4. Copy cell A2’s formula.

to save now.

$$30\,000 = P \left(1 + \frac{0.06}{12}\right)^{60}, \quad \text{or} \quad P = \frac{30\,000}{\left(1 + \frac{0.06}{12}\right)^{60}}.$$

We need a scientific calculator to find P because the exponent is as large as 60.”

“What if I save \$10 000 now at 6 % annually, but compounded daily, and want to know how long to wait until I double my money?” Brian was eager to explore. “Don’t tell me, Janette! I want to figure this one out myself, too. Let t be the number of years needed to double the money. Since the banks use 365 days a year, we will have $365t$ compounding periods. Using the **compound interest** formula,

$$20\,000 = 10\,000 \left(1 + \frac{0.06}{365}\right)^{365t}, \quad \text{or} \quad 2 = \left(1 + \frac{0.06}{365}\right)^{365t}.$$

The quickest way to focus on the exponent is to take logarithm on both sides because

$$\log 2 = 365t \log \left(1 + \frac{0.06}{365}\right), \quad \text{so} \quad t = \frac{\log 2}{365 \log \left(1 + \frac{0.06}{365}\right)}.$$

I hope our calculators have log or ln. Either logarithmic function will give us the answer.”

“I can’t believe that the two of you are turning into mathematicians!” David cackled. “For the sake of completeness, why don’t you, Janette, figure out how to find the rate r when all the other variables in the equation are given? And as for you, Brian, the number of times to compound a year, or m is the only one left?”

Taken as a challenge, Janette dove straight into the scenario. “OK, suppose my mother inherited a first-edition copy of *Silent Spring* [3] by Rachel Carson from her Godmother. After ten years, the value of the book tripled. Assuming that the appreciation of the book was

5. Highlight cells A3 to A11 and paste onto these 9 cells.

Notice that cell A_n has the formula “= A_{n-1} *1.0005”; therefore, numbers from column A match those in the table above. ◀

How can we, as a society, help narrow the wealth gap? Greek economist, Yanis Varoufakis, and Israeli historian, Yuval Harari, warned us of the consequences if the wealth gap was to continue growing and challenged us with ideas to help narrow the gap.

In Janette’s calculation, $P \approx 22\,241.17$.

In Brian’s calculation, $t \approx 11.5534$, so Brian would need to wait over eleven and a half years or 4217 days to double his money.

Compared with David’s Table 4.1 on page 149 compounding annually at 6 %, compounding daily at the same annual rate still requires over 10 years to double the principal.

Visit 3 Blue 1 Brown’s take on Compound Interest through five questions from 3 minutes on.

Can money buy everything in today’s market, like paying obese people to lose weight or children to do homework? Perhaps Michael J. Sandel’s *What Money Can’t Buy: The Moral Limits of Markets* [28] can help us answer some of those questions.

To use the logarithmic function to extract the exponent

compounded **bi-weekly** (every two weeks), at what rate was the book appreciating? Let me identify all the variables in the compound interest equation: $A = 3$ (for tripling in value), $P = 1$ (original value), $m = 26$ (because 52 weeks in a year grouped into 2 weeks for each compounding period yield 26 groups), $t = 10$ (years), and r is unknown.

$$3 = 1 \left(1 + \frac{r}{26}\right)^{26 \times 10}, \quad \text{or} \quad \sqrt[260]{3/1} = 1 + \frac{r}{26},$$

so

$$\frac{r}{26} = \sqrt[260]{3/1} - 1, \quad \text{or} \quad r = 26 \times (\sqrt[260]{3/1} - 1) = 26 \times (3^{1/260} - 1).$$

Here we need a calculator that can take large radicals or we remember that a radical is just a fractional exponent!" Janette felt more than ready to pass the torch to Brian.

Cautiously, Brian began his analysis. "I don't know enough math to do this type of question. When I look at the compound interest formula for the purpose of focusing on m , the unknown,

$$A = P \left(1 + \frac{r}{m}\right)^{tm}$$

m appears both as an exponent and as a denominator inside the parentheses. Taking logarithm on both sides of the equation allows the exponent to come down.

$$\log \frac{A}{P} = tm \log \left(1 + \frac{r}{m}\right), \quad \text{or} \quad \frac{1}{t} \log \frac{A}{P} = m \log \left(1 + \frac{r}{m}\right).$$

Now, I can't solve for m without trial and error. Sorry." Brian shrugged his shoulders.

"I am impressed by your analysis, Brian. I had no idea that it would be this hard to solve for m ." David tried his best to sound encouraging. "Now, we can spend wisely after careful savings. By the way, can we get rich investing in **Cryptocurrency**?"

in an equation $y = a^x$, simply take \log on both sides of the equation to get $\log(y) = x \log(a)$. For example, to solve for x in $45 = 3^x$, take \log on both sides to get $\log(45) = x \log(3)$. Both $\log(45)$ and $\log(3)$ can be calculated by any scientific calculator. Therefore, $x = \log(45)/\log(3) \approx 3.46497$.

With two logarithmic functions \log and \ln available, one must choose one and use the same one all the way through the calculation. The difference between these two functions is their base: \log is base 10 and \ln , the natural log, is base $e \approx 2.718281828$. For more on exponential and logarithmic functions, **Khan Academy** has a thorough exposition. \triangleleft

In Janette's calculation, $r \approx 0.110094$, more than 11%!

One of the rules of **exponents** states that $\sqrt[m]{x} = x^{1/m}$; when $m = 2$, one has $\sqrt{x} = \sqrt[2]{x} = x^{1/2}$. Note that only the square root does not need to be specified. \triangleleft

A **financial calculator** makes comparisons easy.

4.3 BORROWING

“Did I understand correctly from what we just talked about that none of us need to borrow to attend university?” Janette timidly checked while trying not to sound nosy.

“I don’t think it is an accident that those in the Nomads Club do not like to borrow.” Brian answered for David, too. “Except, of course, for my fiasco with the motorcycle.”

“My parents could save for their children and want their children to be debt free after completing a post secondary education. But, in both your families, your grandparents are super generous in supporting their grandchildren to ensure that they remain debt free after university.” David continued, “But how could other students afford such privileges like us?”

Loans or leases—chains and shackles

Janette replied carefully, “Well, when I was deciding which university to attend and what I could afford, I looked into some websites like [Help to young Canadians](#) to see if I could qualify for some [student loan or grant](#). It turns out that there is help for getting an education, just lots of bureaucracy. For those who want to get into an [apprentice program](#) to learn a skill, some employers even collaborate with the government to support the apprentice at many different levels from education to living expenses.” Janette took a cookie from David’s cookie box, and continued, “Apparently, many high school students think that going to a post secondary institution can help them get a well paying job later, but our society lacks a lot of skilled workers in many sectors.”

“Can you imagine borrowing lots of money to attend school just to come out of school unemployed?” Brian started getting worried again. “How could one ever get out of debt without work?”

“This sounds like a completely different degree of trouble: Instead of a growing investment account like my parents own, one is slowly drowning in debt. I understand now the value of a free education

In [Yanis Varoufakis’s](#) book, *Talking to My Daughter about the Economy or How Capitalism Works—And How it Fails* [34], readers learn, among other things, the difference between making profit before borrowing and taking on debt before creating revenue, i.e. from a *society with markets* to a *market society*.

Many ways of mounting an apprenticeship program were explored in both the Work Shift and Rethinking College series by [Paul Solman](#) at PBS NewsHour: 1. [A tech apprenticeship program helping middle America find jobs without colleges](#), 2. [How apprenticeships can bridge the employment gap for workers without college degrees](#), 3. [Jobs requiring college degrees disqualify most U.S. workers—especially workers of color](#).

Suppose a new 2022 Toyota Camry Hybrid

1. costs \$33 235 to own,
2. \$89 weekly at 5.49% to lease for 60 months with \$3900 down, or
3. \$124 weekly at 3.69% to finance for 60 months with \$3900 down,

assuming an interest rate of 0.05% on the cash of \$(33 235 – 3900). Compare the future value in all three cases after 5 years.

With interest compounded annually on \$29 335 for 5 years, one gets $29\,335 \times (1.0005)^5 = 29\,408.41$ while still owning the car after five years.

Using a financial calculator’s [auto lease calculator](#), one gets a future value of \$30 000 leaving a car valued at \$10 000 in the shop after 5 years.

like many of the European countries provide.” David’s horizon slowly expanded to the realization of the crushing weight of debt. “In the past, my parents only showed us the difference in amount from purchasing a car with and without a loan. We learned that by leasing a car, one would need to pay far more than the car’s value. But, getting an education should not be a luxury item.”

Credit/Debit cards—plastic temptations

“While we are on the subject of debt, how many credit cards do you own?” Janette swiftly changed the topic. “It was not so easy to apply for a credit card until I turned 19. Luckily, I found *Canadian Express* which has no annual fees, but higher interest rate than most other credit cards.”

“My parents also found me a credit card, *Student First*, through their financial advisor, no annual fee, with a lower interest rate, but the card required a security deposit to match the credit limit.” David was interested about credit cards, too.

“I know how useful a credit card can be, but knowing myself, it will be too much of a temptation in my pocket, constantly nagging me to spend. To avoid this problem, I decided to use my debit card for purchases instead.” Brian felt confident that his solution was best.

“You know how debt averse my mother and grandparents are, so most of the time, I stay on top of paying off everything before the deadline. This way, I really benefit from having a credit card, building credit score while paying no interest for my purchases.” Janette continued, “However, one time, when I got tempted into buying too many pairs of shoes on a great deal, and forgot to pay the balance that month, I was not happy to see how much *Canadian Express* charged me the following month. You see, as long as I pay the entire balance at the end of each month, my *Canadian Express* lends me money interest free—for that billing cycle. That time I forgot to pay my bill of over \$500, I learned that all the purchases I made the following month were charged interests

Using a financial calculator’s [auto loan calculator](#), one gets a future value of \$40 443 after 5 years. <

When purchasing, one is often asked whether to include an [extended warranty](#). Often, credit cards or manufacturers offer warranty already. As a rule of thumb, when one could afford to replace an item without too much hardship, one should not buy an extended warranty.

The International Monetary Fund, or [IMF](#), summarized data collected from 2020 in a [map](#) showing household debt, loans, and debt securities as a percent of GDP (gross domestic product). Hover your mouse over a country to see its percentage.

Notice that Canadians held 12% more household debt than what they produced in 2020 while Italians only held less than half (45%) of their GDP as household debt. However, on comparing general government debt to the nation’s GDP in another [map](#), Canada had 117.46% while Italy had 155.82% in the same year. Does this show that private citizens in Italy exercised more fiscal responsibility than the Italian government?

The risks and ramifications of over-borrowing are described in the article in *The Guardian* about [Evergrande](#), the Chinese real estate developing giant.

Before getting a credit card, one should understand how [interests](#) are calculated.

also. At a rate of more than 22% per year, even for one month, it felt like highway robbery!”

Brian was intrigued by Janette’s experience. “Do you mean that normally, when you purchase using the credit card, the credit card company lends you money without charging you interest as long as you pay off your balance before the due date?”

“Yes, that is the idea.” Janette replied.

Brian wanted to learn more. “Now suppose you forgot to pay a balance of \$500 from the previous month. During this month, you made \$300 worth of new purchases and returned \$100 worth of goods and paid \$200 to the credit card company. What would be the balance at the end of this month?”

Just then, David was eager to contribute. “I don’t know if you looked into how **finance charges** were computed, Janette. But, when my parents and I were comparing student credit cards, I learned that the one we settled on used the **unpaid balance method** while most other cards used the **average daily balance method**. Let me take Brian’s example and apply the **unpaid balance method**.

1. First, a **finance charge** of $500 \times 0.22 \div 12 \approx 9.17$ is added because the annual rate of 22% is prorated to one month at $22\% \div 12 \approx 1.83\%$.
2. Second, one finds the total purchases of $500 + 300 = 800$ to add.
3. Third, one finds the total of returns and payments, i.e. $100 + 200 = 300$ to subtract from the total.
4. Finally, one computes the balance: $9.17 + 800 - 300 = 509.17$.

Notice that interest is charged only on the *unpaid* portion of the last month’s balance while this month’s purchases still receive free credit.” David took another bite of his mom’s homemade cookie and continued explaining, “However, with the **average daily balance method**,

A **finance charge** is the interest on a loan.

Simple interest formula is used to calculate the balance in the **unpaid balance method**: New balance in the current month is

1. the previous month’s balance P plus
2. finance charge ($I = Prt$) plus
3. purchases minus
4. returns minus
5. payments.

For only one month, $t = 1/12$ with r as the annual rate. The **finance charge** for the next month is the new balance multiplied by rt . ◀



interest is charged on all future purchases the moment a statement carries a positive balance from the previous month. In particular, the dates of each purchase, return, and payment are recorded for averaging the cost of financing. This means that the more you push expensive purchases to the end of the billing cycle, the less interest you need to pay. Let us suppose that we attach the schedule to the activities:

1. A balance of \$500 was carried over from the previous month.
2. On the 10th day, \$300 was spent.
3. On the 15th day, \$100 was returned.
4. On the 25th day, \$200 was paid to the credit card company.

To find the average daily balance, we have to track the number of days on each new balance using 30 days a month: see Table 4.2. Knowing that on average, the account owes the credit card company

Activities	Balance	Lasting days	Weighted balance
Day 1	\$500	1, 2, 3, 4, 5, 6, 7, 8, 9	$\frac{9}{30} \times 500$
Day 10	\$300 + \$500	10, 11, 12, 13, 14	$\frac{5}{30} \times 800$
Day 15	\$800 – \$100	15, 16, 17, 18, 19, 20, 21, 22, 23, 24	$\frac{10}{30} \times 700$
Day 25	\$700 – \$200	25, 26, 27, 28, 29, 30 total averaged	$\frac{6}{30} \times 500$ ≈ 616.67

Table 4.2: Average daily balance

\$616.67 every day for 30 days (or a month), the finance charge is then

$$616.67 \times \frac{0.22}{365} \times 30 \approx 11.15,$$

The **average daily balance method** tracks the duration of the balance during a billing cycle to find the average balance for the month. If a balance of \$400 lasts 10 days, then it is weighted 10/30 using 30 days a month. The sum of weighted balance is the average daily balance.

The **finance charge** from the **average daily balance method** is

$$I = Prt = Pr \frac{30}{365}.$$

Average daily balance finance charge calculation is worked out. ◀

Some **credit cards** marketed to students should be studied carefully, especially for interest rates and conditions.

In addition to charging merchants a service fee, credit card companies make money by keeping credit card users in debt.

Even though an interest rate of 60% and above is considered **usury** in Ontario, some payday loan companies charge \$15 for every \$100 over a two week period, equivalent to 390% annually.

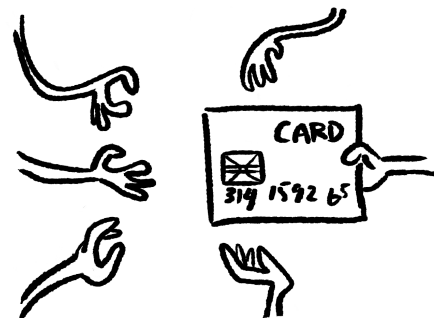
instead of \$9.17 from the unpaid balance method.”

“Oh, my goodness! This explains why I kept seeing interest charges even after I thought I paid all the interest for the forgotten statement. Nobody taught us that not paying down everything would make all our future purchases so much more expensive! Thank you, David, for enlightening me,” exclaimed Janette.

“I only have a small thing to add,” continued David, “about Brian’s debit card spending instead of credit cards. My parents have both and choose when to use which: large purchases and travel arrangements on credit cards, small grocery shopping, debit or cash, since their credit cards have theft and travel insurance as added benefits. Some corner stores are charged a service fee when clients pay with credit cards, so my parents use cash. Some shops do not accept credit cards, so a debit card is the only cashless option. We need to remember though that the moment money leaves an account from a debit card purchase, it is not easy to get it back, like in the case of an **identity theft** or a scam as opposed to a credit card theft where the company would refund the victim in most cases. The most relevant point to you is that when you need a mortgage for a house later in life, there is not much credit built without ever owning a credit card. Brian, in your particular situation, why not apply for a credit card and set up an automatic payment for a recurring fixed expense (like a cell phone bill) from the credit card and match it with an automatic transfer from your checking account to pay the credit card company, then put the credit card in a safe. I am sure with practise, you would be able to optimize on the benefits of the credit card without falling into its traps.”

“Traps, you say!” Brian suddenly recalled a scam his parents fell for. “Once my parents were invited to a concert in Downtown Vancouver where parking was expensive. They were told by their friends that the concert would end by 5pm, then paid for parking accordingly; however, due to the sickness of a performer, the performance started half an hour late. While standing in line, they saw that street parking was available, just needed a **PAYBYPHONE** app. My mother googled PAYBYPHONE on

When giving credit card information online, one must first check that the website is **secure**. Even if the website contains the symbol of a lock to the left of the web address, it may not be secure. In the case of getting the right **PAYBYPHONE** app, one needs to go to the City of Vancouver’s official website or search for *Parking in Vancouver*. Eventually, one arrives at **How to pay for parking**. Signing up for automatic payment programs needs peace of mind and focus, best done at home after a thorough discussion with family members.



All credit card owners should learn *How to Protect Yourself Against Credit Card Fraud in Canada* from Young and Thrifty.

her phone and clicked on a PAYBYPHONE free app link which took her to a site where she signed up and gave my father's credit card number. When my father returned from the restroom, she asked him to complete the application. Without checking, my father completed the application and got to a site totally different from what they expected. Needless to say, they did not enjoy the concert in peace. The first chance my father had, he checked the **top-level domain** name of the website, *.co*, to locate that the company was based in Columbia. Upon checking his credit card record, my father also found an entry from a company in Pennsylvania between the first parking fee and the subsequent street parking extension for the delayed concert. My sister googled *tudum.co* to show my parents that the website was 100% fraudulent, and taught my parents how to lock their credit card. The next morning when my father called his credit card company to cancel the card, he learned that the same company attempted to charge him the second time while the card was locked overnight. Even though my parents checked the entries of all their credit cards religiously on their monthly statements, for the next many months, my parents went on that scammed credit card's website many times a month to look for suspicious activities. That was another reason for my credit card phobia. It is so much worse than losing your wallet of cash!"

Line of credit—a slow descent

"That is an idea! I will think about the pros and cons of having a credit card. Thank you for telling us so much about what you know, David." Brian began thinking about something he overheard his parents discuss one day. "How about having a **line of credit**?"

Janette quietly said, "I remember my mother considering that when she was unable to work for some months due to a serious illness. All three of us were still living at home, but not making much to help with expenses. It is true that the interest rate is *a lot* lower than credit cards, but for large amounts, a **collateral** is necessary, like a house. When my

Advice on how to manage **revolving credit balance** and **interests** can be found from Capital One.

Germany and Japan both lost the war (WWII) in 1945. Both countries rebuilt successfully with big and strong industries and economies. Why would the **general government gross debt as a percentage of GDP** be so different: 69.8% for Germany and 252.3% for Japan? Did debt cancellation play a role in Germany's lower percentage? To understand the crippling effect of debt on Third World countries, learn about the U2 singer, Bono's **campaign**.

The formula for the monthly **instalment** of a loan P using the **add-on interest method** at an annual rate of r , over t years equivalent to $12t$ (or $n = 12t$) months is $P(1 + rt)/n$. <

An **annuity** is a **financial product** that receives a series of fixed deposits in regular intervals to save for the future. The **future value of an annuity** is the total amount in the account after all payments are made.

Depositing \$6000 into a Tax Free Savings Account (TFSA) every year on January 1st for 40 years earning 3% **APR** is saving in an **annuity**. However, one can buy an **annuity** with a lump sum to receive future income as a fixed amount paid out in regular intervals. This form of an **annuity** is a type of insurance often for an income stream after retirement.

grandparents learned about this, they stepped in to be her bank instead of letting us lose housing security to get the line of credit. The moment my mom got better, she paid back that loan so fast with all of us taking on extra jobs after school. I don't ever want to be under that type of financial stress if I can help it."

David added thoughtfully, "My grandma took on debt once for a fur coat before there was anything like a line of credit. Oh, no, not a fancy one, just something thick enough to help her survive the winter in Winnipeg where she had her first job and was too poor to buy a car. She was shocked that the store sent a **debt collector** after her a month later; and my, did she pay off that fur coat fast. Ever since, my grandma has tried to help the poor whenever possible, like organizing groups of volunteers to feed the homeless, donating to charity organizations, or collecting clothing for those in transition houses. It is from her that members of my family learn to live simply and well." David took a piece of Janette's cut fruit and continued, "I guess depending on a line of credit long term is analogous to our society's attitude towards the environment—over consumption is never sustainable. Just look at the deficit carried by countries of the **G20**, the more affluent the country, usually the greater the national deficit and the greater the **energy consumption**. I really hope that our generation will do a better job from the environment to the national debt to income inequality. Unfortunately, Covid raised our **deficit** from 1.8% of GDP in 2019 to 14.8% of GDP in 2020. It will take us a long time to pay off this debt."

"Well, David, we can look at our government's budgets for the past years to compare how much more interest on debt the government would need to pay as a percentage of the budget. A pie chart is really helpful here, say of the \$226.1 billion in **statutory authorities**, our government estimates \$16.17 billion for interest on unmatured debt for 2021–2022. Instead of a pie chart, when there are many items, one can also use a square partitioned into rectangles whose areas correspond to items of the list proportionally." Janette was busy scrolling through Government of Canada's websites to find infographics from the Department of Finance.

Comparison of large numbers can be inconclusive without the right approach. For example, if one wishes to compare the population and military expenditure of Canada and Germany in 2020–2021, listing the numbers is the first step but not helpful: Population of Canada, 38.01 million vs. 83.24 million in Germany; Military expenditure of Canada, 23.3 billion Canadian dollars (CAD) vs. 53 billion Euros (or 75.47 billion CAD) in Germany Which country spent more on the military? Just giving the larger number does not provide the whole picture. Instead, one can ask the expenditure per capita, or use relative comparison to find that Canada spent about \$613 per person whereas Germany spent around \$906 per person, almost 50% more than what Canada spent.

On the other extreme, the United States had 329.5 million in population while spending 766.58 billion USD (978.75 billion CAD) on defence, that is, \$2970 per person.

World military expenditure as a percentage of GDP tracks the spending of many countries from 1960 to 2020. One can select the countries of interest to compare their graphs from the World Bank's Data Bank. Despite being swamped by all the bad news from the mass media, the graph for World military expenditure (% of GDP) showed a downward trend for the last 60 years, only a small number of countries had an upward trend, like China and Russia. At the time of writing, Russia had more than 150 thousand troops amassed at its border with Ukraine, and China continued its routine military exercises over Taiwanese sea and air space.

The **IMF** prepared a video explaining under which con-

Mortgage or rent—saving or spending

“Thinking about such large numbers gives me anxiety attacks.” Brian wanted to shift the focus back to loans of a practical size. “I can tell you how the seller of the motorcycle computed monthly payments for me on the **instalment loan** the company issued. He gave me a 15 % loan on \$10 000 over 4 years, so my monthly payment was

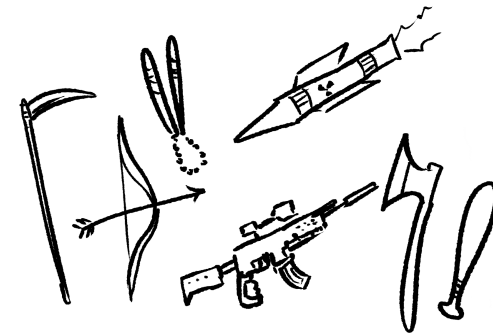
$$\frac{10\,000 + 10\,000 \times 0.15 \times 4}{4 \times 12} \approx 333.33.$$

Now I understand lending rate, borrowing rate, and my savings account interest rate, 15 % was crazy high when I signed the loan agreement. Then when I found the future value of $48 \times 333.33 = 16\,000$, I realized how much my grandmother helped me by forcing me to return the motorcycle. Who would have guessed a \$6000 interest on a \$10 000 purchase? Thanks to both of you, I am going to do all my calculations of **finance charges** from now on.”

“After we move out of home and before we have enough saved for a down payment, many of us need to rent.” David was thinking again. “Owning real estate often feels like having a gift that keeps on giving because the mortgage payments act like a forced savings program. That’s why my parents told us to join the band wagon as soon as we could afford the down payment. For those who can’t put enough aside for a down payment, their rent just disappears as expenses. No wonder the number of the working poor has grown in the last four decades.”

“The environmental movement has helped me feel less self-conscious about being poor.” Janette echoed David’s concern. “My family usually receives hand-me-downs from friends and the extended family for all non-perishables. Now, it is no longer fashionable to follow fast fashion; in fact, visiting second-hand stores is encouraged because of how harmful the fashion industry is to our environment. Regarding education, whatever I am interested in learning, I can probably find a **free course** on that topic. Still, all of us are here at Capilano University because having interactions

conditions it would lend money to countries undergoing sovereign debt restructuring.



An **amortized loan** like a student loan or a mortgage uses the same mathematical principle as an **annuity** (regular interval fixed saving) just in reverse—a big sum is borrowed to be paid back with a fixed amount in a predetermined regular interval.

The United Nations reported on bridging the **income gap**.

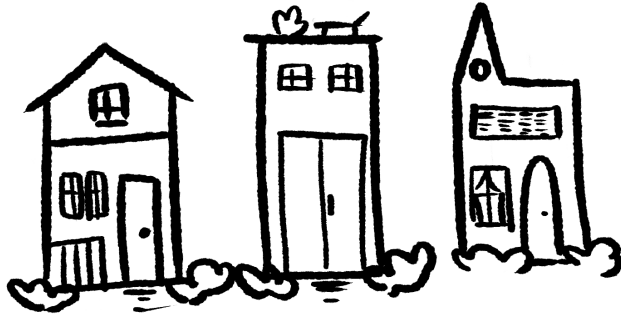
When one is in the mood for a treasure hunt, visiting **Share, Value Village**, or Thrift stores operated by the **Salvation Army** can be entertaining.

David Chilton taught Canadians how to become more empathetic of the needs of fellow Canadians in a **Conference from CBC**.

Many universities offer open courseware. For example,

1. **MIT's open course ware**,

with fellow students and professors in the same room is priceless. With so many free course ware online, I am sure that graduating from fancy universities will matter less and less. I expect to learn a lot of growing and foraging food this summer. Once education and food security are covered, we only need to find housing security in the future. Have you looked into [microhomes](#)?”



Brian, David, and Janette started researching tiny houses, designing green homes, checking the cost and affordability as they went.

4.4 UNINTENTIONAL GIVING

The next time Brian, David, and Janette gathered at the Nomads Club, Janette burst out with lots of questions. “I have been thinking a lot about having a smaller footprint on the environment and curbing my impulse buying tendencies, but our discussion last time only focused on money we control: to save, to spend, or to borrow. How about money total strangers make off us unbeknownst to us? Donating our money and talents to a cause is one thing, but giving unknowingly is another.”

“What do you mean?” Brian looked puzzled, “Merchants need to make a profit to support themselves.”

David jumped in to explain to Brian. “I am guessing that Janette is referring to social media and the advertising associated with that.”

2. [Harvard's free courses](#),
3. [Data Visualization Literacy from U of Toronto](#),
4. [free courses from U of Waterloo](#),
5. [languages from Cambridge University](#), and
6. [Philosophy from University of Oxford](#).

To access over five thousand courses, [Coursera](#) offers short modules.

Visit the first [Tiny House strata in Canada](#).

[Cosmetic revenue](#) fell 8% in 2020 due to the pandemic, i.e., 94.2 billion USD world wide, projected to increase to 177 billion USD by 2025. Does social media shape the ideal look and lead the consumers to an unnecessary expense?

A [timeline](#) of plastic surgery before 1914 showed the first applications.

Upon comparing [life expectancy in the world](#), one may think that in a not-so-distant future, rich people in rich countries could live forever if human cloning is to succeed. The Nobel Laureate in Literature, [Kazuo Ishiguro](#), started writing [Never Let Me Go](#) [17] in 1990, the same year the Human Genome Project was launched. Today, [human cloning](#) research continues for the sake of organ transplantation. Should human organs be cloned for sale?

Janette looked relieved. “Thank you, David, for knowing what I am talking about. How much is your attention worth to you? I noticed last time how the two of you paid attention to our discussion without even once checking your smart phones. But, when I am alone, unwittingly, my attention is constantly stolen by a notification-generating screen. Recently, I learned how people’s *attention* is monetized by an **algorithm**, so as I watch a video from YouTube, for example, my attention translates into money for the owner of the video and the hosting platform. As the artificial intelligence tracks the websites I visit, similar topics begin showing up in my feed. Before long, I am totally cocooned in my own thoughts and beliefs, missing out on thought-provoking growth. I don’t want to sell my attention to social media at the same time undermining our **liberal democracy**. Help!”

“I never thought of it that way. Are you saying that we are selling our souls to the social media?” Brian wondered aloud.

“Every new tool or technology comes with its own pros and cons.” David calmly launched into what he heard his parents talk about before. “The Chinese invented fireworks and the Europeans made cannons out of them to colonize China. Plastic surgery made great strides during the First World War to help wounded soldiers, but now rich countries have more plastic surgeons serving people who are not ill than those who are sick. I am sure you can think of many other examples like these. The Internet is a very new invention, developing faster than we can figure out how to use it properly. Regarding selling your attention unintentionally, my parents told us never to scroll on the infinite feed because artificial intelligence could track the time we spent on a page and deduce what we were interested in. Instead, whatever we want to know, just search directly or visit reliable sources for information, hopefully sources without advertisements.”

“David, you think in very grand terms. In my little world as a student, the screen sucking up all my attention not only affects my relationship with others, but also has direct implications on my studies, and consequently, my grades. Remember how Brian felt that he was



Does one have any control on what the Internet knows about oneself? Perhaps, **George Orwell’s 1984 [23]** is still **relevant** more than 70 years after it was published.

working too much to get good grades at university? When I hold a scholarship, I have a lot of pressure to keep my grades above a certain level to continue receiving the scholarship. Now that I know about our *Big Brother AI* watching me, it will motivate me to be more selective on where I put my attention.” Janette reached for another cookie from David’s box.

Brian, who had been enjoying both cookies and cut fruit, finally asked, “Do you like Sci-Fi movies? Of all the movies I have seen, lots of them involved artificial intelligence one way or the other, mostly in warfare. From what David just said, I am very worried about an impending **AI arms race**. Once a robot is programmed to kill humans, is anyone safe? Is the global community talking about it?”

And so David and Janette promised Brian that the three of them would look into the development of artificial intelligence in the military and find ways of bringing the discussion on AI arms race to the forefront of the political discourse.

4.5 EXERCISES

Fraction, decimal, and percent

4.1 Convert each fraction to decimal by expanding the denominator to a power of 10 following the first example. Sometimes, reducing the given fraction first is necessary.

a. $\frac{1}{2} = \frac{5}{10} = 0.5$

b. $\frac{4}{5} = \frac{\quad}{10} =$

c. $\frac{13}{52} = \frac{1}{4} = \frac{\quad}{100} =$

d. $\frac{3}{8} = \frac{\quad}{1000} =$

The video live streamed on Dec 7, 2021 discussed **international artificial intelligence** developments, implementations, and an urgency for treaties and standards.

According to **David van Weel**, the Assistant Secretary General for Emerging Security Challenges in the North Atlantic Treaty Organization (NATO), de-weaponizing or de-militarizing artificial intelligence may not serve humans best; for example, AI guided weapons might minimize both civilian casualties and infrastructural damages more efficiently than conventional weapons. To learn more, please read **Why Our Values Should Drive Our Technology Choices** and **An Artificial Intelligence Strategy for NATO** published in NATO Review.

A **report** prepared by **American Enterprise Institute**’s senior fellow, Elisabeth Braw, who chaired the panel discussion on Artificial intelligence: *The risks posed by the current lack of standards* addressed many pressing issues awaiting global cooperation.

A calculator helps fraction conversion to decimals. For example, for $13/52$, enter $13 \div 52$.

e. $\frac{4}{50} = \frac{\quad}{100} =$

f. $\frac{63}{84} = \frac{\quad}{\quad} =$

g. $\frac{1}{20} = \frac{\quad}{100} =$

h. $\frac{1}{40} = \frac{\quad}{1000} =$

i. $\frac{3}{50} = \frac{\quad}{100} =$

j. $\frac{7}{25} = \frac{\quad}{100} =$

k. $\frac{1}{80} = \frac{\quad}{10000} =$

l. $\frac{1}{200} = \frac{\quad}{1000} =$

m. $\frac{6}{500} = \frac{\quad}{\quad} =$

n. $\frac{8}{250} = \frac{\quad}{\quad} =$

o. $\frac{35}{400} = \frac{\quad}{\quad} =$

The prime factorization of 10 is $10 = 2 \times 5$ where both 2 and 5 are prime numbers. When one sees 2 in the denominator, expanding the fraction by 5 leads to 10 in the denominator; similarly, a 5 in the denominator needs 2 to expand the fraction to 10 as the new denominator. Using the same reasoning, $100 = 2^2 \times 5^2$, so one sees that a $2^2 = 4$ in the denominator needs $5^2 = 25$ to expand it to 100; similarly, $1000 = 2^3 \times 5^3$, so one pairs $2^3 = 8$ with $5^3 = 125$.

4.2 Compute each of the following.

a. Find 50% of 350.

b. Find 45% of 1000.

c. Find 36% of 50.

- d. Find 30% of 150.
- e. Find 30% of 15.
- f. What % of 150 is 30?
- g. What % of 250 is 5?
- h. What % of 50 is 30?
- i. What % of 500 is 30?
- j. What % of 15 is 30?
- k. 20% of x is 30. Find x .
- l. 20% of x is 3. Find x .
- m. 2% of x is 30. Find x .
- n. 200% of x is 30. Find x .
- o. 12% of x is 12. Find x .
- p. 80% of x is 32. Find x .

4.3 Rewrite each percent as a decimal. a. 29%; b. 58.2%; c. 7.5%;
d. 30%; e. 125.5%; f. 0.25%; g. 0.007%

4.4 Rewrite each decimal as a percent. a. 0.635; b. 0.02; c. 0.45;
d. 3.7; e. 43; f. 2.015; g. 0.0089; h. 0.333 33...

4.5 Fill in the blanks in Table 4.3 on the following page.

4.6 Answer the following.

- a. Find 175% of 139 to the nearest whole number

Fraction	Decimal	Percent	Fraction	Decimal	Percent
$\frac{2}{5}$					$3\frac{1}{8}\%$
	0.06			2.5	
		40%	$1\frac{1}{3}$		
	0.375		$\frac{1}{6}$		
$\frac{3}{200}$			$2\frac{5}{9}$		
	0.3		$\frac{2}{7}$		
		5%	$\frac{3}{11}$		
	0.08		$4\frac{5}{6}$		
		25%	$1\frac{4}{7}$		
$\frac{5}{8}$			$\frac{1}{99}$		

Table 4.3: Fractions, decimals, and percent

- b. How much is twenty fifteenths of 40 %?
- c. How much is 4 % of three elevenths?
- d. If 160 % of a number is 29, find the number exactly as a fraction.
- e. If 56 % of a number is 53 % of 24, find the number exactly as a fraction.
- f. How much is 189 increased by 77 %?
- g. Decrease 190 by 113 %.
- h. Decrease 176 to 83 %.
- i. If a number increased by 7 % is 41, find the number.
- j. If a number decreased by 28 % is 132, find the number.
- k. If a number decreased to 31 % is 12, find the number.
- l. Eighty-nine is which percentage of eighty-six?
- m. Sixty-four is what percent more than fifty-eight?
- n. Which percentage of 210 equals 60 % of 42?

The final sale price is the price before taxes after all the discounts.

4.7 On Brian's motorcycle written test, 112 passed and 81 did not. Find the passing rate in percent.

4.8 On David's written driving test, 33 passed and 28 % did not. How many took that test? [IM]

4.9 Janette bought a bike helmet ticketed at \$30.94 in the Bookstore of Capilano University. If the provincial sales tax (PST) rate was 7 % and the Goods and Services tax (GST) rate was 5 %, find the total cost in dollars.

4.10 A North Vancouver Sports offered a 15% discount on a tennis racket that originally sold for \$61.44. The new price was then further reduced by 50%. Find the final sale price.

4.11 Janette's mom bought a \$188 closet in Dandesign. If she paid \$12.22 in sales tax, what was the tax rate?

4.12 David's father bought a \$319 sofa table. If he paid 9.0% in sales tax, how much did he pay in taxes? [IM]

4.13 Brian's parents bought a \$435 entertainment unit. If they paid 12.0% in sales tax, how much did they pay including taxes?

4.14 David's parents paid \$512.72 for a granite garden bench (including taxes). If they paid \$70.72 in sales tax, what was the tax rate? [IM]

4.15 Janette's grandma paid \$597.38 for a fabric sofa (including taxes). If she paid 12.5% in sales tax, how much did she pay in taxes?

4.16 Janette's grandpa paid \$243.89 for an oak bookcase (including taxes). If he paid 6.5% in sales tax, what was the ticket price before taxes? [IM]

4.17 Janette's older brother bought a \$275 sofa at a "no sales tax"-sale; that is, he paid \$275 including taxes for his sofa. Find the discount rate that Janette's older brother enjoyed if the tax rate was 4.0%. [IM]

4.18 On Wednesday, David's financial advisor bought some tech shares for \$7010.00. By the following Monday, the stock price had increased by 0.9%. However, by Thursday, the price dropped 9.7% based on Monday's price. Find the value of the financial advisor's shares on Thursday. [IM]

Even with the help of software like **Turbotax**, filing a Canadian income tax return may seem daunting. Some young adults making little income often do not file tax returns because they think that they don't make enough to pay any tax. However, it is especially important to learn how to complete a **basic tax return** in this case because one may receive monetary support for low income individuals from the Canadian Revenue Agency.

4.19 On Tuesday, David's mom bought some mutual funds. By Wednesday, the fund price had decreased by 3.9%. However, by Friday, the price regained 2.1% based on Wednesday's price. She immediately sold her funds for \$14 721.53. How much did David's mom pay for her funds on Tuesday?

4.20 On Friday, David bought some shares. By the following Tuesday, the stock price had decreased by 4.2%, and by Wednesday it had dropped a further 4.5% based on Tuesday's price. David immediately sold his shares for \$18 301.48. How much money did David lose on his transactions? [IM]

4.21 Janette answered eleven of twelve questions on a test and got six of them right. What percentage of the questions did Janette skip?

4.22 Janette answered eleven of twelve questions on a test and got nine of them right. What was Janette's success rate? [IM]

4.23 In playing a video game, Brian shot down eight of the fourteen monsters with 49 shots. One successful shot destroys one monster. What percentage of Brian's shots hit the monsters? [IM]

4.24 When 5580 people attended a rock concert, the stadium was 93.0% full. Find the capacity of the stadium.

4.25 In each of four successive years, the cost of living increased by 1.6%. Find the percentage increase over the four years.

4.26 David's parents had a circular patio in their garden. They decided to enlarge it by increasing the radius by 8.9%. What was the percentage increase in area?

Success rate is the number of successful tries out of the total number of tries.

The **cost of living** is based on the **consumer price index**.

The area of a circle is $A = \pi r^2$ where r is the radius of the circle.

The area of a rectangle is $A = l \times w$ where l is the length and w is the width.

4.27 Jannet has a rectangular vegetable garden in her yard. She decides to enlarge it by increasing the length by 3.6% and the width by 5.6%. What is the percentage increase in area? [IM]

Currency conversion

4.28 The British pound was at one time equivalent in value to 46.84 Taiwanese dollars. Find the value of one Taiwanese dollar in British currency at that time.

4.29 The Japanese yen was at one time equivalent in value to 0.2676 Taiwanese dollars. At the same time, the Japanese yen was equivalent in value to 0.007346 Swiss francs. Find the value of one Taiwanese dollar in Swiss currency at that time. [IM]

4.30 The Canadian dollar was at one time equivalent in value to 0.5518 British pounds. At the same time, the British pound was equivalent in value to 14.35 South African rands. Find the value of one South African rand in Canadian currency at that time. [IM]

4.31 Before October 2019, 1500 Lebanese pounds was equivalent to 1 USD. In January of 2022, 30 000 pounds was equivalent to 1 USD. How much value did the Lebanese pound lose since October 2019 (in %)?

4.32 **Hyperinflation in Venezuela** started in 2016. If the annual inflation rate was 800% in 2016, 4000% in 2017, 1 700 000% in 2018, how much was a one-million Venezuelan bolivar note from the beginning of 2016 worth by the end of 2018?

4.33 In March of 2017, one Bitcoin was worth around 1300 Canadian dollars. In February of 2022, one Bitcoin was worth over 47 000 CAD. What was the average annual rate of increase over five years? Apply the compound interest formula. [IM]

Lebanese currency crisis is another destabilizing factor in the Middle East.

The BBC collected **nine charts** from different sources to explain the Venezuelan hyperinflation crisis.

Inflation rates are computed with a **continuously compounding** model; therefore, the future value formula using an inflation rate of r is $A = Pe^{rt}$. For the sake of simplicity, we use simple interest year after year to find the value of one-million Venezuelan bolivar note after three years.

Learn about Cryptocurrency from [Google Finance](#).

Interest

4.34 Find the interest earned by a savings account with \$6000 at a rate of 2% after one year assuming no interest compounded during the year.

4.35 Fill in the blanks labelled a), b), c), d), and e) in Table 4.4 assuming simple interest. [IM]

Interest =	Principal	\times rate \times	t (years)
a)	\$4000	1%	1
b)	\$2400	2.5%	3
\$500	c)	$4\frac{1}{4}\%$	2
\$250	\$80 000	d)	1.5
\$3000	\$600 000	$5\frac{3}{8}\%$	e)

Table 4.4: Interest, principal, rate, and time

4.36 The yearly interest paid on a loan of \$4700 is \$376.00. Find the annual rate of interest expressed as a percent.

4.37 The yearly interest paid on a loan of \$9700 is \$1067.00. Applying the same annual rate of interest, how much interest should one pay on a loan of \$3000 for a month? [IM]

4.38 David's uncle borrowed \$19 000 and a year later paid back the loan plus interest with a cheque of \$21 660.00. Find the annual rate of interest paid in percent for the loan. [IM]

4.39 Fill in the blanks labelled a), b), c), d), and e) in Table 4.5 on the following page assuming simple interest.

4.40 How much would Brian's grandparents need to put in a GIC earning 1.5% to have a million dollars in 30 years? Use simple interest. [IM]

To solve for a specific variable in the simple/compound interest formulas, one needs inverse operations. For example, addition and subtraction are inverse operations of each other, similarly multiplication and division, raising an expression to the n th power and to the $1/n$ th power (equivalently, taking the n th radical).

From the simple interest formula: $I = Prt$, solve for each stated variable.

- To solve for P , divide both sides of the equation by rt to get $P = I/(rt)$.
- To solve for r , divide both sides of the equation by Pt to get $r = I/(Pt)$.
- To solve for t , divide both sides of the equation by Pr to get $t = I/(Pr)$. \triangleleft

Future Value	Principal	rate	t (years)
a)	\$7800	$1\frac{5}{8}\%$	3.5
b)	\$10 500	2.75%	2
\$500 500	c)	$5\frac{3}{4}\%$	10.25
\$250 000	\$80 000	d)	15
\$45 000	\$9000	$8\frac{3}{8}\%$	e)

Table 4.5: Future value, principal, rate, and time

4.41 A few years ago, Janette considered buying a \$5000 GIC with different offers:

- 2% compounded semi-annually, or
- 1.8% compounded monthly, or
- 1.78% compounded weekly, or
- 1.75% compounded daily.

Which option earned the most interest for Janette? Does the answer depend on the dollar amount of the GIC or whether the GIC was for one year or five years?

4.42 David considered buying a \$3000 GIC with different offers from his parents' financial advisor:

- 0.5% compounded biweekly, or
- 0.45% compounded monthly, or
- 0.48% compounded quarterly, or
- 0.47% compounded every minute (using 365 days a year to get $365 \times 24 \times 60 = 525\,600$ minutes a year).

For the future value equation using simple interest: $A = P(1 + rt)$, solve for each stated variable.

- To solve for P , divide both sides of the equation by $(1 + rt)$ to get $P = A/(1 + rt)$. The parentheses, $()$, must be entered in the calculator to avoid getting the answer $A + rt$.
- To solve for r , divide both sides of the equation by P first to get $A/P = 1 + rt$. To get only rt on one side of the equation, subtract 1 on both sides: $rt = \frac{A}{P} - 1$. Finally, divide both sides by t to get $r = (\frac{A}{P} - 1)/t$. Again, the parentheses must be present to avoid the wrong answer of $\frac{A}{P} - \frac{1}{t}$ where $\frac{A}{P}$ is not divided by t .
- To solve for t , follow the steps for solving for r until the last step where one divides both sides by r to get $t = (\frac{A}{P} - 1)/r$. \triangleleft

Since Canadian currency's smallest unit is a cent, one rounds money answers to the nearest cent.

Most banks use 30 days for a month, 90 days for a quarter (of a year), 180 days for half a year, and 360 days for a year in the compound interest formula. However, there is no difference when one uses fractional years to express a month ($1/12$), a quarter, $1/4$, or half a year, $1/2$. The only difference is in daily compounding where most banks use

$$A = P(1 + r\frac{1}{360})^{360t}$$

Which option earned the most interest for David? Does the answer depend on the dollar amount of the GIC or how long the GIC was kept? Does one need to compute for every item on the list to find the greatest effective rate? [IM]

4.43 Brian overheard his grandparents mention that they earned \$5237 in interest on their 5-year GIC compounded monthly at 2.25%. How much was his grandparents' principal for this GIC?

4.44 Janette's grandparents earned \$2579 in interest on their 4-year GIC compounded weekly at 1.75%. How much was the principal for their GIC? [IM]

4.45 David remembered a bank offering students GICs in increments of \$1000 for 10 years worth \$1100 when matured for every \$1000 dollars invested. Suppose the GIC was compounded daily, what was the rate offered?

4.46 Bank of Mom and Pop offered children under 18 3-year GICs worth \$1500 when matured for every \$1200 dollars invested. Suppose the GIC was compounded quarterly, what was the rate Bank of Mom and Pop offered? [IM]

4.47 David wanted to see one of his savings accounts grow to \$50 000 from an initial deposit of \$45 000 compounded monthly at 1%. How long would David need to wait?

4.48 Janette's mom put aside \$15 000 in a savings account some years ago earning 1.5% compounded biweekly. How long would it take to achieve \$18 000 in that account? [IM]

4.49 A few years ago, Janette's grandpa purchased some structured notes upon the recommendation of his financial advisor. The structured

using **ordinary interest**, instead of

$$A = P\left(1 + r\frac{1}{365}\right)^{365t}$$

using **exact interest**. We use exact interest in daily compounding questions. <

Recall from algebra that $1.1189P - P = 0.1189P$; similar to $3P - 2P = 1P$, or just P .

Royal Bank **Structured Notes** offer some products which guarantee the return of the principal with higher rates than GICs, but of course, with some risks.

notes averaged a 6% growth compounded annually and would mature in 7 years. How much would Janette's grandpa expect to have if he purchased a \$65 000 note? [IM]

4.50 Brian's second brother, Patrick, bought some art from an NFT market place for \$100 in 2021. After a year, in order to raise money for an emergency, he sold it for \$85. What was the rate of his loss?

4.51 Brian's sixth brother, Colin, invested in a 0th generation **CryptoKitties** for \$10 in 2018. By 2022, the value had increased to \$50 000. Using the compound interest formula compounded annually, find the annual rate for this investment.

Credit cards and loans

4.52 In 2020, the Canadian national debt reached \$942.5 billion with a population of 38.01 million. Shared equally, how much debt did each Canadian shoulder?

4.53 In 2020, the Canadian household debt reached \$2.1 trillion with a population of 38.01 million. On average, how much household debt did each Canadian shoulder? [IM]

4.54 *Student first* credit card charges 18% on unpaid balance. Using simple interest, how much would it cost in finance charge to borrow \$800 for 40 days? Note: Credit card companies use 365 days a year.

4.55 If *Canadian Express* credit card charges 22% on unpaid balance, how much would it cost in finance charge to leave \$570 unpaid past the due date for 90 days? [IM]

4.56 Janette's mom saw a \$28 888 car advertised at \$888 per month for 5 years. Find the finance charge and the rate on the car.

Non-fungible tokens, or **NFT's**, started in 2014 and gained prominence in 2021 when trading increased both in volume and dollar total.

The Canadian national debt is the debt of the federal government; the Canadian household debt is the debt Canadian citizens incur like mortgages, loans, credit card debts, and so on.

To put Canadian debt into perspective, consider a household of five people, two parents with three children; each person is allotted the combined share of national and household debts resulting in around \$400 000 of debt for the family of five. But the difference between a sovereign debt and a personal debt is that the nation is in control of printing its own money while a private citizen can't print money legally. To understand more, please visit the first two undergraduate courses in economics from MIT's open course ware.

Using the credit card to purchase goods which are not paid off at the end of the statement period is different from using the credit card to draw cash, called cash advance, which is charged at a higher rate than the advertised rate.

4.57 David's grandmother took a taxi from the airport back home after one of her volunteer trips. She used her credit card to pay the taxi driver adding a 12% tip. When her credit card's monthly statement came, the taxi company charged her \$74.52. How much did the taxi ride cost before a 5% GST and the tip?

4.58 Janette's friend, Annika, purchased a bakery and financed \$85 000 at \$1137.28 per month for 7 years. Assuming the add-on interest method, what was the amount of interest paid?

4.59 Brian started a senior's service and took out a loan of \$50 000 for a van with a wheel chair lift. If he paid \$954.75 per month for 5 years, what was the annual rate for the loan? [IM]

4.60 David's friend, Josh, graduated from university with a student loan of \$30 000 at 4% to be paid off in 5 years.

- a. What would the monthly instalment be using the add-on interest method?
- b. How much was the total finance charge on the loan?
- c. After the first instalment, what was the balance on the loan?

4.61 Brian's friend, Tom, wanted Brian's advise on how to pay off a credit card debt of \$2000 at 19%, if Tom wanted to be debt free in a year. Use the add-on interest method.

- a. What would the monthly instalment be?
- b. How much was the total finance charge on the credit card debt?
- c. After the first instalment, what was the balance on the debt?

[IM]

The most common mistake in reversing a total bill of \$74.52 including a 5% GST and a 12% tip to find the price without GST and the tip is

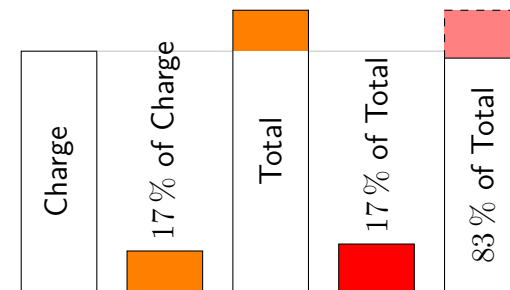
$$74.52 \times (0.05 + 0.12) = 12.6684,$$

so

$$\text{wrong original charge} = 74.52 - 12.67 = 61.85$$

Since both the GST and the tip are calculated based on a smaller charge, the mistake is an over-estimate of the taxes ($0.17 \cdot 1.17x = 0.1989x$, not $0.17x$ where x is the original charge before taxes and tip). Once subtracted, the number would be smaller than the original charge.

Represented as rectangles with red indicating wrong interpretations:



The charge was \$63.69 and the tax and tip was 17% of that, so $0.17 \times \$63.69 = \10.83 . The total bill was therefore $\$63.69 + \$10.83 = \$74.52$. Now 17% of \$74.52 is $0.17 \times \$74.52 = \12.67 , so subtracting 17% from \$74.52 yields $\$74.52 - \$12.67 = \$61.85$ which is \$1.84 too little. Note that the discrepancy is 17% of 17% of the

4.62 Janette took out an add-on interest loan from the Bank of Granny for \$1388 to buy a new computer. The loan would be paid back in 2 years with an annual interest rate of 3%.

- a. How much interest would she pay to Granny?
- b. What was her monthly payment?
- c. Find the effective annual interest rate for the last month of the loan.

[IM]

4.63 David wanted to build a greenhouse to grow vegetables all year round. He borrowed \$1800 from the Bank of Mom and Pop which charged him 1.5% using add-on interest method to compute his monthly payments.

- a. If David wanted to pay off the greenhouse in 36 months, what would his monthly payments be?
- b. If David put down \$300 of his own money, how much would the down payment reduce his monthly payments?
- c. If David wanted monthly payments of \$40 instead, how large should his down payment be for the greenhouse?

4.64 Brian wanted to train for the North Shore mountains search and rescue society and needed special gear. He borrowed \$600 from the Bank of Grandpa which charged him 2.25% using add-on interest method to compute his monthly payments.

- a. If Brian wanted to pay off the loan in 6 months, what would his monthly payments be?

original charge \$63.69, that is $0.17^2 \times \$63.69 = \1.84 . ◀

Note that the add-on interest method requires the same interest every month even though the principal decreases. The effective rate charged is thus more than the advertised rate.

Michael J. Sandel engaged in a [public debate at St. Paul's Cathedral](#) regarding his book, *What Money Can't Buy: The Moral Limits of Markets* [28].

- b. If Brian put down \$100 of his own money, how much would the down payment reduce his monthly payments?
- c. If Brian wanted monthly payments of \$88 instead, how large should his down payment be for the search and rescue gear?

[IM]

Predicting—Hamlet’s dilemma: To bet or not to bet

Graham (Ham) and Samantha (Sam) had three children, 10-year-old Graham Junior (Hamlet), 5-year-old Samantha Junior (Samlette), and an infant a few months old who often kept Sam up many times every night. For this reason, Ham tried his best to keep super energetic Hamlet busy with lots of games; in turn, Hamlet invented games to play with Samlette, including games for chores like assigning health and strength points to each type of clothing while hanging or taking down laundry. Winning was important to Ham, but not as important as having fun with others; therefore, he was always willing to let Samlette win to entice her to play with him. However, he loved playing with Ham the most because Hamlet knew that he could play as hard as he could to try to defeat his dad, and—in the rare occasion that Hamlet won, his victory was so much more memorable.

A typical morning unfolded in Ham’s family...

5.1 PROBABILITY TREE—HAMLET’S GAMES

Choosing one of two

“Wow!” exclaimed Hamlet, his first greeting to a new day every morning. “Rise and shine!” Hamlet yelled at the top of his lungs as he charged

Here are some common terms in probability:

1. An **experiment** is a process, like coin tossing or die rolling, that yields observable results.
2. The different results of an **experiment** are **outcomes**. For example, when the experiment is tossing one coin, the outcomes are a Head (H) or a Tail (T).
3. The collection or **set** of all possible **outcomes** for an experiment forms its **sample space**. The sample space for tossing one coin is $\{H, T\}$. The braces enclosing the outcomes are **set braces**. Order of outcomes does not matter in a set, namely, $\{H, T\} = \{T, H\}$.
4. An **event**, E , is a **subset** of the **sample space**. For example, the event that tossing a coin gives a Tail is $E = \{T\}$.



into his parents' bedroom followed by a kangaroo jump into the middle of the bed between his groggy, sleepy parents. "Hmm, should I hug Dad first or Mom first?" Hamlet gave his father a big hug and moved over to his mom for a bear hug and a kiss. "Should I wake Samlette first or Baby?"

"If your yelling did not wake Baby yet, please let Baby sleep a bit longer. How about having breakfast with me first before waking Samlette?" Dad was now fully awake to stop Hamlet in time from waking Baby. Reluctantly, Dad got out of his warm comforter, put on his bathrobe, and took Hamlet out to the kitchen to give Mom a little more sleep after a colicky night with Baby.

As usual, Dad poured some O's and raisins into a dinosaur bowl for Hamlet and some müsli for himself. Dad then poured Hamlet a glass of milk and put milk in his own müsli bowl while Hamlet sat impatiently waiting. "Would you like me to put down the dinosaur bowl first or the glass of milk?"

5. A **simple event** corresponds to only one outcome from the sample space.
6. Let E denote an event, then $P(E)$ denotes the **probability** or **chance** of the event E , and $0 \leq P(E) \leq 1$.
7. If $P(E) = 0$, E is an **impossible event** corresponding to the **empty set** (the set with no member) of the **sample space** for a **discrete** probability (with countable events, the type studied in this chapter) as opposed to a **continuous probability distribution**.
8. If $P(E) = 1$, E is a **certainty**, or a **certain event** corresponding to the entire sample space also in the **discrete** case. <

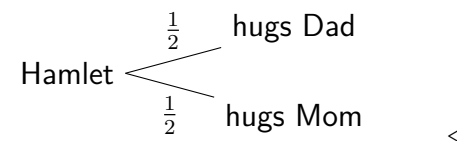


“The bowl! The bowl! I want to count how many of each kind I have today.” Hamlet reached into the bowl to start his daily routine of checking out this morning’s mix. “I will start with the O’s first.” Hamlet began lining up O’s on the dining table, five in a row. He counted 21 O’s. Next came the raisins, also five in a row to a total of 15. “Look how many of each I have today!” Hamlet called to Dad who was busy making tea.

Rolling two dice

“Let me show you a new game, Hamlet, before you start eating.” Dad went to get two dice, a brown die and a purple die. “Roll both dice. If the purple die shows a bigger number than the brown die, then purple wins and you eat one raisin; otherwise, take an O.” Dad was hoping

A **probability tree** is similar to a counting tree diagram except with a probability assigned to every edge of the tree. Suppose Hamlet *chooses randomly* between Mom and Dad with *equal probability*, then a **probability tree** with two edges will have $1/2$ assigned to each edge as its probability. This means $P(\text{Hamlet hugs his dad}) = 1/2 = P(\text{Hamlet hugs his mom})$. Note that $\frac{1}{2} + \frac{1}{2} = 1$. When a sample space contains only two equally likely outcomes, the probability of each outcome is $1/2$.





The same probability tree of two branches also works for the dinosaur bowl and glass of milk experiment proposed by Dad, if the choices are *equally likely*. However, waking Samlette or Baby first is not a random choice from the parents’ perspective for the sake of their self-preservation.

that 36 O's and raisins would take much longer to consume if Hamlet had to roll two dice before eating every single piece. Perhaps Samlette, Baby, and Mom could get more sleep while Hamlet was entertained. "We can draw what you might expect; that way, you can tick the box that shows your rolls." Dad got brown and purple pencil crayons and a piece of scrap paper to show Hamlet how to get started on the table.

Table 5.1: The Sample space for a pair of dice showing 36 outcomes

Hamlet was excited by the table of dice Dad drew for him and started counting the results corresponding to O's winning. "Hey, Dad! Do you know that 21 of these 36 pairs of dice have the brown die showing the same or more number of dots than the purple? I have exactly 21 O's and 15 raisins today. Did you do this on purpose?"

"No, Hamlet, just a coincidence because it is almost impossible to get every single pair of results after rolling two dice 36 times.¹ Try to throw some dice and see how often you get repeated results."

Hamlet got busy throwing dice while carefully ticking off boxes which matched the dots. "Hmm, I wonder how many pairs would show a total of 10 or more. Since the maximum total is 12 (= 6 + 6), it's easier for me to work from the largest or lower right hand corner: , ,

A six-sided cubic die (D6) shows 1, 2, 3, 4, 5, 6 as dots, a different number on each face. If a die is **fair**, then on average, when rolled, each of the 6 numbers has 1/6 probability of appearing as the top face. However, due to the **randomness** of the experiment, say, of throwing a die 10 000 times, one should not expect exactly 1/6 of 10 000 (which is not a whole number anyway) rolls for each of the six numbers.

Let X be a **random variable** from $\{1, 2, 3, 4, 5, 6\}$, meaning that X can take on any of the six values with some probability. Then the probability that a fair die, once rolled, results in a 3 is $P(X = 3) = 1/6$. This is a **theoretical probability**, also called an **a priori** model. ◀

¹The probability is slightly less than 0.000 000 000 035 %.

Dad, see this triangle I am drawing here containing all six pairs at the lower right hand corner of your picture.”

“Can you see that the slant side of your triangle has all pairs whose total is 10? If you change your question to finding pairs with a total of 3 or less, a similar triangle will box the pairs at the upper left hand corner of my table because of the symmetric pattern in and on the slant side. But, what if you want to find the number of odd totals, which pairs would be selected?” Dad felt really hopeful that his table of dice would keep Hamlet busy if the questions could keep coming.

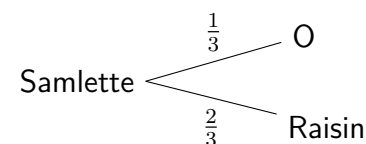
“Look at the checkerboard pattern, Dad! Starting from (1 + 1 = 2), an even total, when I move onto its neighbouring pairs, one of the dice just increases by one, so the total changed from even to odd. This reason passes onto the neighbouring pairs of the odd totals, and so on. It’s clear from the checkered pattern that half the pairs have even totals, and the other half, odd.” Hamlet felt very proud that he could explain the pattern to his Dad.

Drawing from a bag

After Hamlet finished his breakfast, Dad made Hamlet promise to enter Samlette’s room quietly before waking her. As Dad watched, Hamlet overdramatized his sneaking into Samlette’s room. However, the moment he was out of Dad’s sight, Hamlet hugged his sleeping sister tightly with a loud question, “Are you hungry? Let me help you.” Poor Samlette could not figure out whether to cry in protest for the disruption of a sweet dream or to jump up for joy for another day filled with endless games with Hamlet.

Instead of pouring O’s and raisins into his sister’s tiger bowl, Hamlet put one O and one raisin in a thick brown paper bag, shook the bag hard and asked, “Samlette, put your hand in the bag and take out one thing from the bag.”

If a bag contains an O and a raisin, and Samlette picks a piece out **at random**, then each has a probability of $1/2$ for being picked. On the other hand, if a bag contains an O and two raisins, also picked out **at random**, then an O has a probability of $1/3$ while a raisin, $2/3$ for being picked. Note that the total probability is $1/3 + 2/3 = 1$.



When an experiment has only two outcomes: **Success** and **Failure**, then it is a **binomial experiment**. The analysis of probabilities in games begins with the understanding of the simplest game: win or lose from the **binomial distributions** ◀.

Gerolamo Cardano was an Italian polymath who first systematically studied probability before **Pierre de Fermat** and **Blaise Pascal**.



“I love this game!” Samlette squealed with delight. Quickly, she put her little hand into the bag. “I got an O!” And she ate it. Hamlet put in another O, shook the bag. “Try again!” Hamlet offered.

“I got another O!” And she ate a second one.

Hamlet started thinking, “Hmm, if Samlette likes eating O’s today and keeps picking them out of the bag, there is no fun in this. I am going to put them in my hand, one each, and hide my hands behind my back to let her pick a hand.”

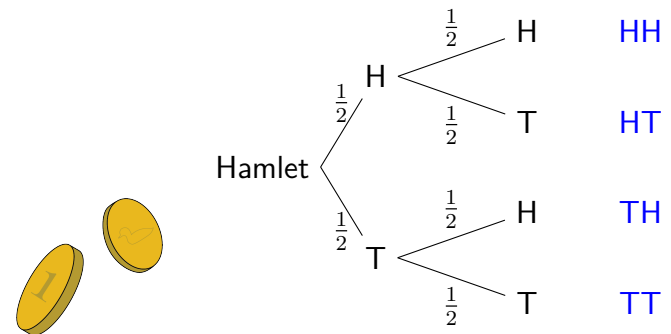
Before Hamlet was even done preparing, Samlette complained, “I don’t want to play this game because I can’t get O’s every time.”

“OK, OK, I will put an O and a raisin back in the brown bag again. But now, I am putting another raisin in the bag.” Hamlet wanted Samlette to have a greater chance of getting a raisin. Little did he know how sensitive Samlette’s little fingers were in picking out an O even with the presence of two raisins.

Samlette eagerly put her little hand into the brown bag. “Hooray! I

The **randomness** from flipping a coin is used to illustrate **Gambler’s Fallacy**. Anyone over 19 can **gamble in B.C.** However, unlike a bar where bar tenders and the establishment are held responsible if an already drunk patron continues being served alcohol and subsequently gets into an accident, many casinos are not accountable if they continue serving gamblers who have lost all possessions. The **responsible and problem gambling program** provides a 24-hour, toll-free gambling support line to counsel and educate those at risk. The Safety Council of Canada also provides **information and contact** for problem gamblers.

A tree diagram for tossing two fair coins: $1/4$ ($= \frac{1}{2} \times \frac{1}{2}$) probability for each outcome shown in **blue**. Because tossing one coin has no effect on the result of tossing the other coin, these two events are **independent**. In this case, getting the first *H* has a probability of $1/2$; subsequently, $1/2$ of the chance the second coin toss also ends with an *H*. Taking $1/2$ of $1/2$, one obtains a probability of $1/4$ for 2 Heads. The concept of **independent** events will be defined using conditional probabilities in the next section.



Both a sample space table and a probability tree dia-

got another O!”

Tossing coins and rolling dice

“Of course you do—because you can feel it in the bag without looking.” Hamlet replied curtly. “I am going to prepare another bag with only raisins. Let’s toss two coins: if we have head and head or tail and tail, you pick a bag to get your O; if we don’t get the same results, then I pick a bag for you.”

“Wow! Two bags and two coins. What fun! Let’s try.” Samlette clapped enthusiastically. It didn’t take long for her to figure out that roughly half of the time, Hamlet got to choose a bag, and he always gave her the raisin bag. “I want a third bag full of O’s.”

“I am tired of this game now. You can eat as many O’s as you want.” Hamlet gave both bags to Samlette and showed her the dice Dad gave him. “Dad showed me a game using these two dice. Do you want to play?”

“I will if you replace the brown die by a pink one.” Samlette mumbled with a mouth full of O’s. “But, I don’t want the same game that Dad showed you. Make me a new game!”

“How about this? You can eat an O when both the pink and purple dice show the same number of dots and a raisin when they don’t.” Hamlet suggested.

“Yes. That’s fun!” Samlette reached for the dice and began rolling. “Different—a raisin. Different again—a raisin. Different the third time—a raisin. I want to switch O’s and raisins in this game.”

5.2 COMBINATIONS OF EVENTS—HAMLET’S AND SAMLETTE’S CHALLENGES

Complements

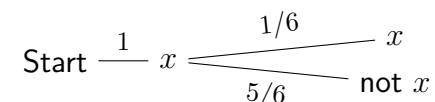
“Dad, I don’t want to play with Samlette anymore. Can I wake Baby now?” Hamlet called for his father hoping for more fun games.

gram show that tossing two coins gives four outcomes: $\{HH, HT, TH, TT\}$. The probability of getting the same results, namely, $\{HH, TT\}$, is $2/4 = 1/2$. However, if instead of choosing the raisin bag every time Hamlet won, he chose at random with equal probability for each bag ($P(\text{a bag}) = 1/2$), then $P(\text{raisin bag}) = 1/2 \times 1/2 = 1/4$ because $1/2$ of the chances after tossing two coins Hamlet got to choose a bag followed by $1/2$ the chance when the raisin bag was picked. \triangleleft

Coin tossing generates random binary sequences. Numberphile shows a few coin-tossing games related to the choice of short sequences to explain winning strategies in games, like [Coin flipping](#) and [Penney’s game](#).

Using the same sample space from [Table 5.1 on page 181](#), only 6 of the 36 outcomes from rolling two dice contain the same number. Let X_1 be the random variable for the result of rolling the brown die, and X_2 , the purple die. Then $P(X_1 = X_2) = 6/36 = 1/6$. The rest, or the **complement**, $1 - 1/6 = 5/6 = P(X_1 \neq X_2)$ is the probability for two dice showing different results.

Note that it does not matter what the first die shows, i.e. true for any $X = x$ (a random variable X equal to a specific value x) for $x = 1$, or 2, or 3, or 4, or 5, or 6, so [Figure 5.1 on the following page](#) can be simplified to:



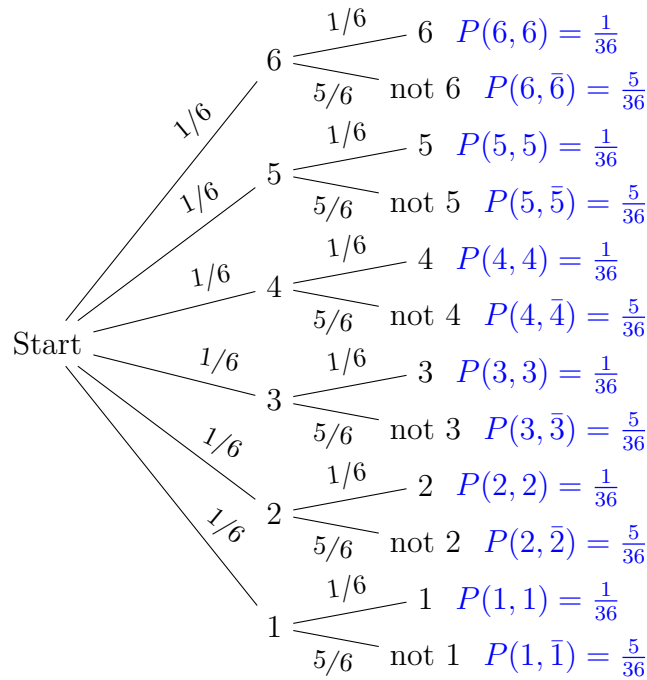







Figure 5.1: Two dice showing the same or not the same

“Just a moment, Hamlet. Please bring the drawing of two dice to me. Let’s play another game.” Dad was trying to have his tea in peace while reading this morning’s news on his tablet. When Hamlet showed up next to him with the piece of paper, Dad had to think very quickly to come up with a new game. “Remember that you counted 36 pairs of dice in this picture showing all different results? How many show  and ?”

Hamlet pointed to   in the middle of the paper. “That’s a boring game.”

“Not so fast, Hamlet. Now how many of the pairs do not have ?”

Given a sample space, where E denotes an event (a subset of the sample space), and $P(E)$ denotes the probability of the event E , let $P(\bar{E})$ denote the probability that event E does not occur. Then $P(\bar{E}) = 1 - P(E)$. We call \bar{E} the **complement** of event E .


According to **Lotto 649**, 1 in 6.6 tickets win prizes (most of which is another free ticket), what is the probability that a ticket does not win?



Let $P(W) = 10/66$ be the probability of a winning ticket, then $1 - 10/66 = 56/66 = P(\bar{W})$, or approximately 85% of the tickets does not win prizes. ◁

The **odds** against getting a raisin in a 2-raisin and 1-O bag is 1 : 2, while the **odds** against getting an O in the same bag is 2 : 1. In general, the probability for computing **odds against** an event E is $n(\bar{E}) : n(E)$ (the number of outcomes not in E versus the number of outcomes in E), or equivalently, $P(\bar{E})/P(E)$. The **odds in favour** of an event E is the ratio of success to failure, namely, $n(E) : n(\bar{E})$, i.e. $P(E)/P(\bar{E})$. ◁

Note that Hamlet was counting the number of pairs of dice containing specific results whereas probabilities are fractions less than or equal to 1. Counting is fundamental in studying discrete probability.

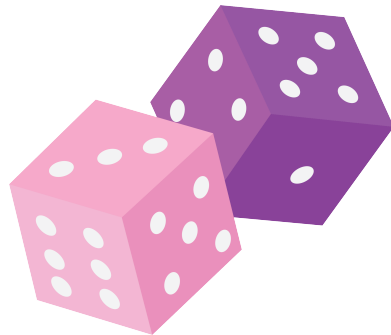
In the experiment of rolling a purple die and a brown die, let L be the event that a purple die shows a 3; similarly, let B be the event that a brown die shows a 3. Let E be the event that the result contains at least one 3 from rolling two dice. Then $E = L \cup B$ where \cup denotes **set**

and  together?” Dad tried to push his brain to spin faster.

“I see only one box with  , so the other 35 pairs do not.” Hamlet felt insulted by his father’s feeble attempt, just getting him to subtract 1 from 36.

“I overheard your dice game with Samlette. Why do you think she was getting so many raisins in a row?” Dad asked gently.



“Well, the game I invented gave her an O when the dice showed the same number. From your picture, only 6 pairs out of the 36 have the same number while 30 pairs show different results. Of course, Samlette was getting a lot more raisins when there are many more pairs with different results.” Hamlet explained to his Dad unwillingly.



Unions and intersections

“OK, Hamlet. I see you don’t want to do any more subtractions. How about this? Pick your favourite number from 1, 2, 3, 4, 5, 6,” Dad suggested.

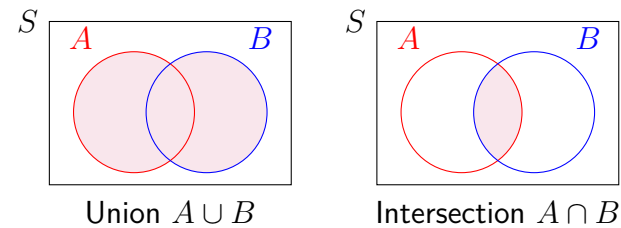
“3,” Hamlet quickly said.

“How many of the results have at least one 3, either  or ?” Dad asked.

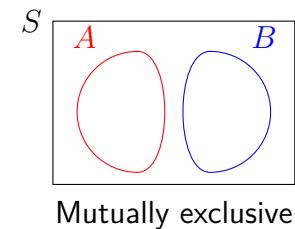
union. Let D be the event where both dice show 3. Then $D = L \cap B$ where \cap denotes **set intersection**. One can also think of \cup and \cap in logical terms as **or** and **and**. This logical **or** means one or the other or *both*.



Venn diagrams can be used to represent the union and intersection of sets: the sample space is an enclosing rectangle; sets A and B are bubbles which may or may not overlap.

Union and intersection of sets A and B as shaded regions.





When the intersection of two sets is empty, namely, the bubbles don’t overlap, they are **mutually exclusive**.



“But, Dad, your picture shows 6 pairs on the third row each with a , and 6 pairs down the third column each with a . So, 6 + 6 = 12?” Hamlet asked.





















































“That’s pretty fast how you picked out the third row and the third column. Why don’t you count them out to see if you really get 12?” Dad went back to reading his news, but not for long.

“It’s 11. I should not have counted   twice.” Hamlet tried to get Dad’s attention. “I want to play another game. No more picture of two dice.”

“OK. Would you please bring me a deck of cards?” Dad pointed at a deck in the bookcase. “Do you know how to play **Elevens**?”

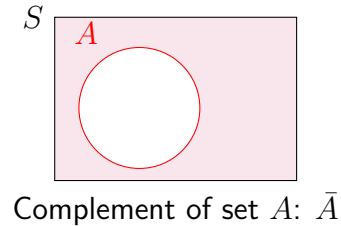
Hamlet ran to get Dad’s favourite bicycle deck. “A-má” taught us how to play **Sevens**, but not Elevens. How do you play?”

“First, we need to remove jokers from the deck to make it a 52-card deck. Second, I must shuffle the deck well before drawing 9 cards from the deck, facing up, arranged in a 3 × 3 grid. The remaining cards form the draw pile, facing down. From the top of the pile, you draw one card at a time to see if you can add up to 11 with any existing card on the grid. If so, you remove the pair. The royals need to come as a triple to be removed. You win when you remove all cards.” Dad wasn’t quite sure if he remembered the rules precisely, just enough for Hamlet to get started.

	K	Q	J	10	9	8	7	6	5	4	3	2	A	
Club														Black
Diamond														Red
Heart														Red
Spade														Black

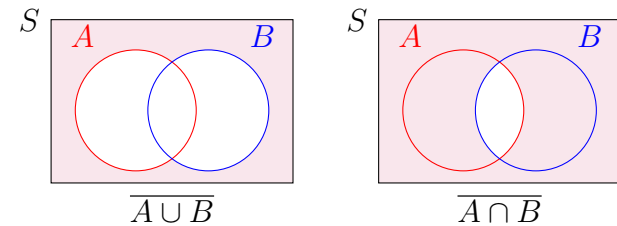
“How many possible ways are there for you to take 9 cards out of 52? What if I am stuck from the beginning?” Hamlet began to ponder. “If it matters how the 3 × 3 grid looks like, then 52 × 51 × 50 × 49 × 48 ×

The complement of set A , \bar{A} shaded in the Venn diagram (set A is not shaded):



One can also take the **complement** of union or intersection of sets. For example, $\overline{L \cup B}$ is the event where rolling two dice results in no 3, whereas $\overline{L \cap B}$ is the event where rolling two dice does not result in a double 3.

Complements of union and intersection of sets:



One of De Morgan’s laws of probability states that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Applied to the probability of the event E where at least one of the two dice shows a 3, $P(E) = P(L \cup B) = P(L) + P(B) - P(L \cap B) = 6/36 + 6/36 - 1/36 = 11/36$. ◁

A deck of cards can provide great **entertainment** from

$47 \times 46 \times 45 \times 44$ is a huge number of ways to choose 9 cards out of 52.” Hamlet’s eyes opened so wide just imagining the size of the product. “But, if we don’t care how to line the 9 cards up in the grid, then we need to divide by $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ because there are this many ways of arranging any chosen 9 cards.”



Figure 5.2: A full deck of 52 cards

©2013 SyLindsley

Dad put down the first 9 cards from his shuffled deck.

8♣	4♣	5♦
7♥	A♠	2♥
J♦	9♦	K♣

“How many cards are left in the draw pile which will make an 11 pair or a royal triple?”

Hamlet began writing what he already had: 2, 4, 5, 7, 8, 9, Ace, Jack, and King, no repeats. He knew that a complete deck contained 4 of each, and a 2 matched a 9 to make a sum of 11, and so on; therefore,

games to magic shows.

An enormous tree diagram helps count the number of ways one can arrange 9 cards out of a 52-card deck—52 branches (or choices) for the first card, 51 branches (or choices left) for the second card, 50 branches (or choices) for the third card, and so on. The total number of arrangements is called the number of **permutations** of 9 distinct objects out of 52 distinct objects. ◀

When the order of 9 distinct objects is immaterial, 52 choose 9, or ${}_{52}C_9$, is the number of ways, also called **combinations** of choosing 9 distinct objects out of 52 distinct objects. Mathematicians use the binomial coefficient from Pascal’s triangle, $\binom{m}{n}$ to denote m choose n .

To find the number of combinations for ${}_mC_n = \binom{m}{n}$, one counts permutations in two steps—first to arrange n distinct objects out of m distinct objects (the big number, $m \times (m - 1) \times \dots \times (m - n + 2) \times (m - n + 1)$), second to remove the order of arranging n distinct objects (thus dividing by $n! = n \times (n - 1) \times \dots \times 2 \times 1$). The exclamation mark is read as **factorial**. ◀

Draw a card at random from a 52-card well shuffled deck. Find the probability that the card drawn is as specified.


1. A Spade? $P(\spadesuit) = 13/52 = 1/4$
2. An eight? $P(X = 8) = 4/52 = 1/13$
3. A Face card? $P(\text{Face}) = 12/52 = 3/13$

in order, the draw pile still had three 9's, three 7's, four 6's, three 4's, four 3's, three 2's, four 10's, ten royals to match the two royals for a triple. "The draw pile has $52 - 9 = 43$ cards, and 34 cards I can use. The chance that I am not stuck with the first card is 34 out of 43."

"Wow, Hamlet! When did you learn about chance in a game?"

"Oh, you taught us to think about it all the time when playing games. Like here you put down 5 red cards and 4 black cards, so there are 21 red and 22 black cards in the draw pile. The chance of drawing either red or black is about the same, slightly better for black cards." Hamlet tried to humour Dad, not quite sure if Dad was reading his news or listening.

"OK, now you know the cards in the draw pile, what is the chance that you either get a red card or a royal card?" Dad wondered.

"I am not going to make the same mistake again, like counting  twice. The chance for a red card is 21 red out of 43 cards in the draw pile. The chance for a royal is 10 royals out of 43. I know that red royals belong to red and royal, so I should not count them twice—and there are 5 red royals. Like I did before by counting boxes, here I can calculate using chances: $10/43 + 21/43 - 5/43 = 26/43$, still more than $1/2$." Hamlet was ready to test out his luck. "Enough questions, Dad. Let me play."

"Just one more thing, Hamlet," Dad pleaded. "You will be happy to see this, I promise." Dad grabbed a piece of paper and began drawing. "Suppose you have a full deck of cards from which you draw two cards, one at a time, without putting any card back. What is the chance that you get at least one diamond out of the two cards drawn? Look at this—I put your name at the start of the tree before it branches out to the first card—one branch leads to the first diamond card while the second leads to other cards. Because there are 13 diamonds out of 52 cards, or 1 out of 4 suits, the chance of getting a diamond as the first card is $13/52 = 1/4$, and the probability for non-diamond cards is $1 - 1/4 = 3/4$. As for the second card, after one card is drawn

4. A Queen of Heart? $P(Q \cap \heartsuit) = 1/52$
5. Not a seven? $P(X \neq 7) = 12/13$, also $P(X \neq 7) = 1 - P(X = 7) = 1 - 1/13 = 12/13$.
6. Not a Club? $P(\bar{\clubsuit}) = 1 - P(\clubsuit) = 1 - 1/4 = 3/4$
7. A Jack or a diamond? $P(J \cup \diamondsuit) = P(J) + P(\diamondsuit) - P(J \cap \diamondsuit) = 4/52 + 13/52 - 1/52 = 16/52 = 4/13$ \triangleleft

Draw two cards from a deck. Describe the probability each statement finds.

1. $P(\text{no } \clubsuit) = \frac{19}{34}$
2. $P(\text{at least one } \spadesuit) = \frac{15}{34}$
3. $P(\heartsuit \cap \heartsuit) = \frac{1}{4} \cdot \frac{12}{51} = \frac{1}{17}$ (Also describe the multiplication of fractions.) \triangleleft

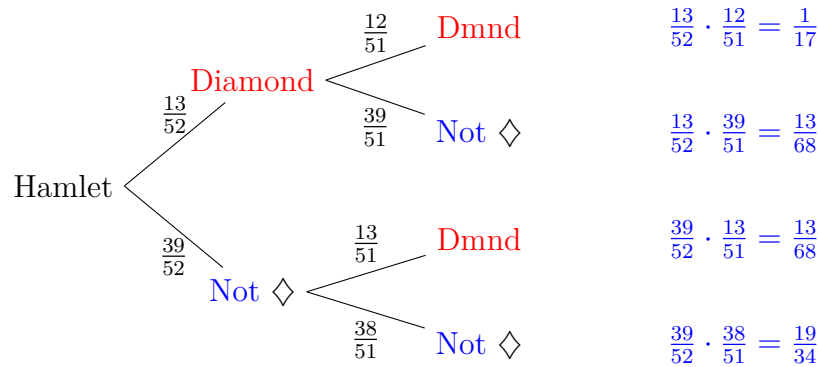


Figure 5.3: Drawing a \diamond or not a \diamond

from the deck, there are only 51 cards left; that’s why you see the new denominator. When the first card drawn is a diamond, then only 12 diamonds are left among 51 cards, and so on for all probabilities on the branches.”

Hamlet could not wait to answer Dad’s original question. “I can see the answer: The probability of drawing at least one diamond in this case is the total of all probabilities in these top three cases, (\diamond, \diamond) , $(\diamond, \text{not } \diamond)$, $(\text{not } \diamond, \diamond)$: $\frac{1}{17} + \frac{13}{68} + \frac{13}{68} = \frac{30}{68}$.”

“Good for you, Hamlet, for adding three fractions by first reducing each to lowest terms before finding a common denominator. Do you remember that the total of probabilities of all cases is 1? If you only have one fraction left, then a faster way of getting the probability could be, $1 - \frac{19}{34} = \frac{15}{34}$ because having at least one diamond is the same as not having both non-diamond cards.” Dad checked that Hamlet’s fraction, when reduced to lowest terms, matched his $\frac{15}{34}$.

Conditionals

In the meantime, Samlette finally finished her breakfast so that she could join Hamlet and Dad in the living room. “What can I play?”

The number of combinations for choosing 2 different bills out of 4 different bills is ${}_4C_2$ —first permute 2 objects out of 4 to get $4 \times 3 = 12$, then lining up \$1 before \$20 is a different permutation than \$20 before \$1, but immaterial for combinations. Therefore, 4 choose 2 is $12/2 = 6$. \triangleleft

Out of 6 combinations of taking 2 bills out of 4, the best case scenario **with replacement** (i.e. putting back what is taken out) is done with two draws to identify all four bills while the worst case scenario is an infinite sequence of identical draws, having the same two bills appear again and again with a probability of $1 \times \frac{1}{6} \times \frac{1}{6} \dots$, a positive number approaching 0 in the limit.

Taking two bills simultaneously out of a bag containing four bills is equivalent to taking two bills, one at a time, **without replacement** (i.e. not putting back what is taken out), out of the bag when the order does not matter. The table for drawing two bills **without replacement** has no diagonal entries, but tracks the order of the bills. Notice a total of 12 entries. Since $(10, 1)$ and $(1, 10)$ yield the same two bills, $12 \div 2$ gives the number of combinations for 4 choose 2. \triangleleft

“Go to find the Monopoly box. I need Monopoly money to play a game with you.” Dad pointed at a box at the bottom of the bookcase. When Samlette returned with the box, Dad took out four different bills from the box and put them in a brown paper bag.

“Oh no, Dad, you are not giving me the same game Hamlet invented for my breakfast!” Samlette pouted.

“Just a minute now. I put in four bills, and they all feel the same, unlike an O and a raisin. Instead of picking one out, you are to pick two bills out and add up the amount.” Dad coaxed Samlette into playing. “Let me know when you draw the biggest amount, and how you know that it is the biggest possible.”



		2nd Bill			
		\$1	\$10	\$20	\$50
1st Bill	\$1	–	(1,10)	(1,20)	(1,50)
	\$10	(10,1)	–	(10,20)	(10,50)
	\$20	(20,1)	(20,10)	–	(20,50)
	\$50	(50,1)	(50,10)	(50,20)	–

Samlette got busy right away, recording the total as she went. She noticed that her first draw had \$20 and \$50, then Dad put both bills back in the bag; second had \$10 and \$20 (bills back in the bag again), and third had \$1 and \$20. “The largest comes from a \$20 and a \$50 because I have seen all four bills after only three tries.” Samlette was eager to show her Dad what she recorded.

“That was fast! Now instead of putting both hands into the bag to take out two bills, you take one at a time. What is your first bill?” Dad asked.

“A \$20!” Samlette showed Dad.

“Do you think your second bill will make the total more than \$25 or less?” Dad asked.

“I bet more than \$25,” Hamlet interrupted.

The probability of event B given that event A has already occurred, denoted by $P(B | A)$, is the **conditional probability** of B given A . In the example of drawing two bills out of four, $P(X_t > 25 | X_1 = 20) = 2/3$ where X_1 is the random variable for the amount of the first bill, X_2 , the second, and $X_t = X_1 + X_2$. Compare that with $P(X_t > 25) = 4/6 = 2/3$, the same as the conditional probability. Also, $P(X_1 = 20 | X_t > 25) = 2/8 = 1/4$ and $P(X_1 = 20) = 3/12 = 1/4$. When $P(B | A) = P(B)$ and $P(A | B) = P(A)$, events A and B are **independent**. It is not so with other first bills, like \$50, $P(X_t > 25 | X_1 = 50) = 1$ (certainty) whereas $P(X_t > 25) = 2/3$, clearly not **independent** events. ◀

The misuse of probability in courts could result in wrongful convictions of mothers accused of murdering their babies. According to the article, [Ten Percent of SIDS Cases are Murder—or are They?](#), before the 1990’s, the courts followed Meadow’s Law: one cot death, a tragedy; two cot deaths, suspicious; three cot deaths, murder until the contrary was proved. However, assuming probability independence of Sudden Infant Death Syndrome (SIDS) in children of the same parents may cause the court to neglect medical reasons of infant deaths as in the case of lawyer Sally Clark’s diseased infants.

“Hey! This is Dad’s question for me, Hamlet.” Samlette complained loudly before getting back to Dad’s question. “I know that the bag only has three bills left—a \$1, a \$10, and a \$50; if I don’t draw a \$1, my total would be more than \$25.” Samlette smiled at her Dad. “I see why Hamlet bet for more than \$25. What if I am super lucky and get a \$50 as my first bill?” Samlette was the eternal optimist.

“Ah, in that case, no matter what the second bill is, with a \$50, you already have more than \$25.” Dad encouraged Samlette to continue exploring.

“Can I play with Hamlet’s deck of cards, too?” Samlette was worried that Hamlet was having more fun with the cards.

“Yes, Samlette, you can help me finish making 11’s with the rest of the cards here.” Hamlet wanted to switch to Samlette’s game of money.

“You want to bet some more, Hamlet? In this brown paper bag, I am leaving only one bill. You don’t know the amount, just that the bill with an equal chance is either a \$20 or a \$50. Now I add a \$50, into the bag and randomly draw a bill out of the bag. Wow, you see that it is a \$50. Do you bet that the bill still in the bag is also a \$50?”

Hamlet scratched his head looking very puzzled. “Are you trying to confuse me, Dad? You told me that the original bill in the bag had a 50–50 chance of being a \$50. If the bill you took out was the same as the one just put in, then the chance of the first bill would still remain the same, or $1/2$, right?”

“Well, let’s see if we can draw pictures for this.” Dad took another piece of scrap paper to draw bags of bills for Hamlet. “First we draw two brown bags: one with a \$20 and the other with a \$50 because you know it was a 50–50 chance of having either bill in the bag. Second, I add a \$50 to both bags. Third, I draw an arrow from each \$50 to mean the removal of a \$50. Now count how many of the cases still have a \$50 left in the bag.”

Formula for conditional probability: Let $n(E)$ denote the number of outcomes in event E . Let A and B be events in a sample space S . Then

$$P(A | B) = \frac{n(A \cap B)}{n(B)}.$$

Dividing both the numerator and the denominator by $n(S)$, one also has

$$P(A | B) = \frac{P(A \cap B)}{P(B)}.$$

Applying a little algebra to the formula for conditional probability, one has

$$P(A \cap B) = P(A | B)P(B).$$

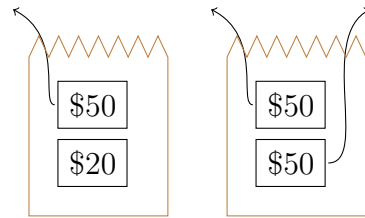
In the case A and B are independent, so $P(A | B) = P(A)$, one has

$$P(A \cap B) = P(A)P(B). \quad \triangleleft$$

Conditional probability on a deck of cards.

1. Find the probability of drawing a diamond given that the card is red.

$P(\diamond | \text{red}) = 1/2$ because half of the red cards are diamonds while the other half are hearts. Compare with $P(\diamond) = 1/4$.



“Well, the bag with 2 \$50’s wins 2 to 1, so the chance that the remaining bill is \$50 has changed to 2/3 from 1/2. How weird! So I should have bet that the bill in the bag was a \$50.” Hamlet thought the picture was playing tricks on him.

When Samlette finished completing pairs of elevens for Hamlet, she brought two brown paper bags for Dad, saying “Could you put money in my paper bags, too? I like to take bills out.”

“Sure, Samlette. Let’s put lots of bills in the bags for you. Which bills go into this bag?” Dad offered the box of Monopoly money to Samlette.

“Five \$500’s, three \$200’s, and two \$100’s for this bag. Four \$1000’s, six \$50’s for that bag. That’s ten fingers and ten toes.” Samlette showed Dad her spread-out fingers and toes.

“Wow, Samlette. What a lot of money! I am putting the bags behind my back, one in each hand. Now, pick a hand.” Dad mixed the bags behind him quickly before Samlette pointed to a hand. Then, from his back, Dad brought out the bag held by the chosen hand. “OK, take out one bill from this bag.” Samlette reached into the bag. “Before you look at the bill, Samlette, do you guess that it is more than 499 or less?”

“I bet smaller.” Samlette quickly answered.

“Let’s take a look. Oh, it is actually a \$1000 bill; you are very lucky then. Remember which bills you put into each bag? Given that a \$1000 bill came out of this bag, can you take a bill from the other bag that is greater than \$500?” Dad asked.

- Find the probability of drawing a spade given that the card is red.

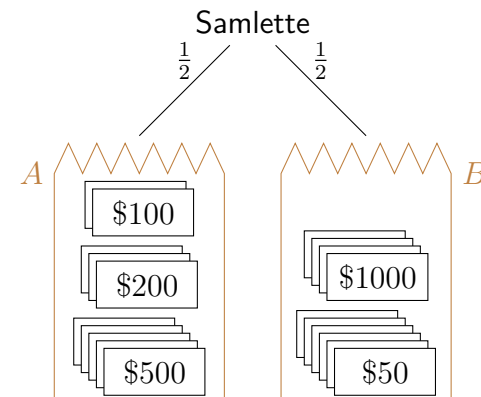
$P(\spadesuit \mid \text{red}) = 0$ because a spade is not a red card, but $P(\spadesuit) = 1/4$.

- Find the probability of drawing a Queen given that the card is a face card.

$P(\text{Queen} \mid \text{Face}) = 1/3$ because there are three types of face cards: Jack, Queen, and King. Note that $P(\text{Queen}) = 1/13$. ◀

Dad’s two-bill problem in a brown paper bag is inspired by No. 16 of [Lewis Carroll’s Pillow problems \[29\]](#) which is linked to [Bertrand’s Paradox](#) and the [Monty Hall problem](#) from a game show.

A probability tree to support Samlette’s claim:



“You know that I didn’t put any bill greater than \$500 in the other bag, so I can’t.” Samlette was not amused by Dad’s impossible question.

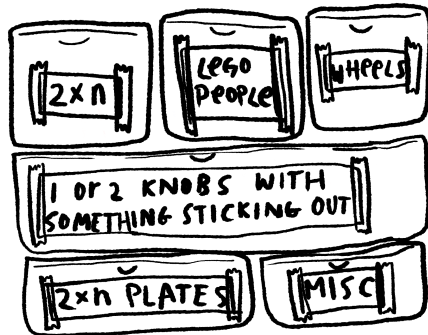
“I see you know the bills well in each bag. Now the bags are behind my back again. This time, I draw a bill from one of the bags and tell you that it is more than \$200. Do you think it comes from the \$1000 bag?” Dad asked Samlette.

“I bet not.” Samlette said.

Hamlet stared at Samlette with his mouth open wondering how his sister bet like a professional with a kindergartener’s reasoning power.

5.3 TABLES AND TREES—HAMLET AND SAMLETTE BUILDING

When Hamlet and Samlette had enough with cards and Monopoly money in bags, Hamlet was ready to wake Baby. “Is it time for Baby to wake up now or should I play with Lego instead?”



“You know that playing Lego will wake Baby in less than five minutes. But, it is late enough now. Go ahead!” With a big sigh, Dad relented.

Rephrase Dad’s questions to Samlette as probability questions and answer them using probability notations:

Let A be the bag (set) containing 5 \$500’s, 3 \$200’s, and 2 \$100 bills. Let B be the bag (set) containing 4 \$1000’s and 6 \$50 bills. Let X_A denote the bill randomly drawn from A , and similarly, X_B from B .

1. Randomly pick a bag and randomly draw a bill from the chosen bag. Find the probability that the bill shows an amount less than 499. Since each bag is equally likely to be chosen, $1/2$ is assigned to each probability. Since $P(X_A < 499) = 5/10$, and $P(X_B < 499) = 6/10$, including the choosing of the bag in the first step, one obtains $\frac{1}{2}(\frac{5}{10} + \frac{6}{10}) = \frac{11}{20} = \frac{11}{20} > \frac{1}{2}$. This supports Samlette’s claim that more than half the time, the amount would be smaller than 499.
2. $P(X_A > 500 \mid X_B = 1000) = 0$ describes the scenario where Samlette saw her first bill to be \$1000, thus identifying the bag as B , then was asked whether it was possible to draw a bill worth more than \$500 from the other bag, namely, A .
3. $P(X \in B \mid X > 200)$ denotes the probability where Dad knew that a randomly drawn bill was greater than 200, and asked Samlette whether the bill might have come from B . The symbol, \in is *belongs to or an element of* (a set). To find the conditional probability, one uses $P(X \in B \mid X > 200) = P(X \in B \cap X > 200) / (P(X \in B \cap X > 200) + P(X \in A \cap X > 200)) = \frac{1}{2} \cdot \frac{2}{5} / (\frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{1}{2}) = \frac{4}{9}$. This is less than 50%, thus supporting Samlette’s claim. \triangleleft

“I want to play, too.” Samlette ran after Hamlet. “Can we continue building your Virus? I want to find more red and blue blocks for it.”

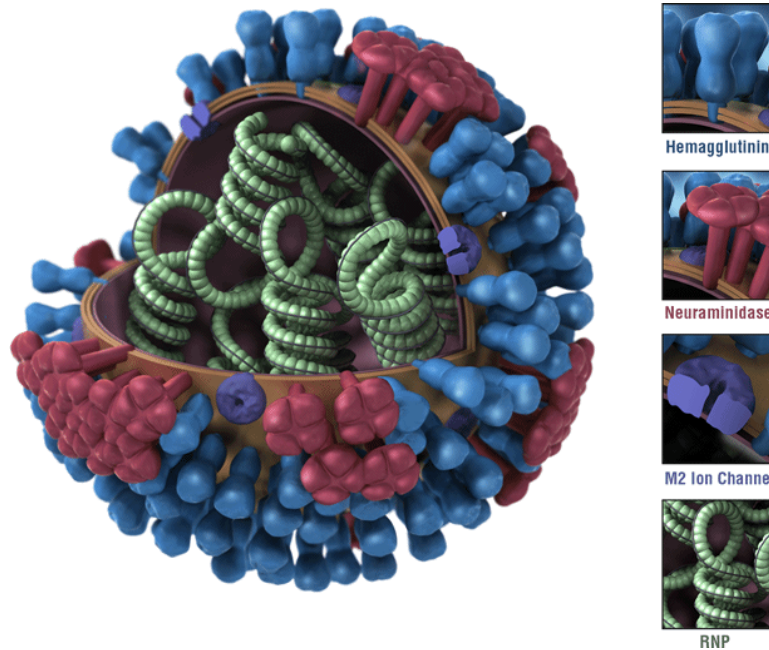


Figure 5.4: A virus model

“You can make your own ball of Virus with different colours. See in this picture of a virus: I am using red blocks for this very long word beginning with the letter **N** and blue blocks for this other long word beginning with the letter **H**. You have to pick two other colours so that I have enough blocks.” Hamlet warned Samlette.

“OK, I will take yellow and white blocks, then.” Samlette grudgingly settled for the colours Hamlet didn’t want. “But, I am going to get Dad to help me build the ball to put these blocks on.”

“Samlette, see our big box of $2 \times$ *something* blocks. I have been taking a lot of red and blue blocks while white and yellow ones are

Four types of influenza viruses with their variants are explained by the Centers for Disease Control and Prevention (CDC).

Nanoparticle flu vaccines which block seasonal and pandemic strains were invented by biochemists at the University of Washington from the National Institutes of Health, USA.

The surface of Influenza A viruses contains two proteins: hemagglutinin (H) and neuraminidase (N).

A seasonal flu vaccine contains four strains to protect against influenza viruses known to cause epidemics, like A(H1N1), A(H3N2), B/Victoria, and B/Yamagata. Covid-19 is not an influenza virus.

When Dad organized the number of Lego blocks according to colour and dimensions to compute probabilities based on the data, he found **empirical probabilities** (also called **a posteriori** models). <

Lego is not just for children. Many adults also enjoy **playing with Lego** blocks, like building a **grand piano** which autoplays.

not used yet. Let’s take turns taking out one block at a time without looking. If we get our own colours, we win and get to use the block; otherwise, we lose and return the block. You see, you should win more times than I do when your yellow and white blocks have not been picked out yet.”

“Let me sort the blocks according to colour just to check that Hamlet is right.” Dad was trying to make sure that Samlette really had enough blocks to build her virus model. “See what I found for you. From my Table 5.2,

	colour			
	red	blue	yellow	white
2×4	89	58	94	82
$2 \times \text{not } 4$	156	74	169	123
total	245	132	263	205

Table 5.2: Lego blocks by shape and colour

Samlette should expect to get a block of her colours more than half the time. Since the two-by-four blocks are most useful, I also sorted them out from two-by-something else. Hamlet, can you see that my probability tree diagram (Figure 5.5 on the following page) shows the probabilities that each of you can expect to get a 2×4 block of your own colours?”

“Wow, Dad. I see that you just multiplied fractions to get to the blue fractions. You know how Samlette doesn’t really take things out at random, like her picking out O’s all the time from the bag. I expect her to do the same with the Lego blocks. If she only picks out 2×4 blocks, what is the chance that she gets a block of her colours?” Hamlet started testing the usefulness of Dad’s probability tree.

“If you know already that Samlette is only picking out 2×4 blocks, then there are $147 + 176 = 323$ such blocks in the box, and 176 of which

The question about block colours and sizes from a Lego box is similar to a drug testing question applying Bayes’ Theorem for which 3 blue 1 brown offers an intuitive visualization. The application of Bayes’ Theorem in drug testing problems is explained in The medical test paradox: Can redesigning Bayes rule help? via Bayes factor and odds.

The Cochrane Library published How accurate are rapid tests for diagnosing COVID-19? on March 24, 2021. For people with confirmed COVID-19, the antigen tests correctly identified the infection in 72% of people with symptoms, compared to 58% without symptoms. For those confirmed without COVID-19, antigen tests correctly ruled out infection in 99.5% of people with symptoms and 98.9% of people without symptoms.

For a health screening procedure, four outcomes are possible.

		Screening	
		Have disease	No disease
Test results	+	Specificity	False +
	–	False –	Sensitivity

Hamlet’s comment about multiplication of fractions along the branches from the root of the tree to the leaf is due to **independent** events. ◀

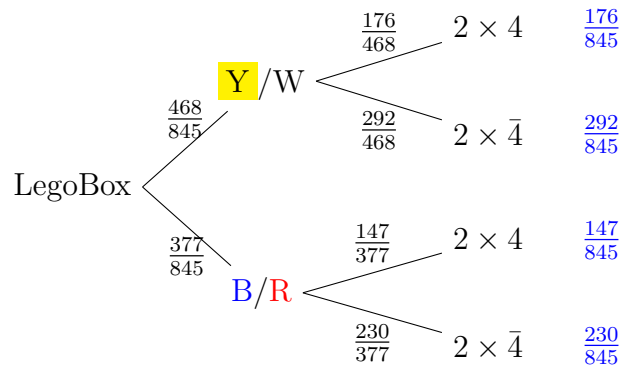


Figure 5.5: Probability tree for Lego blocks

are either yellow or white. Therefore, the chance that Samlette picks out a 2×4 block of her colours is $176/323$, still more than half the time.” Dad pointed at the first row of numbers to Hamlet.

5.4 PHYLOGENETICS TO FLU SHOT DESIGNS—MUTATING VIRUSES

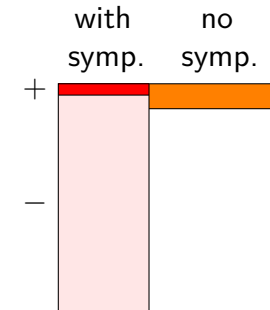
“Oh, no, Dad!” Hamlet screamed. “Samlette is pulling out my blocks on the virus and replacing them with her colours!” Hamlet grabbed his virus ball out of Samlette’s hands.

To de-escalate the situation, Dad suggested, “You know sometimes viruses mutate. Check and see how many blocks mutated.” While talking, Dad got busy building a Lego ball for Samlette to let her begin her own virus model.

“Can I draw a tree to show how my virus mutated?” Hamlet wanted to build a tree more complex than Dad’s tree.

“Yes, try to imagine a tree accounting for all these blocks Samlette changed on your virus. Instead of probabilities assigned to branches, a tree tracing mutated viruses back to a common ancestor actually has probabilities (actually, likelihoods) labelled on the nodes to signal how

Consider a room full of people confirmed to be without COVID-19 where $\frac{2}{5}$ of them show symptoms. Using data given by the article from the Cochrane Library, find 1) the **false positive rate**, the probability that a person in the room has a positive result, 2) a person in the room has no symptom (NS) given a positive result.



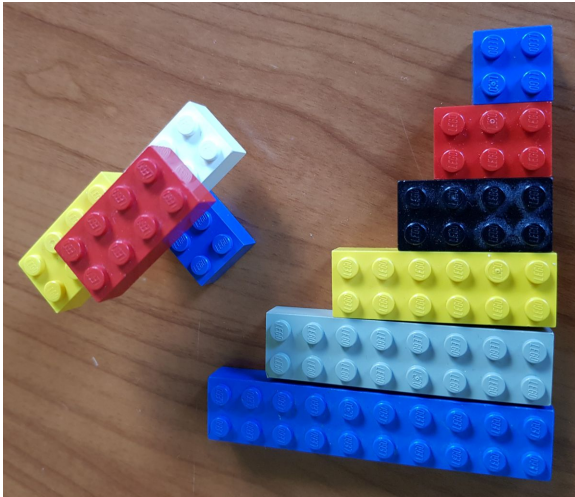
$$P(+) = 0.4 \times 0.005 + 0.6 \times 0.011 = 0.0086 = 0.86\%,$$

and

$$P(NS | +) = \frac{0.6 \times 0.011}{0.6 \times 0.011 + 0.4 \times 0.005} \approx 0.7674,$$

or 76.74%. ◁

How to read a phylogenetic tree explains the likelihood nodes sharing the same parent node (in red with a number) came from that parent. When Hamlet examined the changes Samlette made to his virus model, Hamlet was working from the original to the mutated new virus. However, phylogenetic trees of viruses are built from the bottom up by comparing genes from many viruses, putting closely related viruses at the same level, and tracing (guessing with a likelihood supported by data) the



likely the mutated viruses shared a particular ancestor on each node. When Samlette mutated your virus, she replaced a block by her block. Think of that as a forward developing probability tree. Now, as you work backwards, imaging looking at many different mutated viruses, trying to figure out which two viruses are most closely related. The detective work you do is developing the tree backwards from its leaves back to the root.” Dad went to get more scrap paper for Hamlet to start designing.

“Hamlet, let us imagine that each cart of your Lego train has one of the four amino acids (the coding blocks for our DNA): $\{A, C, G, T\}$. If there are four trains encoding

1. Virus1: AGTTG**C**TGAAAT,
2. Virus2: AGTTGATG**A**GAC,
3. Virus3: **T**GTAGATGA**A**CG, and
4. Virus4: **T**GCTGA**A**GATCT,

parent of the children viruses.

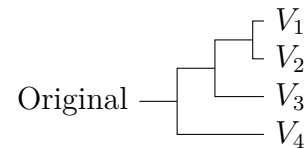
With the help of Next-Generation Sequencing (NGS) providing DNA sequences of viruses in a short time, the mutated viruses, once analyzed, allow genomic epidemiologists to localize the spread of each mutated virus and to trace the origin of the mutated virus.

In the case of COVID-19 which mutates on average in every **two transmissions**, the interpretation of phylogenetic trees helps shape public health policies to slow down the spread.

For an introduction to **phylogenetic trees**, Khan Academy provides a series of articles. **How to interpret phylogenetic trees** uses COVID-19 as an example to cover ten topics with ten pictures. For video lectures from experts, please watch **Phylogenetic trees** in Module 1.3 from the CDC’s Advanced Molecular Detection (**AMD**).

Epidemic simulations from 3 **blue** 1 **brown** assigns different probabilities to transmission rates to study the speed of spread.

upon comparing the carts, you may feel that Virus1 and Virus2 are most closely related, together, they are related to Virus 3, and Virus 4 is the farthest from the rest. The phylogenetic tree to draw would look like this.”



“But, Dad, if viruses mutate all the time, how do you know what type of flu shot to get?” Hamlet noticed how his tree kept splitting and getting larger.

“Luckily for us, the **World Health Organization** (WHO) Global Influenza Programme studies the mutations every year and chooses the most likely strains to include—separately for the northern and southern hemispheres because of the timing of their flu seasons. When I receive my flu shot every fall, I only know that it contains four strains, two flu A’s and two flu B’s. Because the mix is just WHO and Health Canada’s best guess, some years work better than others.” Dad looked at Hamlet’s tree of mutated viruses, and continued, “Remember the tree I drew for the Lego blocks? I am going to draw a similar one for having flu and testing for the flu with percentages instead of fractions. One winter 75 % of Canadians on the West Coast got the seasonal flu for which a test was administered: 10 % of people with the flu showed a negative result whereas 5 % of those without the flu showed that they got the flu, i.e. a positive result.” Dad drew **Figure 5.6 on the next page** for Hamlet. “From this tree, suppose someone tested positive for the flu, what was the chance that the person didn’t have the flu?”

Hamlet examined the tree carefully and collected the probabilities with positive results: $0.75 \times 0.9 + 0.25 \times 0.05$. He then found the

To understand different types of artificial learning from machine learning, deep learning, to **sequential learning**, one first understands **graphs** of neural networks.

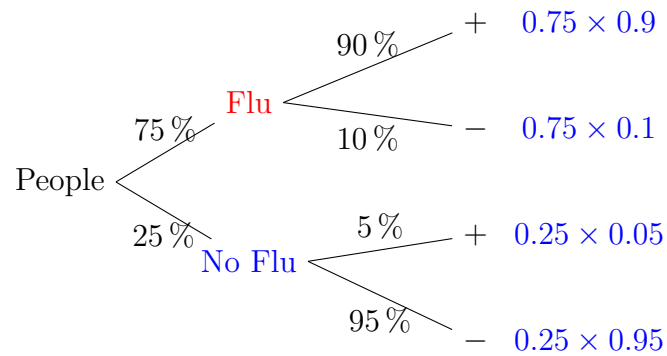


Figure 5.6: Testing for the flu

conditional probability,

$$P(\text{no flu} \mid +) = \frac{0.25 \times 0.05}{0.75 \times 0.9 + 0.25 \times 0.05} \approx 0.01818,$$

or 1.82%.

And with that, Dad suddenly got out of his reading chair because Baby just woke up with a distressing cry. On his way to Baby’s room, Dad glanced quickly at what Hamlet drew for his virus mutation tree and said, “I will show you a more complex picture tomorrow morning for **neural networks**.”

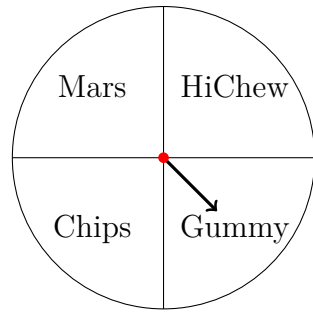
5.5 EXERCISES

Theoretical probability

5.1 Dad constructed a spinner out of a piece of cardboard, a paper clip, and a thumb tag for Hamlet and Samlette the morning after their trick-or-treat trip on Halloween. Assume that the arrow (made from the paper clip) has the same probability of landing in each sector, but never on the border, answer the following questions.

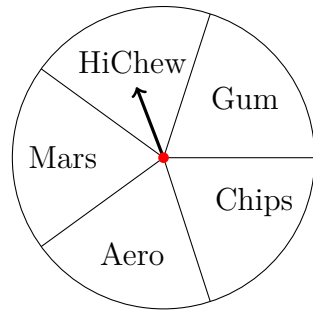
Many search engines use **graphs** to serve clients efficiently. The **PageRank algorithm** from Stanford University’s Info-lab discusses how Google’s search engine parsed queries according to relevance using the example of *Bill Clinton Joke of the Day: April 14, 1997*. Section 4.1 of the paper shows a **directed graph** representing the high level Google architecture.

Search engines try to direct users to relevant websites. However, scammers whose primary task is to lure the unsuspecting customers devise web addresses closely related to popular sites. For example, 3 **blue** 1 **brown**’s *.com* might be mistyped as *.co* where *.co.uk* could be a familiar address for commercial websites in the United Kingdom. However, when the mistyping leads to an unsecured Columbian (*.co*) website claiming to sell Clothing, Flights, and Dresses, the safest next step is to exit the website immediately.



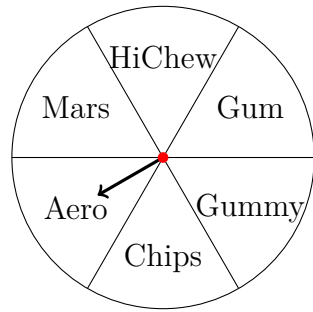
- a. Draw a tree diagram for spinning the spinner once.
- b. Find the sample space for the experiment of spinning the spinner once.
- c. Draw a tree diagram for spinning the spinner twice.
- d. Find the sample space for the experiment of spinning the spinner twice.
- e. Write the event of Chips appearing exactly once when Hamlet spun the spinner twice. Express as a set of outcomes.
- f. Write the event of Gummy appearing at most once when Samlette spun the spinner twice. Express as a set of outcomes.
- g. Write the event of HiChew not appearing at all when Hamlet spun the spinner twice. Express as a set of outcomes.
- h. Write the event of Mars (Bar) appearing exactly twice when Hamlet spun the spinner three times. Express as a set of outcomes.

5.2 Dad constructed a second spinner with five sectors the second morning after Halloween for Hamlet and Samlette. Assume that the pointer never lies on a border line, answer the following questions.



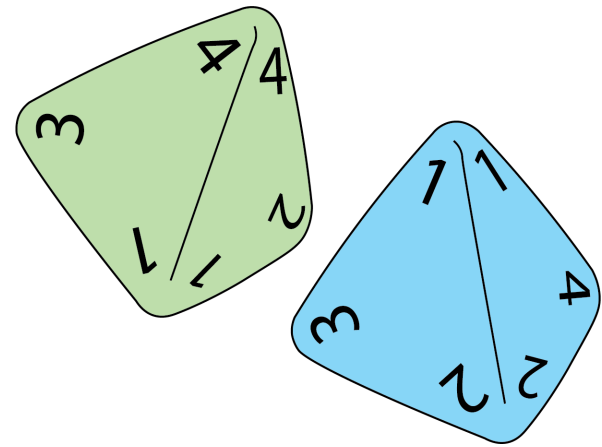
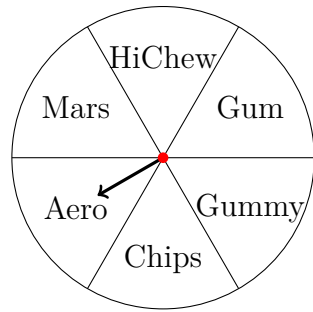
- a. Draw a tree diagram for spinning the spinner once.
- b. Find the sample space for the experiment of spinning the spinner once.
- c. Draw a tree diagram for spinning the spinner twice.
- d. Find the sample space for the experiment of spinning the spinner twice.
- e. Write the event of Gum appearing at least once when Hamlet spun the spinner twice. Express as a set of outcomes.
- f. Write the event of Aero appearing more than once when Samlette spun the spinner three times. Express as a set of outcomes.
- g. Samlette was still too young to have gum. Write the event of Gum not appearing at all when Samlette spun the spinner twice. Express as a set of outcomes.
- h. Write the event of Chips appearing at most twice when Samlette spun the spinner three times. Express as a set of outcomes.
- i. Write the event of either a Mars or an Aero (or both) when Hamlet spun the spinner twice.

5.3 Dad constructed a third spinner with six sectors the third morning after Halloween for Hamlet and Samlette. Assume that the pointer never lies on a border line, answer the following questions.



- a. After one spin, find $P(\text{HiChew})$.
- b. After one spin, find $P(\text{Mars} \cup \text{Chips})$.
- c. After two spins, find $P(\text{Mars} \cap \text{Chips})$.
- d. After two spins, find $P(\text{Mars} \cup \text{Chips})$.
- e. After two spins, find $P(\text{no Gummy})$.
- f. After two spins, find $P(\text{at least one Mars})$.
- g. After two spins, find $P(\text{exactly one Chips})$.

5.4 Dad used the third spinner with six sectors made on the third morning after Halloween for Hamlet and Samlette. Assume that the pointer never lies on a border line, answer the following questions.



A D4 has three different numbers on each face, a number per corner each of which has three identical numbers. Once a D4 is rolled, one reads the number at the tip of the tetrahedron.

- After two spins, find $P(\text{no HiChew})$.
- After three spins, find $P(\text{Mars} \cup \text{Chips})$.
- After three spins, find $P(\text{Mars} \cap \text{Chips})$.
- After three spins, find $P(\text{Mars} \cup \text{Chips} \cup \text{Aero})$.
- After three spins, find $P(\text{no Gummy})$.
- After three spins, find $P(\text{at least one Mars})$.
- After three spins, find $P(\text{exactly one Chips})$.

[IM]

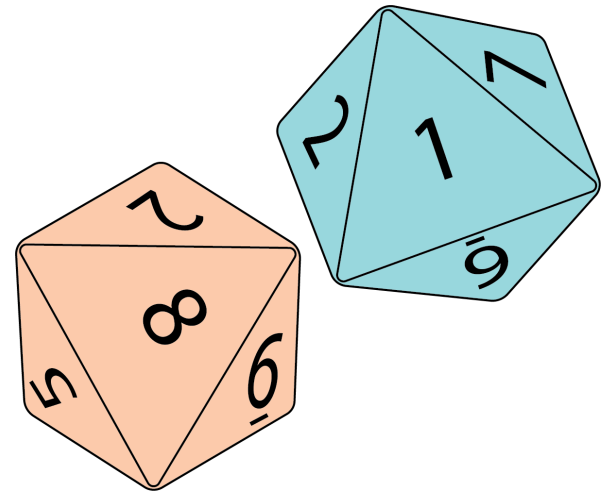
5.5 Dad gave Samlette a four-sided die (D4) and a coin. Draw a sample space for rolling the die and flipping the coin. [IM]

5.6 Dad gave Hamlet 4 old pennies. Find the probability that Hamlet got at least one tail when all 4 coins were flipped.

5.7 Dad gave Samlette 10 old pennies. Find the probability that Samlette got at least one head when all 10 coins were flipped. [IM]

5.8 Dad found two four-sided (D4) dice for Hamlet and Samlette. Find the probability for each of the following for the experiment of rolling these two dice.

- $P(X_1 + X_2 = 3)$
- $P(X_1 + X_2 \geq 5)$
- $P(X_1 + X_2 < 4)$
- $P(|X_1 - X_2| > 2)$
- $P(X_1 + X_2 = 2n)$ for some integer n . This is another way of saying that the total is even because an even number is divisible by 2, or equivalently, an even number is a multiple of 2.
- $P(X_1 + X_2 = 2n - 1)$ for some natural number n , or the total is odd because an odd number is not divisible by 2.
- Find the probability that the total is either 2 or 5. Express the probability using proper notations.



5.9 Dad found two eight-sided (D8) dice for Hamlet and Samlette. Find the probability for each of the following for the experiment of rolling these two dice.

- $P(X_1 + X_2 = 10)$
- $P(X_1 + X_2 \leq 4)$
- $P(X_1 + X_2 > 5)$
- $P(|X_1 - X_2| \geq 6)$
- $P(X_1 + X_2 = 4n)$ for some integer n .
- $P(X_1 + X_2 = 3n)$ for some natural number n .

- g. Find the probability that the total is neither 2 nor 5. Express the probability using proper notations.

[IM]

5.10 Samlette rolled a green die and a violet die (both D6's). Find the probability that the sum was less than 7.

5.11 Hamlet rolled a black die and an orange die (both D6's). Find the probability that black die showed 6 less than the orange die. [IM]

5.12 Ham, Hamlet, and Samlette each rolled a six-sided die. What's the chance that the total from all three dice was 4? [IM]

5.13 Dad found his favourite deck of cards and asked Samlette to draw one card. Find the following probabilities.

- a. $P(4\clubsuit)$
- b. $P(4)$
- c. $P(\clubsuit)$
- d. $P(5 \cup Q)$
- e. $P(8 \cup \text{Face})$
- f. $P(\heartsuit \cup K)$

5.14 Dad asked Hamlet and Samlette to give the number of hands for the following.

- a. Royal flush (10, J, Q, K, and A, all in the same suit).
- b. Other Straight flush (consecutive numbers, all in the same suit).

Sample space tables (2-dimensional) and probability tree diagrams (multi-draw) help answer questions visually. Another approach is through counting using permutations and combinations.

Then Dad gave Hamlet and Samlette the number of hands for the others: • 624 hands of Four of a kind (number); • 3744 hands of Full house (3 of the same kind and 2 of another kind); • 5108 hands of Flush (same suit but not in sequence); • 10 200 hands of Straight (consecutive numbers, not all of the same suit); • 54 912 hands of Three of a kind (and two other cards of different kind); • 123 552 hands of Two pairs (two of one kind and two of another kind); • 1 098 240 hands of One pair; and • 1 302 540 other hands.

5.15 Dad drew a tree for Hamlet and Samlette to show them the number of ways of getting 5 cards out of 52 regardless of order, namely, $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48/5! = 2\,598\,960$, then asked Hamlet and Samlette to compute the following probabilities using a calculator each.

- a. $P(\text{one pair})$
- b. $P(\text{royal flush})$
- c. $P(\text{full house})$
- d. $P(\text{straight flush})$ including the Royal flush
- e. $P(\text{three of a kind})$
- f. $P(\text{flush})$
- g. $P(\text{two pair})$

5.16 Dad continued playing cards with Hamlet and Samlette. Answer each question.

- a. Hamlet was asked to draw two cards from a shuffled deck without replacement. What is the probability that neither card was black?
- b. Samlette was asked to draw three cards from a shuffled deck without replacement. What is the probability that exactly one card was a spade?



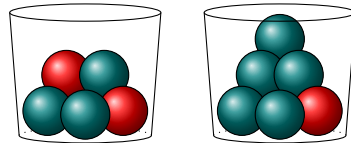
- c. Find the probability of drawing a jack, a heart, and a club in that order from a shuffled deck without replacement.

[IM]

5.17 Dad washed two beach sand pails to place 2 red and 3 teal balls in one and 1 red and 5 teal balls in the other. Suppose Hamlet had a 50–50 chance of choosing either pail before picking one ball out of the chosen pail, answer the following questions.

- $P(\text{red ball})$
- $P(\text{teal ball})$
- Find the odds against drawing a red ball.

[IM]



5.18 For the two pails Dad prepared, one containing 2 red and 3 teal balls and the other, 1 red and 5 teal balls, Samlette took out two balls without replacement by choosing a pail at random (with equal probability), then a ball out of the pail, and again for the second ball. Answer the following questions.

- $P(\text{red ball in the first draw})$
- $P(\text{teal ball in the second draw})$
- $P(\text{two balls of the same colour})$
- Find the odds in favour of getting two red balls.

5.19 Samlette dumped all the balls together in one big pail, i.e. 3 red and 8 teal balls, and asked Hamlet to take out two balls, one at a time, without replacement. Find the following probabilities.

- a. $P(\text{two red balls})$
- b. $P(\text{balls of the same colour})$
- c. $P(\text{balls of different colour})$

[IM]

5.20 Ham found 8 old ping-pong balls and marked each with a digit from the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ before putting them all into a pot. Hamlet drew two balls from the pot without replacement. What is the probability that the sum of the numbers was 10?

[IM]

5.21 Ham found 5 old ping-pong balls and marked each 0, 4, 7, 8, and 8 before putting them all in a box. Samlette drew three balls from the box without replacement. What is the probability that the sum of the three numbers was 19?

[IM]

5.22 Ham found two identical buckets, one with ping-pong balls numbered 1, 2, 2, 3, 3, 5, 5, and 8, another, the numbers 0, 0, 1, 7, 9, and 9. Samlette randomly picked a bucket and drew a ping-pong ball from it. What is the probability of drawing an 8?

5.23 Ham used the same buckets to put the numbers 1, 3, 6, 6, 7, 7, 9, and 9 in one, and the numbers 2, 3, 4, 7, 8, and 9 in the other. If Samlette drew a number from each bucket, what is the probability that the sum of her numbers was 15?

5.24 Ham used the same buckets to put the numbers 0, 6, 8, and 9 in one, and the numbers 1, 2, 4, and 8 in the other. If Hamlet drew a number from each bucket, what is the probability that the sum was 17?

[IM]

5.25 Ham used the same buckets to put the numbers 0, 2, 3, 8, and 8 in one, and the numbers 0, 0, 1, 3, 6, 8, and 10 in the other. If Hamlet drew a number from each bucket, what is the probability that the sum of his numbers was at least 6? [IM]

5.26 Ham found three identical looking pots and many marbles for Hamlet and Samlette. Answer the following.

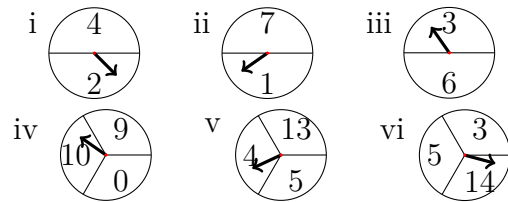
- a. A pot contained a black marble and a cyan marble; another pot contained two black marbles; and a third pot contained three black marbles and a cyan marble. Hamlet randomly selected a pot and drew a marble. What is the probability that the marble Hamlet took out was black?
- b. A pot contains four red marbles; another pot contains four brown marbles; and a third pot contains a brown marble and a red marble. Hamlet randomly selected a pot, drew a marble, and put it aside. Then Samlette drew a second marble from the same pot. What is the probability that both marbles were red?

[IM]

5.27 Ham found digits $\{1, 2, \dots, 9\}$ on magnets and asked Hamlet to form 3-digit numbers using distinct digits. What's the probability that Hamlet assembled an odd number?

5.28 Samlette also wanted to play with magnetic digits so Ham gave her four digits, $\{1, 2, 3, 4\}$. If Samlette formed a 2-digit number using only distinct digits, find the probability that the number formed was larger than 40. [IM]

5.29 Dad drew six spinners for Hamlet and Samlette to play against each other. For each set of choices, decide who has the higher chance of winning.



- Hamlet picked spinner **i** while Samlette picked spinner **ii**.
- Hamlet picked spinner **iii** while Samlette picked spinner **i**.
- Hamlet picked spinner **i** while Samlette picked spinner **iv**.
- Hamlet picked spinner **vi** while Samlette picked spinner **v**.

[IM]

5.30 Sam's favourite hockey team had 22% chance of winning last season. What were the odds against the team?

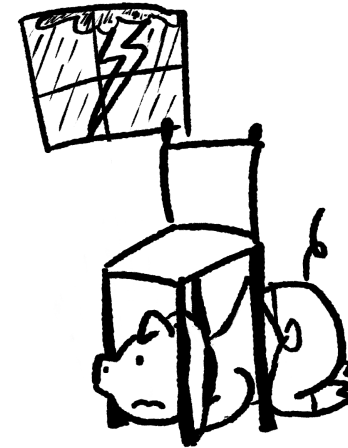
5.31 Ham's favourite volleyball team had 57% chance of winning last season. What were the odds in favour of the team? [IM]

5.32 The odds against winning the World Cup for Sam's favourite soccer team was 8 to 5 last season. Find the probability of winning.

5.33 The odds in favour of winning the World Series for Ham's favourite baseball team was 4 to 9 last season. Find the probability of winning.

5.34 Ham looked up the chance of being **struck by lightning in Canada**. If the odds of getting struck by lightning was less than one in a million, what was the probability of getting struck?

In the US, the chance of being struck by lightning is **more than double** that of Canada's.



Empirical probability

5.35 Ham and Sam loved bringing their children to **Vancouver Aquarium**. In 2017, they participated in a web-survey regarding the keeping of cetaceans. Answer the questions according to the data shown in the table.

	Issue in Survey	
	Keep Cetaceans	Free Cetaceans
Live in Vancouver	4235	8275
Live in suburbs	7829	6098

- From the group participating in the web-survey, find the probability of a randomly chosen person supporting the keeping of cetaceans.
- From the group participating in the web-survey, find the probability of a randomly chosen person not being a resident of Vancouver.
- From the group participating in the web-survey, find the probability of a randomly chosen person from Vancouver supporting the keeping of cetaceans.
- From the group participating in the web-survey, find the probability of a randomly chosen person from the suburbs opposing the keeping of cetaceans.

5.36 Ham and Sam selected the streaming services monthly for Hamlet and Samlette depending on the shows they wanted their children to watch. The following table shows the result of a survey according to age group and the number of streaming services used on average per month.

Age	Streaming services			
	1	2	3	4+
16–30	280	435	205	78
31–45	380	129	198	54
46–60	223	552	352	452
61+	121	673	236	22



- According to the survey, how many streaming services did most people settle on?
- If a participant was randomly selected, what is the probability that the participant came from the 31–45 age group?
- If a participant was selected from the 61+ group, what is the probability that the participant signed up for more than 4 streaming services?
- Knowing that a participant randomly chosen signed up for 2 streaming services, what is the probability that the participant came from the 46–60 age group?

Use a spreadsheet to find column and row totals. Google Sheets offers sum as the first suggestion when = is entered in the cell. [IM]

5.37 Baby woke up with a high fever and a bad cough. When Sam brought him to the doctor, a rapid antigen test from a throat swab sample was performed. From the table below, draw a probability tree with probabilities assigned on the branches to answer the probability of having a strep throat given a positive test result in two ways, directly from the table and from the tree.

		Streptococcus	
		Have Strep A	No Strep A
Test	+	235	26
Results	-	45	4342

5.38 The next morning, Samlette also woke up with a bad cough and sore throat. Using the same table for testing Streptococcus, find the probability of not having a strep throat given a negative test result for Samlette after the doctor administered the same swab test. [IM]

5.39 Ham and Sam's family doctor recommended the fecal immunochemical test (FIT) for them every two years. One year, Sam received an abnormal result and was told by her doctor that **abnormal FIT** results occurred between 10% to 15% on average. She was reassured by her doctor that over 96% of those with an abnormal FIT result would be found to have no colon cancer. Assume that Canadians had free access to FIT, provide a range in percentage where the population in Canada might have colon cancer.

5.40 No test is perfect. The fecal immunochemical test (FIT) is not 100% accurate, i.e. it may miss some colon cancer. Suppose 13% of those taking FIT gets abnormal results, of those, 4% has colon cancer while of those with normal FIT results, 0.2% has cancer. Starting with a FIT branching to Normal and Abnormal results first, then cancer or no cancer second, find the probability that given a patient with colon cancer, the patient gets a normal result (negative) for FIT.

5.41 When a population of 1013 are asked to self-assess if they have hypochondria, 4 say yes. Suppose it is known that 10 of the 1013 actually suffer from hypochondria, and that the chance of a false positive self-assessment is 0.30%. What is the chance of a false negative?



Symptoms of colon cancer may include blood in the stool, abdominal pain, change in bowel habits, and unexplained weight loss. **Colonoscopy** helps determine whether there is cancer with a risk of missing a significant abnormality in less than 1 in 10 cases.

Hint: Make a table like the one below. Then figure out the blue, olive green, purple, and red numbers in that order.

	hyp.	not hyp.	Total
Yes	■	■	4
No	■	■	■
Total	10	■	1013

5.42 According to *Opioids and Pain Management*, a poster prepared by Canadian Centre on Substance Use and Addiction, 1 in 5 Canadians lives with chronic pain, and 1 in 8 Canadians is prescribed opioids to manage pain. If half of those living with chronic pain is prescribed opioids, what percentage of Canadians with opioid prescriptions do not have chronic pain? [IM]

5.43 *Every 49 minutes*, by Michael Friscolanti published on September 1, 2021 in Maclean’s magazine, describes how the opioid crisis affects every walk of life by focusing on one week in July, 2020 during the height of Covid lockdown. If 12.5% of Canadians are prescribed opioid to control pain, and 5% of them becomes dependent on opioid, what percentage of Canadians needs to go to the street to find opioids after their doctors stop prescribing opioids? [IM]

5.44 According to *Mother’s Against Drunk Driving* (MADD), some Canadians drive while believing they are impaired by alcohol, cannabis, or other drugs.

Among the 67% of drivers who used alcohol in the past 30 days, 6% drove at least once in the past six months believing they were impaired, with just over a third (35%) driving with passengers.

Hypochondria is an illness anxiety disorder.

How to help Canada’s Opioid Crisis is a poster for all residents of Canada. The first item states that anyone can get a free naloxone kit from the pharmacy without a prescription. Everyone should learn how to use a naloxone kit like knowing how to administer basic first aid.



A **Toolkit** to support employers and employees in the trades states that half of people experiencing a substance use disorder do not seek help because of stigma around addiction.

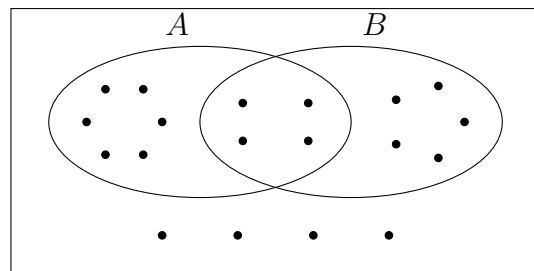
Drug abuse facts from Teen Challenge Canada claims that the average age drug use begins is 15.7 years old.

Alcohol abuse among teens tends to leave permanent scars; therefore, early intervention through **addiction and mental health support** can make a big difference.

What percentage of drivers believing that they were impaired drove at least once with passengers? [IM]

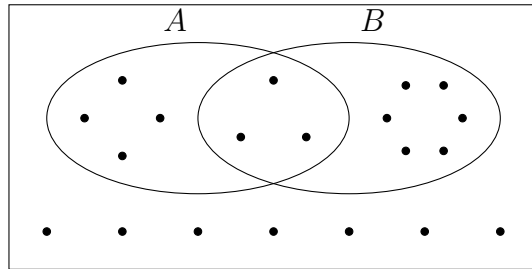
Venn diagrams

5.45 Dad drew a big rectangle representing a sample space containing two events, A and B . Assume that the outcomes (as dots) were all equally likely, answer the following questions.



- $P(\bar{B})$
- $P(A \text{ and } B)$
- $P(B | \bar{A})$
- $P(A | B)$
- $P(A \text{ and } \bar{B})$
- Are A and B independent?

5.46 Dad drew a big rectangle representing a sample space containing two events, A and B . Assume that the outcomes (as dots) were all equally likely, answer the following questions.

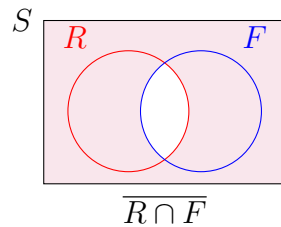


- a. $P(A)$
- b. $P(\bar{A} \text{ or } B)$
- c. $P(\bar{B} \mid A)$
- d. $P(B \mid A)$
- e. $P(\bar{A} \text{ or } \bar{B})$
- f. Are A and B mutually exclusive?

[IM]

5.47 Ham told Hamlet that there was a 70 % chance of rain and 40 % chance of fog on Samlette's birthday. Because rain and fog were not independent events, Hamlet supposed a 30 % chance of both rain and fog.

- a. What was the chance of not getting both rain and fog to occur on Samlette's birthday? See the Venn diagram below.
- b. Find the probability of not getting rain on Samlette's birthday.
- c. Find the probability of having either rain or fog (or both) on Samlette's birthday.



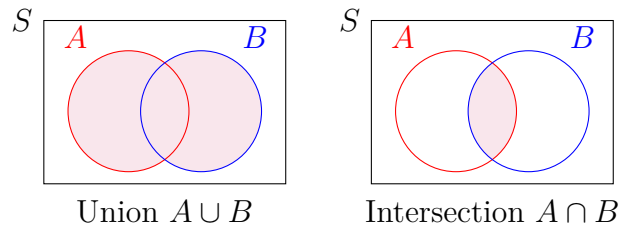
For the sake of simplicity, assume **independence** of events for weather questions.

5.48 Ham and Sam travelled to Seattle one autumn. During the month of their trip, the chance that a rainy day was followed by another rainy day was 70%, and the chance that a sunny day was followed by a rainy day was 60%. What was the probability that a rainy day was followed by three rainy days?

5.49 Ham and Sam drove the children to Prince Rupert one summer. During the month of their trip, the chance that a rainy day was followed by another rainy day was 80% and the chance that a sunny day was followed by a rainy day was 60%. What was the probability that a rainy day was followed by a sunny day two days later? [IM]

5.50 Samlette liked puzzles. Ham showed her a puzzle of a sample space S as a rectangle containing two events A and B . Samlette knew that $P(S) = 100\%$, $P(A) = 90\%$, and $P(B) = 70\%$, with their overlap, $P(A \cap B) = 65\%$. Help Samlette find the following probabilities.

- a. Find $P(A \cup B)$.
- b. Find $P(\overline{A \cup B})$.
- c. Find $P(A | B)$.



5.51 Ham showed Samlette a second puzzle of a sample space S as a rectangle containing two events A and B . Samlette knew that $P(S) = 100\%$, $P(A) = 55\%$, and $P(B) = 40\%$, with their union, $P(A \cup B) = 75\%$.

- Find $P(A \cap B)$.
- Find $P(\bar{A})$.
- Find $P(\overline{A \cup B})$.
- Find $P(A | \bar{B})$.

[IM]

Games

5.52 Project topic suggestion: Study the game **Old maid** and recommend a strategy.

5.53 Project topic suggestion: Study the game **Pass the Pigs** and recommend a strategy.

5.54 Project topic suggestion: Study the **Science and Gambling** from the Royal Institution with Adam Kucharski on best gambling strategies from the United Kingdom's lottery and the mathematics of betting on the roulette table.

5.55 Project topic suggestion: Study *The Game of Life* by John H. Conway to discuss which initial square patterns lead to infinite activities.

5.56 Project topic suggestion: Draw sample spaces to explain the *Hat problems*.

Understanding Data—Florence and Hans’s Better World

6.1 ORGANIZING AND VISUALIZING DATA

Introducing Florence and Hans

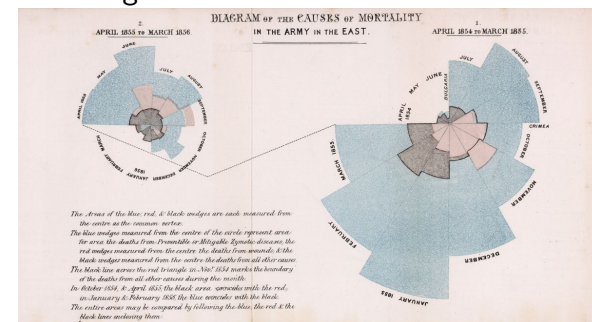
The Software Engineering Department at Capilano University designed Florence and Hans to be co-directors of CapU’s Virtualverse to provide refuge, healing, and inspiration to students, staff, and faculty.

Like artificial intelligence designed to imitate both human emotions with corresponding facial expressions and emotional responses during interactions, Florence was fashioned after **Florence Nightingale** (1820–1910), the statistician most renowned for her pioneering work applying statistics to medicine, in particular, documenting the link between hygiene and the survival rate of wounded soldiers during the **Crimean War** (1853–1856). Her superior intellect, academic excellence, amazing networking skills combined with a tireless work ethic and immense compassion for the poor and the sick were qualities the software engineers worked hard to code into Florence in the Virtualverse where she could welcome, comfort, support, heal, and inspire CapU community.

Hans was designed to imitate **Hans Rosling** (1948–2017), a TED

Not only did Florence Nightingale found the first nursing school in Britain, but the **Florence Nightingale Foundation**, more than a century after her death, also continues to support future nurses. The **Florence Nightingale museum** shows a pie chart drawn by Nightingale when she studied the causes of mortality while serving in the Crimean War.

Data visualization aide drawn by Florence Nightingale—a polar area diagram:



talks star who used statistics to bring optimism to a world that often feels powerless and pessimistic about the problems of the world because of biased media coverage—an overemphasis of bad news. As a professor of public health, and previous to that, a physician who served in Mozambique, Hans Rosling’s concern for public health often extended beyond his home country, Sweden, to the whole world. Using women’s health as a measure for the standard of living, Hans Rosling promoted inventions (like the [washing machine](#)) which freed women from long days of menial labour for the education of their children, and birth control methods to allow for smaller families to enhance the success of the next generation. Above all, his passion for sharing animated data made by [Trendalyzer](#) from [Gapminder](#) corrected many misconceptions and prejudices about the developing world and emerging markets. In CapU’s Virtualverse, Hans would share his experience and knowledge to bring forth future policy makers for shaping a more generous, compassionate, and equitable global community.

Creating the Virtualverse full of data

Florence and Hans began a survey on CapU’s use of virtual reality (VR) in gaming or hybrid work environment via [Metaverse](#) and other similar platforms. As Florence combed through material about VR gaming, she was shocked by what she learned. “Hans, since you were born more than a century after me, and lived in the era of the internet, could you explain why so many popular VR games involve violence and/or sex? If people feel so restricted and stressed by their physical world, why would they escape to a world so much worse? Look at all the shooting and killing in what I found, even a [sexual assault](#) in the Metaverse. Can you imagine that every seven minutes a violent assault occurs somewhere in the Metaverse?”

“I guess that is where you come in; together, I am sure we can create a safe Virtualverse (VIV) where all who join to escape their physical world will be nurtured to be the best version of themselves. People are



[Don’t Panic](#) by Hans Rosling is a video about world population growth from the [Gapminder Foundation](#).

[Carmine Gallo](#), a Forbes magazine contributor, wrote about TED talks star, Hans Rosling, and made a 99-second video about how to emotionalize data to make it memorable using disposable K-cups as an example.

[Neal Stephenson](#), an American novelist, first coined the term *Metaverse* as the virtual reality where people lived and worked after the physical world had collapsed.

The fear of the unknown commonly shared among humans often paralyzes them into a state of hopelessness and inaction. Simply by looking at the body of young adults’ literature centred around dystopia since 2000, one needs to revisit visionaries like [Ursula K. Le Guin](#) to help one embark on a new journey of freedom in a larger reality.

such complex creatures—the good and the bad all packaged into one body. Who would ever imagine that a TV comedian by profession like [Volodymyr Zelenskyy](#) who was elected president of Ukraine to everyone’s surprise would become such a galvanizing global symbol of democracy against the [Russian invasion of Ukraine](#)?”

“Hans, I know your extraordinary bravery in studying the [konzo](#) outbreak in the Democratic Republic of the Congo. At the time when your wife and children were also in Mozambique, making a decision whether to enter a village afflicted by the mysterious paralytic disease not knowing if you would get back out alive, or if you would infect your own family members must have been truly agonizing. But compared to casualties of war I witnessed from the Crimean War in the 1850’s, at least a quarter million lost on each side, I am so sorry to learn that after more than a century and a half, people still haven’t learned the devastation of war. The world naturally would side with Ukrainians, their displaced refugees, their casualties of war, their fighters, and so on, but how about the Russian soldiers, more than ten thousand of them who did not want to fight but ended up maimed or dead anyway? How do we build CapU’s Virtualverse to enhance peaceful conflict resolution skills?”

Tally and frequency

“Just out of curiosity, between the Crimean War and before your death in 1910, how many wars made it to the list of [Encyclopædia Britannica](#)? How many took place in Asia, in Europe, in Africa, and in America?” Hans inquired.

“When we are designed as artificial intelligences living in VIV, we have a quick way of answering all fact-based questions, like those you just posed—downloading the list of wars, then connecting to their links for geographic locations would answer your questions in no time. However, we need to practise how to be human before we can help those we are designed to help, so how would humans answer these questions based

Thoughts on [The Virtual and the Real](#) by [David J. Chalmers](#) [4] from New York University supplemented by [The Virtual as the Digital](#) engage the readers to anticipate possibilities and problems of virtual reality.

Statistics is a mathematical science pertaining to the collection, organization, analysis, interpretation and presentation of data.

Statistical Studies comprise of two main types: observational and experimental. An observational study measures variables of interest without modifying the subjects, whereas an experimental study first treats the subjects before measuring variables of interest.

The **population** in a statistical study is the collection of all the subjects of interest.

A **census** is the procedure of systematically collecting and recording information about all members of a given population.

A **sample** is a smaller collection of the population.

A sample is **biased** if it does not accurately reflect the population as a whole with regard to the data gathered. Gathering an **unbiased** sample is the first step to good statistical analysis.

A collection of information either quantitative or qualitative is called **data** in statistics. [Infographics](#) [9] [33] is the communication of data via organization and presentation

on this list of wars?

- Crimean War (Europe)
- Bleeding Kansas (North America)
- American Civil War (North America)
- War of the Triple Alliance (South America)
- Seven Weeks' War (Europe)
- Selangor Civil War (Asia)
- Franco-German War (Europe)
- Acehnese War (Asia)
- Red River Indian War (North America)
- Serbo-Turkish War (Europe)
- Anglo-Zulu War (Africa)
- War of the Pacific (South America)
- Gun War (Africa)
- Sino-French War (Asia)
- Serbo-Bulgarian War (Europe)
- Sino-Japanese War (Asia)
- Spanish-American War (Asia, America)
- Philippine-American War (Asia)
- South African War (Africa)
- The War of a Thousand Days (Central America)
- Boxer Rebellion (Asia)
- Moro Wars (Asia)
- Russo-Japanese War (Asia)
- Pig War (Europe)
- Mexican Revolution (North America)

I propose the following strategy.

- First, I would make four groups to represent four continents. Here, I have combined both North America and South America as *The Americas* for simplicity.

with the help of technology, a very active field of development. Nightingale's polar area diagram is an example of an infographic. Chicago Tribune's summary of [Oscar nominees/winners](#) for best actor/actress categories is a more recent example. ◀

Organization of data into **groups**, also called **classes** or **categories** is the first step in understanding data for a specific purpose. The question Hans posed requires the organization of wars according to continents; therefore, the classes are Asia, Europe, Africa, and America.

To **tally**, one puts a stroke down for each war in its respective continent. For example, |||| | for 6 wars in Europe from 1853 to 1910.

Once tallying is complete, the number summarizing the tally is the **frequency**. For example, 6 is the frequency of wars taking place in Europe from 1853 to 1910. ◀

A larger [list of wars](#) since 1300 BCE can be found in Encyclopædia Britannica.

- Second, I would **tally**.
- Third, I would summarize the **frequency**, see Table 6.1.

Continent	Tally	Frequency
Asia		9
Europe		6
Africa		3
The Americas		8

Table 6.1: Frequency table for wars by continent

Have you noticed, Hans, how many catastrophes European colonizers brought to other continents just from causing war alone? Even worse, how few indigenous wars made it to the list in Encyclopædia Britannica because it is too Eurocentric? Just look at the aftermath still suffered by people all over the globe because of the **Doctrine of Discovery**.”

Stem-and-leaf plot and bar graphs

“As you know, my home country, Sweden, has been neutral with respect to armed conflict since the Napoleonic Wars, but the people we are designed to serve may have experienced the horrors of war first-hand. It would be important for us to know from CapU’s survey the number for each age group with war trauma.” Hans brought up some CapU statistics from VIV to show Florence, see Table 6.2 on the next page.

“Looking at this list, the best tool for organizing the data is a **stem-and-leaf plot**, Figure 6.1 on the following page. The stem collects decades of the ages; for example, age 37 will fall in the group with 3 as its Stem and 7 in the Leaves place. Just to help people see the distribution, let me organize the leaves from smallest to largest. The next entry would have 4 in the Stem and 9 in its Leaves. The first three



A **stem-and-leaf plot** contains two columns: The stem (left) column records the tens (or tens and hundreds) digits while the leaf (right) column records the ones digits including repeats. ◀

War	Age(s)
The Balkans war in the 1990's	37, 49, 55
Iraq war both 1991 and 2003	27, 33, 40, 56
Russian-Georgian war in 1992	46
Rwandan genocide in 1994	35
Syrian civil war from 2011 to present	19, 23, 23, 23, 25
Crimean war in 2014	25, 30
Yemen civil war from 2014 to present	42
Tigray of Ethiopia from 2018 to present	33
India-Pakistan border conflict	22, 23, 23, 48
Afghanistan-Taliban	23, 25, 31, 31, 37, 41, 45
Palestinian-Israeli	20, 22

Table 6.2: Wars and ages of trauma survivors

entries from the Balkans war are in blue.” Florence just loved using different tools to organize data.

Stem	Leaves
1	9
2	02233333335557
3	01133577
4	0125689
5	56
6	

Figure 6.1: A stem-and-leaf plot of ages

Hans continued for Florence, “I know next you would summarize the stem-and-leaf plot in a frequency table and probably draw a **histogram** to help people compare visually. Let’s see: the classes will be age groups: 10–19, 20–29, 30–39, 40–49, 50–59, 60–69 with frequency 1, 13, 8, 7,

All CapU statistics are fictional in the story part of this chapter.

A **histogram** is a graph consisting of bars (without gaps between different bars) to represent the frequency distribution visually. The data may be quantitative or qualitative. If the data is discrete, we can make a class for each possible value. When the data is continuous, or consists of a big range of values, we may group the data into classes. Some textbooks use **bar** graphs for qualitative data with some gap between bars. Typically, frequency or relative frequency is on the vertical axis while classes or values are listed on the horizontal axis. ◀

2, 0 respectively. People with poor eyesight can easily make mistakes while counting those ones digits all clustered together. I double check by adding the total of the frequency: $1 + 13 + 8 + 7 + 2 + 0 = 31$ which equals to the number of ages listed. In the frequency table, I add a row for **relative frequency** to show the proportion as a percentage in each age class.” See Figure 6.2.

Age group:	10–19	20–29	30–39	40–49	50–59	60–69
Frequency:	1	13	8	7	2	0
Rel. Freq. (%):	3.23	41.94	25.8	22.58	6.45	0

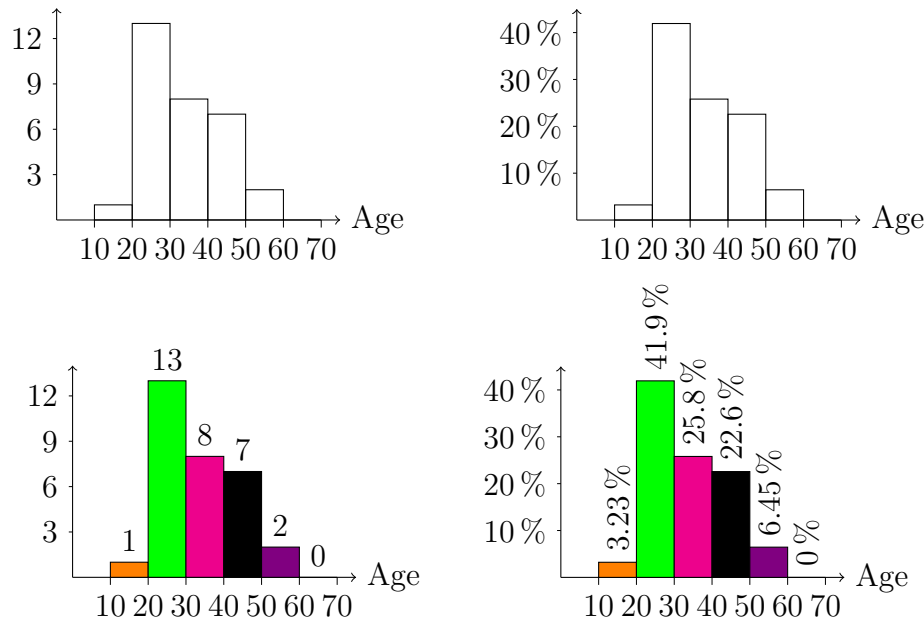
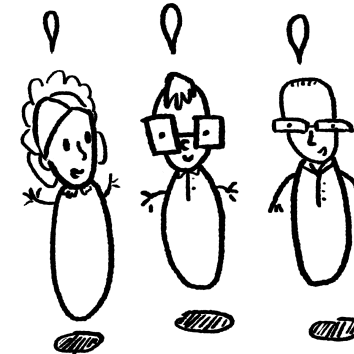


Figure 6.2: People with war trauma by age groups in 4 histograms

Florence smiled at Hans for *reading* her mind. “I have great faith in the resilience of human beings, not only from nursing traumatized soldiers most of whom recovered, but also from learning about heroic

A **relative frequency** of a certain event is obtained by taking the frequency of that event divided by the total number of events in the sample space. It can be expressed as a fraction or a percentage.



Historian, Yuval Noah Harari, warned of aggressors of war sowing the seeds of hatred in his TED interview on the **war in Ukraine**. Harari also gave a talk as a digital avatar on **why fascism is so tempting and how your data could power it**.

recoveries of five Canadians in Catherine Gildiner’s book, *Good Morning, Monster* [16]. Can you imagine that up to 80% of families in North America are considered **dysfunctional**? Nevertheless, many children from dysfunctional families refuse to continue with the dysfunction they experienced by working hard to heal themselves while building much healthier and more loving families. In VIV, we can provide unlimited psychological support for people to put closure to their trauma so that they can move on to live meaningful and fulfilled lives.” The frequency table and the bar graphs led Florence to the next phase of challenges.

Just in time, Hans pointed her to the statistics she needed. “I think we can start working on phobias, or **anxiety disorders** first before tackling war traumatized visitors in VIV. For example, social phobia is one of the most common anxiety disorders and affects between 8% and 13% of Canadians. When Capilano University encourages team work among students, those suffering from social phobia may fear judgment from others. Practise has shown that VR can be a great tool for those with object specific phobias like **arachnophobia** to help them overcome their phobias. Specifically, we can gradually lead those afflicted with the phobia through different stages of simulations to get them ready for the encounter of a real object, a spider, say, in the case of arachnophobia. In particular, with a visitor who wants to overcome social phobia, let us invite the visitor into a group with other designed VIV **avatars** who are sincere and inviting to build trust first. When the visitor is ready to try other avatars, we can introduce challenges—one at a time—to help the visitor learn how to cope with potentially problematic interactions.”

Line graphs

“Hans, you just reminded me that we should look at the record from Counselling/Wellness Centre to see the number of students using the service in the last twenty years. Pulling up the record, we see from 2002 to 2021 the total number of visits per year to the Counselling/Wellness Centre: Table 6.3 on the following page.

Dysfunctional family relationships do not end after one leaves home. It is important to distinguish a healthy one from a dysfunctional one in order to stop perpetuating the dysfunction.

Dr. Bob Maunder and Dr. Jon Hunter’s book, *Damaged: Childhood Trauma, Adult Illness, and the Need for a Health Care Revolution* [26] is leading Canadians to rethink *universal* health care. An interview of the authors by Dr. Brian Goldman from CBC’s *White Coat, Black Art* not only explained the importance of addressing childhood traumas for healing, but also how deep friendship carried both doctors through challenging times.

Cognitive behavioural therapy has been successful in helping many overcome anxiety disorders. To learn more, please read *CBT—An information guide* by Neil Rector.



Jonathan Haidt’s popular article in *The Atlantic*, *Why the past 10 years of American life have been uniquely stupid* points to a link between social media’s prominence and an epidemic of depression suffered by young people.

2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
324	458	470	563	599	626	802	694	701	760
2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
798	858	897	964	912	957	978	1010	1259	981

Table 6.3: Visits to the Counselling/Wellness Centre per year

When we plot these twenty data points as a **line graph** (see Figure 6.3), the ups and downs from year to year are so clear, useful in tracking trends.” Florence deduced that the general trend of increase in the number of visits to the Counselling/Wellness Centre could be either due to the university’s population increase or the recognition of the usefulness of counselling.

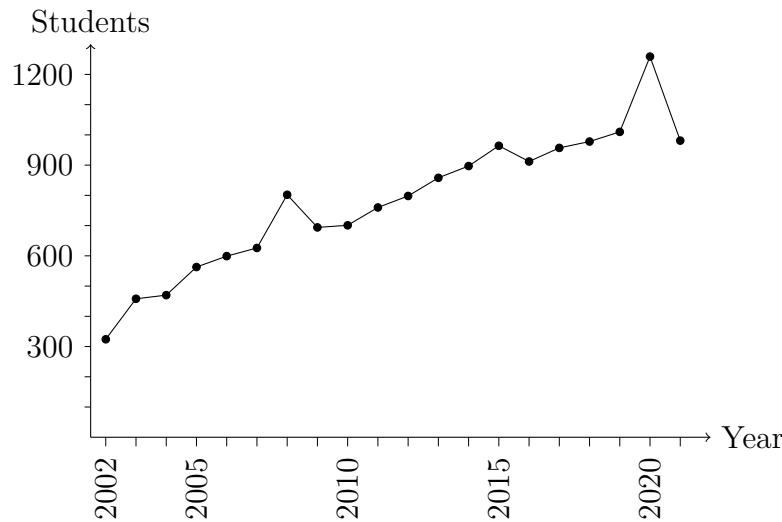
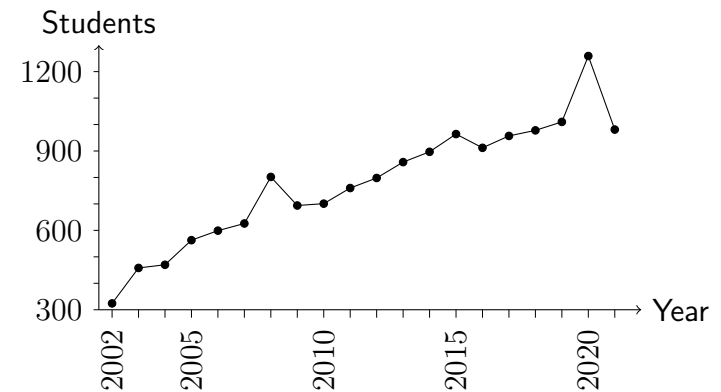


Figure 6.3: Line graph showing usage of the Counselling/Wellness Centre

A **line graph** or **line chart** in Google Sheets uses line segments to join data points sequentially from the left to the right relative to the horizontal axis to show changes in quantities. The labels on the vertical axis should begin with 0 unless clearly stated to exaggerate changes between neighbouring data points.

Notice that the vertical axis does not start from 0 in the line graph below.



Pie charts

Hans wasn't so sure and decided to look more closely at the reasons for visits. "Looking at the line graph, I agree with your proposed reasons for the general upward trend. Upon examining the reasons for the visits for 2019 and 2020 in Table 6.4, I see a significant group asking for Accessibility Centre's service—so refreshing for me, a dyslexic, to see how students with different challenges are now accommodated, for example, given more time for tests to show their true ability. CapU is modelling a kinder community than what universities used to be."

Reasons	2019	2020
relationships	320	293
study skills	444	337
anxiety disorder	125	538
accessibility service	121	91
Annual total	1010	1259

Table 6.4: Breakdown of visits to Counselling Service

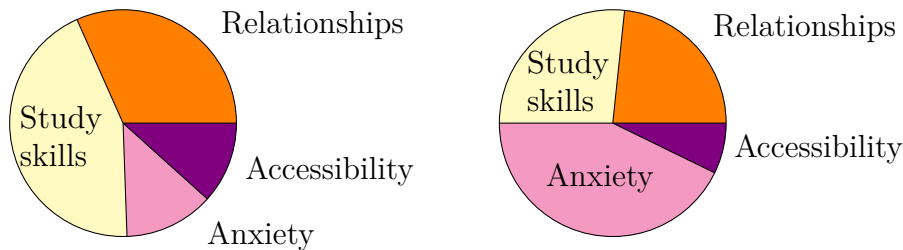
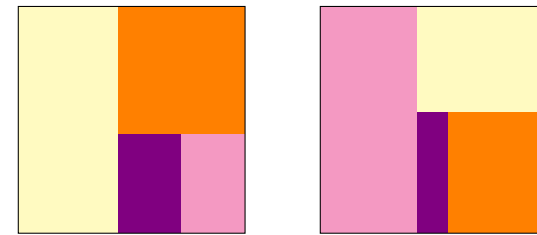


Figure 6.4: Pie charts for types of counselling in 2019 (left) and 2020

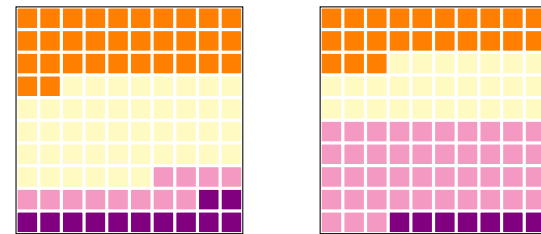
"Hans, I would instinctively arrange the data set from each of these two years into a **pie chart** for each year, like Figure 6.4, but studies

A **pie chart** consists of a circle divided into sectors where each represents a proportion of the whole. Because of human's limited ability to estimate **angles or arc lengths**, one can replace a pie chart by a **square pie chart** or a **waffle chart** which subdivides a square into a 10×10 grid so that each cell represents 1%. For an example, please see **the square pie chart** from ResearchGate.

Square pie chart (also called **tree map chart**) version of Figure 6.4, still 2019 (left) and 2020,



and a waffle chart version,



Diagnoses of learning challenges are more prevalent among the rich, as high as 18% in wealthy districts compared to a national average of 2.7% as explained in **Need Extra Time on Tests? It Helps to Have Cash**. Perhaps education reforms need to begin with the removal of standardized tests.

like *Judgment error in pie chart variations* have shown that a **square pie chart** is much easier for people to estimate correctly.”

Multibar and stacked bar graphs

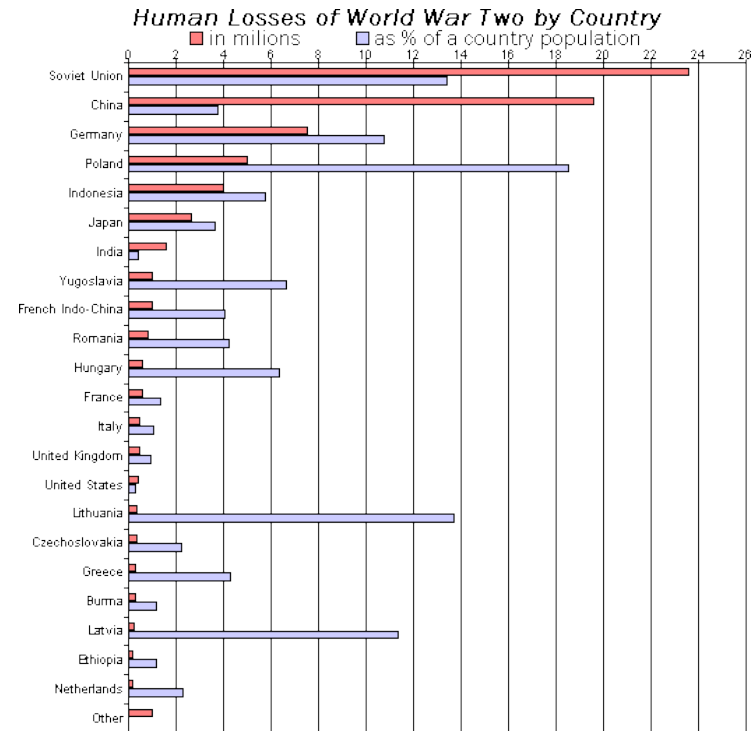
“Hmm—interesting. I wonder if that is also true for circle area comparison versus square area comparison. In most of my talks where comparison of many countries were involved, the size of the circle’s area often represented population, the bigger the dot, the larger the population. Should *Gapminder’s Trendalyzer* have used square dots instead in graphs? About the charts, if we want to fine-tune the study to separate mature students (age 26+) from those 25 or younger in each group (see Table 6.5), we would need to employ a double bar graph, a bar for each year, combined with two stacked bars, one for each age group.”

Reasons	2019	2020
relationships	320 = 231 + 89	293 = 252 + 41
study skills	444 = 413 + 31	337 = 311 + 26
anxiety disorder	125 = 67 + 58	538 = 415 + 123
accessibility service	121 = 70 + 51	91 = 63 + 28

Table 6.5: Age 25– and age 26+ visiting Counselling/Wellness Centre

The stacked double bar graph, Figure 6.5 on the following page, showed a change which concerned Florence greatly. “Can you see, Hans, how hard the pandemic was on students most of whom just tried to survive, therefore, resulting in so many more counselling visits to overcome anxieties? I hope as life slowly returns to normal, students can focus on their studies again.”

A **multibar graph**, also called a **grouped bar chart** or a **clustered bar chart** has more than one bar for each group, with each bar coloured in its own shade, consistently for all groups.



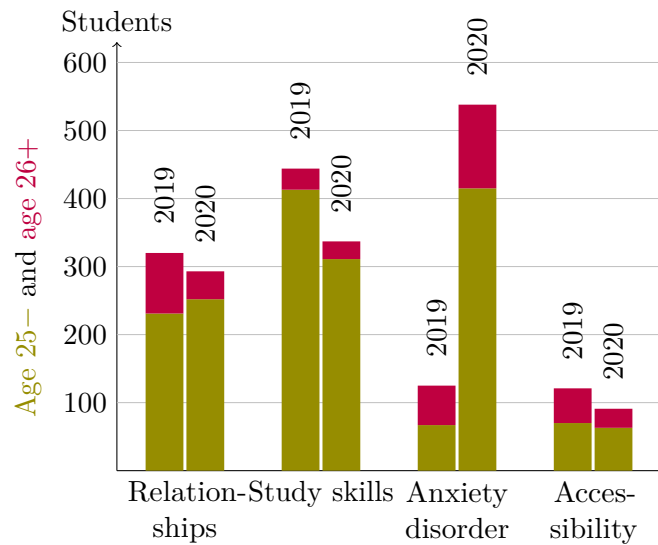


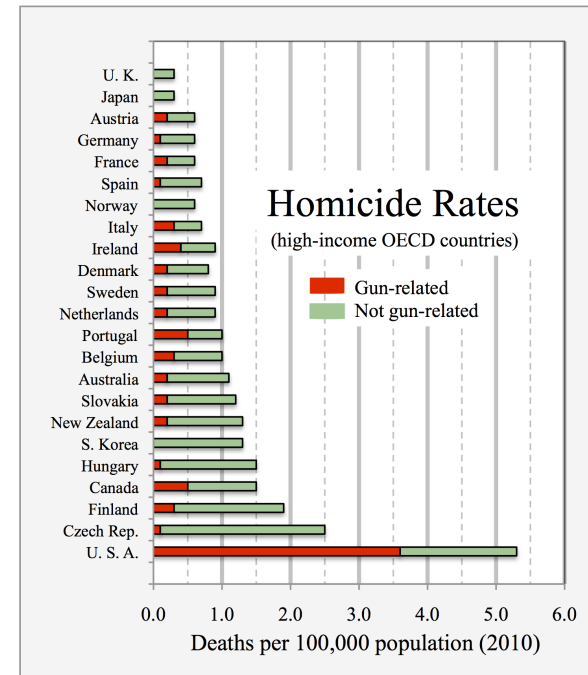
Figure 6.5: Age/type visits to Counselling Service

6.2 DESCRIBING DATA—CENTRES AND SPREAD

Helping instructors in the Virtualverse

Florence was impressed by what Hans showed her, but ready to move on to staff and faculty. “Hans, look at this session with Naomi, a staff member of the Pottery Department. She was concerned about a long-time friend whom the police charged for the possession of child pornography, and wondered if she could invite him to her home again because she had six children, five of whom still under ten. According to the record, Naomi learned about the charge second hand from a common acquaintance and telephoned the friend right away. From their conversation, she learned that the friend bought some foreign DVD’s from a store which also sold child pornographic material. When police searched the store and obtained a list of customers, her friend was the

A **stacked bar graph** stacks bars on top of each other to focus on the total effect.



most prominent on the list. To make an example of him, the police charged her friend without any evidence and publicized it in newspapers and on the internet. Because of the publicity, his job was terminated; all his business travels, banned; almost all his friends, gone overnight.”

Hans thought quietly before answering, “When Naomi’s friend lost his job, friends, and reputation overnight because of an impulsive, baseless police charge, treating him with suspicion would be adding insult to injury. Naomi should not worry about inviting the friend to her home again because, in the worst case, if the accusations turned out true, as a guest, he would have everyone’s attention anyway. Just think how important it would be for her friend to have some support from friends who were brave enough to continue their friendship during the police investigation? By the way, how old were Naomi’s children then?”

Centres—mean, median, and mode

“Let me check—,” Florence searched the file. “Two, four, six, nine, nine, and thirteen.” “A set of twins! The **average** age would be $(2 + 4 + 6 + 9 + 9 + 13)/6 \approx 7.2$ years old—a very busy mom indeed. The **median** age among these six children would be $(6 + 9)/2 = 7.5$ years old; and finally, the **mode** would be 9 years old from the twins’ age. Sorry, I just can’t resist getting all the versions of **centres** in this small data set of six numbers.” Hans smiled his signature wide, generous smile.

“Returning to child pornography though, from Chart 5 of the report, **Crime stats 2020 Canada**, a dramatic increase in child pornography from 2018 to 2020 is making parents’ job of protecting their children even harder.” Florence shook her head in disbelief examining the line graph.

Hans interrupted Florence, “Just to wrap up the case of Naomi’s friend. It turned out that the friend was wrongly accused, proven innocent, but the police refused to acknowledge their wrongful charges and incurred damages. All her friend got was the police dropping all

The **mean** or **average**, denoted by \bar{x} , of a set of numbers is the sum of the numbers divided by the number of numbers:

$$\bar{x} = \frac{\sum x_i}{n},$$

where \sum denotes taking the sum of all data values ($x_1, x_2, x_3, \dots, x_n$) with n as the number of data. This is one way of measuring the **centre** of a data set.

The **median** and the **mode** are two other ways of describing the centre.

The **median** of a set of numbers is intuitively speaking the middle number when the data is arranged in order. It is well-defined if the size of the set is odd because there is a clear middle. When the size is even, then the middle two numbers are averaged (i.e. by adding them, then dividing by 2).

The **mode** of a set of numbers is the number(s) which occurs the most. The mode does not need to be unique. In the case when the frequency is the same for every data value, then there is **no mode**. ◀

The increase in crimes related to **child pornography** is partly due to ease in reporting. The **US Justice Department** tries to educate the public about the victims of child pornography. Unlike pornography involving adults where consumers may reason that porn stars like what they are doing and choose the profession willingly, children have no political voice and are overpowered physically by adults abusing them.

charges while making him sign a contract for not suing them. What's the justice in that?"

That led Florence to the book, *Just Mercy* [32] by Bryan Stevenson of the *Equal Justice Initiative*. "Hans, I have so much respect for your work promoting public health for poor countries. Nevertheless, in rich countries like Canada and the USA, the poor not only get substandard health service if any, they also get almost no justice. Look at how many of the wrongfully convicted are poor because the rich can afford expensive lawyers to work the legal system. The infographic on *Homicide in Canada, 2020* shows a double bar graph comparing homicide rate for every 100 000 people between 2019 and 2020, and consistently, Alberta had a higher rate than B.C. or Quebec. Does Alberta have more poor people?"

Hans checked with *Statistics Canada*, but found answers on *statista* from a bar graph on median annual total income in Canada in 2019 by province. "This is a surprise, Florence. Even amid a downturn in Alberta's oil sand industry with many unemployed, Alberta still had a higher median annual income compared to Ontario, British Columbia, and Quebec. In addition, Alberta is the only province among these four without a sales tax. Perhaps we need to look at the income gap."

Florence jumped in, "The table above the bar graph in *Homicide in Canada, 2020* showed a disproportionate rate for the indigenous population [31]. Maybe comparing the sizes of indigenous population in each province can answer our question. As a nurse, I believe in prevention: good health, good home, and good education will liberate the poor from generational shackles of dysfunction while saving the government money. Your home country Sweden has done a great job helping the public see the importance of income equity. Unfortunately for North America, the *Tyranny of Merit* is really endangering the *common good* with the rich thinking that they deserve the wealth they possess while the poor, doomed to feel that they have only themselves to blame when in fact both groups forget the role chance plays in life. Here, Capilano University can be an example for the other universities

Open the data table for Chart 5 in the report, *Crime stats 2020 Canada* to find the average rate of child pornography reported from 2008 to 2020 inclusively.

The **average** rate is $(4 + 4 + 5 + 6 + 6 + 8 + 11 + 12 + 18 + 18 + 16 + 24 + 29)/13 \approx 12.4$ crimes involving child pornography per 100 000 of the population. Notice that when **rounding** answers in statistics, one gives one more significant digit than the data. In this case, one decimal place. With Google Sheets, one can begin a new column by entering one number per cell for the set of 13 data points. When = is entered in Cell 14, SUM (=SUM(A1:A13)) automatically appears to suggest computing the total of the previous 13 entries. When =AVERAGE(A1:A13) is entered in Cell 15, one finds the **mean**. This is easier than using the calculator.

The data from 2008 to 2020 are almost ordered in ascending order except for 2018 where the crime rate dropped to 16 from 18 in 2017. With 13 numbers ordered

4, 4, 5, 6, 6, 8, 11, 12, 16, 18, 18, 24, 29,

one can find the **median**, namely the middle entry, or the 7th number, i.e. 11 crimes involving child pornography reported per 100 000 people.

This set has three **modes**, all doubles: 4, 6, and 18 crimes involving child pornography reported per 100 000 people. <

Statistics Canada's *Table 2: Median after-tax income, Canada and provinces from 2016 to 2020* shows that Alberta consistently outnumbers the other provinces' median after-tax income by far.

again: Ed Lavallo, an instructor for 44 years, advocated for full benefit to part-time permanent faculty because he knew many of the faculty worked part-time to bring up children; in particular, the extended health benefit would enhance both mental and physical health of the family. Compare that to the number of *permanent* contract university lecturers (adjunct professors in the US), minimal benefit, no job security, barely making enough to live after many years of education while the university makes a lot of money on mega classes.”

“Talking about mega classes, many first year university courses are so large that even a big lecture theatre with a 400+ capacity in some universities still cannot hold the entire class of registered students, so many of the students need to watch the lecture on screen in a different lecture hall. Often, those money generating mega classes are taught by underpaid adjunct professors. In contrast, Capilano University caps its class size in the 30’s to enhance interaction between professors and students. What a privilege to receive such personal attention! Let’s look at the age distribution of a class as a frequency table (Table 6.6) and the age distribution of the same class earning minimum wage as a bar graph (Figure 6.7 on the following page).”

Ages	Frequency (f)
18	14
19	10
20	7
21	3

Figure 6.6: Class age distribution table

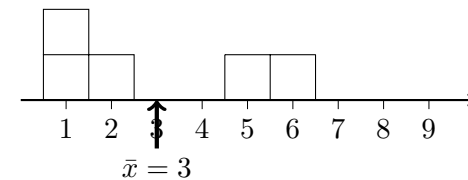
Florence couldn’t resist finding centres of the data Hans just showed her. “Just like what you did with the ages of Naomi’s children, I want to find centres of this age distribution, too. For the average age of the class, I can add the age, 18, fourteen times because the table shows

Michael Sandel wrote *The Tyranny of Merit: What’s Become of the Common Good* [27] during the Covid pandemic.

North American universities are not free from extreme income inequity; for instance, the *Adjunct professor movement* shone light on a *homeless professor* and other *plights of adjunct professors* as explained by the *rise of adjunct faculty*. University service staff earning minimum wage which is far below a living wage in big cities is a similar problem to underpaying the adjunct or temporary faculty. If universities endeavour to bring forth students with vision for a better world, they must first model such a better world in their campus economy.

Even after nearly one decade of awareness campaigns and struggles, in March of 2022, the University of California in Los Angeles (UCLA) still advertised for an *adjunct position with no pay* which raised an uproar beyond the academic community. Three days later, UCLA *reversed the policy* and began paying all its adjunct faculty.

1. The **mean** is the *balance point* or *middle* of the histogram or data set. Note that the “outliers” (abnormally large or small values) will affect the mean.



14 students at age 18, or use multiplication as a shortcut, 14×18 ; similarly for the other ages, we get a total of $\sum x_i = \sum (f \cdot x)_j = 14 \times 18 + 10 \times 19 + 7 \times 20 + 3 \times 21 = 645$. When we add the frequency $\sum f_j = 14 + 10 + 7 + 3 = 34$, we get the total number of students in the class. Thus, the average age is $645/34 \approx 19.0$ years. As for the median in a class of 34, we need to average the 17th and the 18th ages, but they are both 19 years, i.e. the median is 19 years old. The age that appears the most is 18 years, so the mode. How much easier it is to find the median and the mode with a frequency table!"

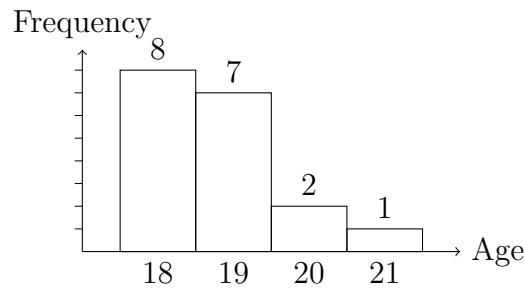
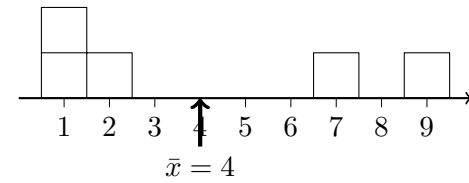
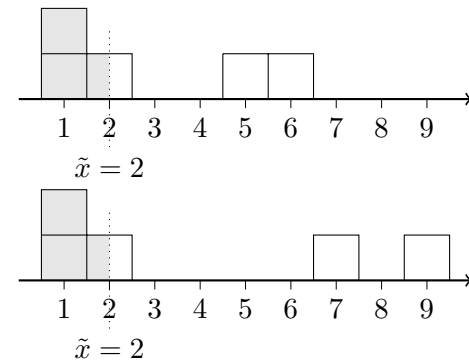


Figure 6.7: Minimum wage age distribution histogram

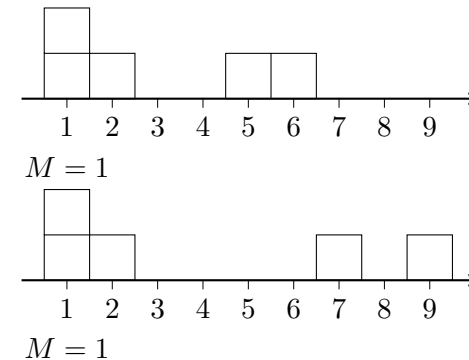
Hans continued with the bar graph. “A bar graph is similar to a frequency table: with the shortcut, $\sum (f \cdot x)_j = 8 \times 18 + 7 \times 19 + 2 \times 20 + 1 \times 21 = 338$, and $\sum f_j = 18$, the total number of students earning minimum wage, we have the mean age of $\sum (f \cdot x)_j / (\sum f_j) = 18.8$ years. For the median, averaging the 9th and the 10th number, both 19, one still gets a median of 19 years. The mode is 18 which appears 8 times. Talking about minimum wage, Sweden doesn’t have such an idea, but most of our citizens earn a living wage. When I compare **minimum wage by province**, I am surprised to see such a spread. It doesn’t seem enough to look at centres only in a data set. I calculate the mean minimum hourly wage to be 14.197 dollars per hour using $n = 13$ for the number of provinces and territories.”



- The **median** will divide the histogram into two equal areas. Note that “outliers” do not affect the median.



- The **mode** gives the location of the “peak” of the histogram where outliers do not affect the mode either.



Dispersion

Florence was also curious about the **spread** or **dispersion** of the data, so she began a column for the **deviation**. “Since the spread of the data regards both sides of the mean equally, I will square each deviation to get the sum, $\sum(x_i - \bar{x})^2 \approx 22.624$; the **variance** of all provinces and territories is the average, i.e. $(\sum(x_i - \bar{x})^2)/13$. The **standard deviation** is, then, the square root of the variance, or $\sqrt{(\sum(x_i - \bar{x})^2)/13} \approx 1.32$ dollars per hour.”

Province/ Territory	Minimum Hourly Wage (\$)	Deviation $x_i - \bar{x}$	Square of Dev. $(x_i - \bar{x})^2$
Alberta	15.00	0.803	0.6449
B.C.	15.65	1.453	2.1114
Manitoba	11.95	-2.247	5.0487
New Brunswick	13.75	-0.447	0.1997
Newf. & Labr.	13.20	-0.997	0.9939
NW Territories	15.20	1.003	1.0062
Nova Scotia	13.35	-0.847	0.7173
Nunavut	16.00	1.803	3.2511
Ontario	15.50	1.303	1.6980
P.E.I.	13.70	-0.497	0.2469
Quebec	14.25	0.053	0.0028
Saskatchewan	11.81	-2.387	5.6974
Yukon	15.20	1.003	1.0062
Average	$\bar{x} \approx 14.197$	0	$\sigma^2 \approx 1.32$

Figure 6.8: Minimum wage by province effective October 2022

“Before the invention of calculators, such computation was tedious. My students in Sweden could just enter data into a calculator, a spreadsheet, or a statistics program to get the **mean**, the total $\sum x_i$, the sum of the squares, $\sum(x_i^2)$, the **population deviation** σ , the **sample deviation** s , the size of the data set n , the

If a distribution is **symmetric**, the mean and the median are equal. ◀

The **deviation** from the mean of a data value x is $x_i - \bar{x}$. This is a signed value: positive for data values greater than \bar{x} , and negative for those less than \bar{x} .

The calculation for variance and standard deviation of minimum wage for provinces and territories is for the *entire population*. In the case of a **variance of a random sample**, it turns out that one needs to divide by $n - 1$ instead of n where n is the size of the data set. Statisticians use $\sigma^2 = (\sum(x_i - \bar{x})^2)/n$ to denote the **population variance**, and $s^2 = (\sum(x_i - \bar{x})^2)/(n - 1)$ to denote the **sample variance**; analogously, $\sigma = \sqrt{(\sum(x_i - \bar{x})^2)/n}$ for **population standard deviation** and $s = \sqrt{(\sum(x_i - \bar{x})^2)/(n - 1)}$ for **sample standard deviation**.

The **range**, i.e. maximum minus minimum data value, is the quickest way of finding the spread of a data set.

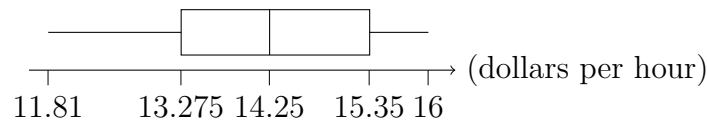
Using the data sets from the study of centres on pages 235–236:

Set 1: 1, 1, 2, 5, 6 versus Set 2: 1, 1, 2, 7, 9.

we compute sample standard deviation: for Set 1,

$$s = \sqrt{\frac{(3 - 1)^2 \cdot 2 + (3 - 2)^2 + (5 - 3)^2 + (6 - 3)^2}{5 - 1}} \approx 2.3$$

minimum, the **first quartile** Q_1 , the median (or the **second quartile**), the **third quartile** Q_3 , and the maximum. When I scan through the data set, the first approximation of dispersion is the **range**, here, $16 - 11.81 = 4.19$ dollars per hour. Of course, when we use a **box plot** to show a visual summary of these five numbers: minimum, Q_1 , median, Q_3 , maximum, we can communicate the spread of data efficiently.” Hans started drawing a box extended by whiskers on either side, left to the minimum, and right to the maximum. The box began with a left edge at 13.275, which marked the first quartile $Q_1 = 13.275$, i.e. one quarter (equivalently, 25 %) of the data less than Q_1 , then a middle edge marked by the median at 14.250, and ended with a right edge at $15.350 = Q_3$, the third quartile, all in dollars per hour.



“What a beautiful idea! The box plot is so compact.” Florence was curious about how her home country was treating their poor. “Hans, tell me about the poor in the British Empire.”

“The British Empire is no more. I can tell you about the Commonwealth, but even that is crumbling. During the Iron Lady, Prime Minister Thatcher’s rule, the United Kingdom, like the USA under President Reagan believed that less government regulation and more tax cuts for corporations would stimulate the economy. They argued that when the rich made money, it would **trickle down** to the poor through job creation. This policy widened the income gap irreparably, and condoned the corporations’ ruthless pursuit of profit above all costs—human and environmental alike. When socially conscious politicians wanted to raise minimum wage, conservatives would argue that such an increase, no matter how minuscule, would hurt businesses and the economy while increasing unemployment rate, thus ultimately hurting the poor. Fortunately, through **natural experiments** used in the

compared to Set 2,

$$s = \sqrt{\frac{(4 - 1)^2 \cdot 2 + (4 - 2)^2 + (7 - 4)^2 + (9 - 4)^2}{5 - 1}} \approx 3.4,$$

indicating a larger dispersion than Set 1. ◁

If using the spreadsheet, say Google Sheets, one first needs to have a column of just plain data, then click on DATA followed by COLUMN STATS to get a frequency bar graph with n , sum of data, \bar{x} , median, minimum, and maximum.

The first **percentile** marks the value where 1 % of all data falls below the value. One can find the median of the data less than the median to obtain the first **quartile**, i.e. the value above 25 % of the data values. When a baby’s height is **off the chart**, parents understand that the baby’s height, when plotted, is greater than the 100th percentile. ◁

A **box plot** or **box-and-whisker plot** is a visual representation of the **five-number summary** of a data set. It contains the minimum, the first quartile, the median (also called the second quartile), the third quartile, and the maximum. More advanced box plots contain **outliers**, namely, extreme values represented by dots. Such plots may be drawn horizontally or vertically. Vertical box plots with only four numbers, i.e. without the median, are called **candlestick charts** in Google Sheets. ◁

research of Card (a Canadian), Angrist, and Imbens, the winners of Nobel Memorial prize in Economics in 2021, one can now conclude that raising the **minimum wage** benefits the economy because the poor tend to put the money from their increased wages back into the economy in contrast to the rich who just save more, thus reducing economic activities. Just look at how poverty rate dropped in Canada during the pandemic because of the government's emergency response benefit (CERB)."

Florence reminded Hans of the data for income per month from **Dollar Street** of Gapminder. "Let's compare the mean and standard deviation of the data collected by Gapminder for income per month of South Africa which is a Commonwealth country and Sweden, both of which show many data values."

S. Africa	90	116	149	232	332	448	556	751	3755
Sweden	1289	1740	2074	2224	3057	3384	3439	4884	

Figure 6.9: Income per month in US \$

Right away, Hans announced, "I have an average of $\bar{x} \approx \$714.3$ per month for South Africa which is not quite satisfactory for me because the highest income of \$3755 per month looks like an outlier in the set. However, keeping it, I have the sample standard deviation to be approximately $s \approx \$1161.3$, very wide spread. As for Sweden, I get an average of $\bar{x} \approx \$2761.4$ per month while $s \approx \$1159.6$, almost the same standard deviation. To gain more insight, we should compare each s relative to its mean, \bar{x} , namely, compute the **coefficient of variation**, CV, which is 163% for South Africa and 42% for Sweden. That is a more convincing method for quantifying income inequity: the larger the CV, the wider the income gap."

"Thank you, Hans. Technology really makes quantitative statistical analysis easier. With such help, students no longer need to spend hours crunching numbers, but rather can focus their time and energy

A 2003 Canadian documentary produced by TV Ontario, **The Corporation** based on the book, **The Corporation: The Pathological Pursuit of Profit and Power [2]** by UBC law professor, **Joel Bakan**, explains how personhood when granted to corporations makes them psychopaths (explained by Bakan from 17:00 on the video) who have insatiable greed, no moral standards, and feel no guilt or shame in exploiting the environment and people for their personal gain. Equipped with the financial power to lobby governments, corporations acting as psychopaths are not only threatening the environment, assaulting the health of people but also upending liberal democracy unless people actively seek policy changes.

More examples from natural experiments like the **association between education and income** and **when to be born** for higher earnings may not be relevant anymore post the pandemic. As labour shortages continue, hard working, skilful people regardless of their degree credentials will be able to attain financial security.

On Gapminder's **Dollar Street** page, one can select specific countries for comparison.

The **coefficient of variation** is a comparison of the standard deviation relative to the mean.

$$CV = \frac{s}{\bar{x}} \cdot 100\% \quad \triangleleft$$

on the understanding, interpretation, and discussion of the analysis.” Florence expected wonderful social changes to happen as a result of more statistical analyses.

Hans could not wait to show Florence a treasure trove of data to analyze and interpret, but before diving into that, he returned to the Counselling/Wellness Centre. “Many techniques in psychotherapy have been developed since the time you treated war casualties. With VIV, I would love to make these methods available to all. Imagine how we can use, for example, [eye movement desensitization and reprocessing \(EMDR\)](#) on avatars to help the traumatized! I am sure some software engineers and neuroscientists are collaborating in a laboratory somewhere in the world to make this happen.”

6.3 MAKING SENSE OF BIG DATA—TAKING ADVANTAGE OF OPEN SOURCE

“Hans, how did you collect data for your studies?” Florence’s curiosity was bottomless. “I had to do all recording by hand.”

“That is something I would love to show you. Since 2002 when global information storage turned more than 50 % digital, humans have been trying to figure out what to do with [big data](#). Those who believed that more data would make better predictions allocated large resources into the collection and analysis of big data; however, [big data investments](#) were very expensive without its matching benefit because data alone were no model for human behaviours. At Gapminder, we either compile data manually or get high quality data directly from organizations like the [World Bank](#) to analyze with our [tools—bubble charts, line chart, age pyramids, and mountains](#).” Hans showed Florence some the tools.

Florence could not believe the amount of data so readily available all in one place. “You are showing me data for global comparisons. Now suppose a student wants to focus on Canada, then the [StatsCan](#) app will make available Canada’s data. Whatever people want to know, information is at their fingertips *instantly*.”

In 2001 when [Robert Bilott](#) convinced 69 000 Americans from West Virginia to participate in a study of the effect of perfluorooctanoic acid (PFOA) and perfluorooctanesulfonic acid (PFOS) dumped by the giant chemical corporation, DuPont, it took 7 years to analyze their blood samples. Now Gapminder synthesizes over 10 000 data values into one graph in minutes.

[UK Biobank data](#) hosts open source data for researchers, most of whom are in medicine. More [open source tools](#) are available for data analysis.

[Tools](#) from [Gapminder](#) allow one access to different sets of big data for further exploration. Click on Tools on the page: *Teaching with Gapminder* to see available options.

[Our World in Data](#) and [USA Facts](#) provide more data for in depth analysis on a broad spectrum of topics.

The [F-scale](#) addresses upbringings which make people susceptible to fascist ideals; however, interviews of the participants in Ottawa’s February 2022 [freedom convoy](#) which the app Zello helped organize, showed that most participants would not be classified as fascist leaning, just no longer trusting authorities, governments, and science.

Hans felt compelled to jump in to warn Florence of the dangers of information. “Information *and* misinformation are both at their fingertips, so consumers need to discern whether the source of information is trustworthy. Another issue with big data is breach of privacy. For instance, when corporations take control of customers’ purchase history and browsing patterns as their own big data to bombard consumers with advertisements, it is too late for these customers to protect their personal data. Like any new invention, humans can choose to use it for good or bad. When Google switched from traffic cameras to live reporting of GPS’s for tracking **current traffic** via Google Maps, all the GPS’s in cellphones contributing to the big traffic data work for good. When social media cocoon users in their own echo chambers and help them organize hate crimes and violent social unrest, big data may eventually topple liberal democracy. My eternal optimism gives me hope that humans will figure this out eventually; just look at the work from **Transparency International** fighting corruptions around the world, **World Economic Forum**’s rethinking of corporations, the awareness of war-zone sexual violence against women leading to an open conversation on **justice for women in war**, **crowdsourcing to evacuate** women and children out of Ukraine, and the United Nations’ **sustainable development goals**.”

Florence completed Hans’s deliberation for him. “We *will* make the Virtualverse a haven for learning, sharing, and research so that our students can go on to change the world for the better.”

6.4 EXERCISES

Tally and frequency tables

6.1 When serving in Mozambique, Hans once visited a village to study family sizes. Among the ladies surveyed, he found the following data

Tesla’s autopilot program continuously gathers data from Tesla’s own cars on the road to make self-driving vehicles a reality sooner and safer.

Covid data from Canada guide Canadian public policy makers to slow the spread or relax restrictions cautiously. The **analysis of poverty** from the World Bank brings attention to those made destitute by Covid in countries without social welfare.

As part of World Economic Forum’s Davos Agenda from 2022, private equity (private investors) are encouraged to lead the charge on **environmental, social, and corporate governance** (ESG) strategy. Because money speaks louder than people, by investing only in companies committed to improving ESG matters, more companies will be forced to implement good ESG practice. Investors need to demand transparency in reporting with globally comparable metrics.

The International Monetary Fund hosts **world economic outlook database** for interested visitors.

Data scientists are in demand in today’s data driven world. For students interested in exploring the field, **Allen Downey** offers many free introductory books from **Green Tea Press**. For condensed learning, **Kaggle** provides both the code and data needed to do data analysis.

for the number of children each birthed.

10, 11, 4, 8, 12, 12, 10, 10, 6, 8, 11, 9, 9, 7, 9, 10, 11, 11, 13, 15,
11, 8, 10, 12, 10, 5, 8, 6, 3, 0, 0, 2

- Tally the data into four classes.
- Construct a frequency table with four classes.
- Draw a histogram with four bars from the frequency table.

6.2 One year in the nursing school, Florence conducted a survey for a class of students regarding the number of alarm clocks each needed in the morning to get out of bed. The results are as follows:

0, 1, 1, 0, 1, 2, 0, 0, 2, 2, 2, 0, 0, 0, 0, 0, 1, 1, 3, 1, 1, 1, 0, 2, 0, 0, 2, 2, 3, 0

- Tally the data.
- Construct a frequency table.
- Draw a bar graph.

[IM]

6.3 Capilano University student society is organizing a tree planting event called Tree for Life. The ages of the volunteers are as follows.

18, 18, 21, 23, 20, 18, 19, 19, 19, 23, 19, 21, 21, 22, 23, 24, 37, 48, 25, 23,
19, 19, 20, 21, 20, 22, 22, 27, 23, 24, 28

- Draw a stem-and-leaf plot.
- Construct a frequency table with four classes. Also show relative frequency.



Almost all data sets from examples in this chapter are gathered without treatments, namely, from **observational** studies. The testing of effectiveness for different Covid vaccines requires **experimental** studies or **double blind** designs. Please watch the 10-minute video on **types of statistical studies**. The video also addresses the difference between **correlation** and **causation**.

- c. Draw a histogram using relative frequency on the vertical axis.

6.4 When serving in Sweden, Hans devised a 30-point capacity assessment system for the elderly. One day in a geriatric ward of the hospital, the following scores were gathered.

19, 11, 24, 18, 22, 12, 16, 19, 26, 18, 21, 19, 9, 17, 19, 10, 11, 11, 23, 15,
21, 18, 14, 14, 19, 25, 18, 26, 13, 20, 10, 12

- a. Tally the data into five classes.
b. Construct a frequency table with five classes.
c. Draw a histogram with five bars from the frequency table.

[IM]

6.5 While serving in the Crimean War, Florence tracked for one month the number of times she got up during the night to check on the patients. The results are as follows:

3, 1, 1, 3, 1, 2, 3, 3, 2, 2, 2, 4, 4, 4, 3, 4, 1, 1, 3, 1, 1, 1, 4, 2, 3, 1, 2, 2, 3, 3

- a. Tally the data.
b. Construct a frequency table.
c. Draw a bar graph.

6.6 One day while serving in the Crimean War, Florence collected the ages of wounded soldiers under her care. The results are as follows:

20, 21, 18, 20, 19, 23, 30, 20, 22, 22, 23, 20, 22, 20, 20, 20, 18, 18, 31, 19,
17, 41, 30, 25, 30, 20, 32, 42, 43, 40, 24, 20, 21, 21, 51, 25, 30, 35, 27, 27

- a. Draw a stem-and-leaf plot.

- b. Construct a frequency table with six categories.
- c. Draw a histogram with six bars.

[IM]

Line graphs

6.7 Hans found in [Statistics Canada's Table 18-10-0001-01](#): Monthly average retail prices for gasoline and fuel oil, by geography. Under Vancouver, B.C., prices of regular unleaded gasoline in cents per litre from April 2021 to March 22 are as follows:

152.2, 155.6, 161.1, 169.1, 165.6, 156.7,
162.4, 159.8, 157.2, 170.1, 177.1, 198.3

Under Edmonton, Alberta, during the same period, he found the following.

119.9, 123.4, 125.6, 133.0, 132.3, 129.5,
138.3, 136.0, 130.6, 133.7, 142.2, 161.7

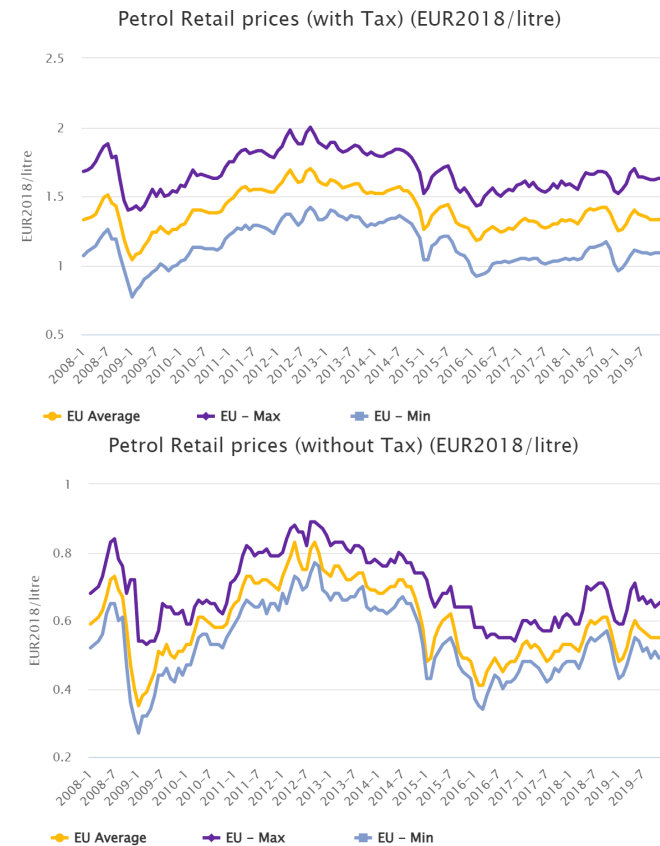
- a. Draw a line graph for each metropolis on the same coordinate system.
- b. Explain the jump of around 20 cents from February to March.
- c. Find the percentage increase of gasoline prices from February to March for each metropolis.

6.8 Hans visited [Eurostat's Energy statistics](#) page and found two charts for gasoline prices with and without taxes. Estimate the tax rate on gasoline in Europe before 2020.

[IM]

When data is not provided in exercises, students are expected to find information directly from the source to answer questions. Students learning to become engaged global citizens need to develop proficient digital literacy.

From Eurostat's Energy statistics dashboard:



6.9 Hans visited [United Nations' Energy Statistics](#) and found a free download of Excel spreadsheet under Energy statistics dashboards. The line graph of Energy highlights up to 2019 showed an increase in energy consumption for many sectors. Which sector had the largest percentage increase from 2018 to 2019?

Pie charts

6.10 Florence found Capilano University's [Executive Compensation Disclosure Statement](#) showing the total compensation for 2020/2021:

- President: \$266 450
- VP academic & provost: \$208 461
- VP Strategic planning, assessment & institutional effectiveness: \$193 341
- VP finance & administration: \$98 934
- VP (2nd) finance & administration: \$63 095
- VP, University relations: \$185 240

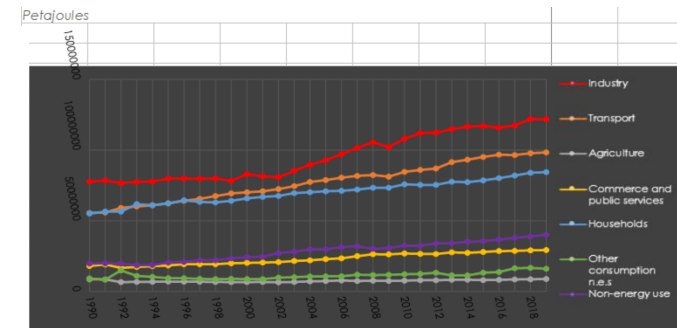
a. Draw a circle pie chart for the data.

b. Draw a square pie chart.

6.11 Florence found Capilano University's 2020 [Financial Report](#) for the year ended on March 31, 2021. On page 2 of the report under Revenue for the actual column 2021 beginning with the provincial government's contribution of \$46 542 936, design a circle pie chart for actual revenue in 2021. [IM]

6.12 Florence found Capilano University's 2020 [Financial Report](#) for the year ended on March 31, 2021. On page 2 of the report under Expenses for the actual column 2021 beginning with Instruction and student support of \$69 744 057, design two square pie charts for actual expenses in 2021 and in 2020.

From UN's Energy Statistics:



6.13 Hans showed Florence a circle pie chart of the **market value of the endowment** as of June 30, 2020 from Harvard University (p.8 of the financial report). How much was in the endowment for the faculty of art and sciences? [IM]

6.14 Florence noted on page 9 of the **financial report** from Harvard University that the endowment had a return of 7.3%. From the table under Performance on page 9, draw a circle pie chart for asset allocation.

6.15 Hans found the **financial report** from the University of British Columbia which stated that the endowment had a return of 16.8% in 2021. From the table of asset allocation on page 2, draw a circle pie chart for items greater than 10%. [IM]

6.16 Hans showed Florence the **financial report** from the University of British Columbia with a circle pie chart on page 1 showing the number of endowed funds by designation. What percentage of such endowed funds was under Student Awards?

Multibar and stacked bar graphs

6.17 Florence found an article from the University of Alberta's China Institute: **Canada-China Trade: 2021 Year in Review** and was surprised by the record setting export of \$28.84 billion to China in 2021. According to the first double bar graph: Canadian Exports to China, 2020 vs. 2021, answer the following questions.

- Which month(s) did 2021 have smaller exports than 2020?
- Which month was the difference in exports between 2020 and 2021 greatest? By how much?
- Which month was the difference in exports smallest?

Capilano University's **financial report** showed an endowment of \$11 869 771 for 2021, compared to over \$2.59 billion in UBC endowment and over \$41 billion USD for Harvard University's endowment.

Darren Choi from University of Alberta's China Institute prepared the following two graphs.



- d. Notice that for July and August, the billion dollar amounts are switched between 2020 and 2021. Why is the percent change year-over-year (YoY) different regardless of the + or – sign?

6.18 Hans joined Florence in studying both double bar graphs from the University of Alberta’s China Institute: [Canada-China Trade: 2021 Year in Review](#) to study Canadian export to and import from China in 2020 and 2021. To understand Canadian [trade deficit](#) with respect to China.

- a. Make a double bar graph to compare monthly exports (first bar) with imports in 2020.
- b. Make a double bar graph to compare monthly exports (first bar) with imports in 2021.
- c. Compare these two double bar graphs. What can you infer about Canadian trade with China during the pandemic?

[IM]

6.19 Hans showed Florence an April 2019 report from Girl Guides Canada: [Girls on the job: Realities in Canada](#). Answer the following questions.

- a. What do the percentages mean in the double bar graph on page 9 of the report under *Gender matters at work*?
- b. List all the wage gaps stated on page 10 of the report under *The gender wage gap starts young*. Which wage gap is most relevant for you?
- c. If this double bar graph is presented as a stacked bar graph, what characteristic of the data is emphasized? What is lost from the double bar graph?

6.20 Florence showed Hans a poster from the United States Census Bureau: *Women's Earnings* from a 2019 American Community Survey. Answer the following questions.

- a. According to the graph, which occupations have nearly equal pay for men and women?
- b. Which occupation has the greatest gender pay gap?
- c. Both professional athletes and movie actors are not on the graph. Either for Canada or the US, find the gender pay gap in either one of the professions.

[IM]

6.21 Florence showed Hans a poster from the United States Census Bureau: *Changes in Women's Participation in Selected Occupations Since 2000* from 2000 Census and 2019 American Community Survey. Answer the following questions.

- a. According to the graph, which occupation has the greatest increase in women's participation?
- b. Which occupations have a decrease in women's participation?
- c. Which occupations that used to be predominantly male now have a female majority?

[IM]

6.22 Florence showed Hans two triple-stacked bar charts (pages 7, 8) on Blood Alcohol Concentration (BAC) from *Mother's Against Drunk Driving*. Assume that one is five times more likely to get into an accident in the green zone than if one had no drinks, and 25 times more likely in the red zone, answer the following questions according to the charts.

The **gender pay gap** has been getting more attention since more people made their salaries public through platforms like **Glassdoor**.

- a. Suppose Jacqueline weighs 130 pounds and has two drinks in 2 hours. If she then drives, estimate her blood alcohol concentration. How much more likely is she to have an accident than if she had refrained from drinking or driving?
- b. If Johannes weighs 180 pounds and has five drinks in 2 hours. If he then drives, estimate his blood alcohol concentration. How much more likely is he to have an accident than if he had refrained from drinking or driving?
- c. Suppose Rita weighs 100 pounds and has two drinks in 3 hours. If she then drives, estimate her blood alcohol concentration. According to the chart, is she committing a provincial infraction or a criminal offence?
- d. Suppose Bobby weighs 225 pounds and has seven drinks in 3 hours. If he then drives, estimate his blood alcohol concentration. According to the chart, is he committing a provincial infraction or a criminal offence?
- e. If Wayne weighs 180 pounds and drinks a six-pack of beer during a 3-hour football game. Can Wayne legally drive after that?

6.23 Hans found a stacked bar graph combined with a line graph on **Police-reported incidents of human trafficking** from Statistics Canada's report on Trafficking in persons in Canada from 2019. Use Chart 1 of the report to answer the following questions.

- a. Using the data table for Chart 1, compare the percentage increase of the total human trafficking incidents from 2014 to 2015, and from 2018 to 2019. Which one is larger?
- b. Draw the graph as a double bar graph. What characteristic is lost from the stacked bar graph? What is gained?

Before drinking, many groups decide their own designated driver or use a **designated driver service** like **DADD**, Dial A Designated Driver.

Mothers Against Drunk Driving (MADD) also inspired the same acronym *Mathematicians Against Drunk Deriving*.

Human trafficking is a complex problem, specifically regarding how to free runaway teens and young adults because the traffickers are expert psycho-manipulators who often succeed in trapping young people by offering them housing security, emotional support, and material comfort often including illegal drugs. Learn more from a Canadian social worker employed by the United Nations, also a **victim of human trafficking**. When these young people have no housing or employment possibilities, what choice do they have?

- c. Explain the differences in the labels for the vertical axes between the left and the right.

6.24 Florence found an article from the Washington Free Beacon on [What We Talk About When We Talk About Deaths of Despair](#) where a triple stacked bar graph broke down the causes of [deaths of despair](#) into over-consumption of alcohol leading to liver failure, suicide, and drugs. How many deaths of despair occurred in the United States in 2017?

Centres

6.25 Florence visited instructors from the Department of Mathematics and Statistics and gathered their ages:

27, 35, 42, 42, 44, 45, 46, 47, 49, 51, 56, 59.

Answer the following questions. Hint: Include units.

- Find the average age.
- Find the median.
- Find the mode.

6.26 Hans went into a second year statistics course to measure the heights of students in centimetres.

177, 165, 159, 182, 179, 170, 175, 188, 169,
156, 176, 180, 181, 179, 183, 166, 171, 183

Answer the following questions. Hint: Include units.

- Find the average height.
- Find the median.

The total number of US soldiers and military contractors [killed in Iraq](#) between March 2003 and October 2018 is $4550 + 3793$; In Afghanistan between October 2001 and October 2018, the total number is $2401 + 3937$. The combined number of deaths spanning over 15 years from both wars is 14 681, about one tenth of the deaths of despair from just one year, 2017.

Does the new policy in [Medical Assistance in Dying](#) or MAID open the door to those Canadians trying to escape intolerable lives? Listen to Goldman from CBC's [White Coat Black Art](#) interviewing Dr. Naheed Dosani, a palliative care physician and health justice advocate.

[Diseases of despair](#) inspired by the book: [Deaths of Despair and the Future of Capitalism](#) [1] from economist [Anne Case](#) and Nobel Prize winner [Angus Deaton](#) gave voice to the plight of the working class after they suffered four decades of economic globalization through free trade. David Introcaso wrote a commentary on [Deaths of despair: The unrecognized tragedy of working class immiseration](#) published on Dec 29, 2021 from STAT news's First Opinion.

c. Find the mode.

[IM]

6.27 Florence visited a class of engineering students and asked for the number of siblings each had.

0, 1, 1, 2, 0, 0, 2, 2, 0, 1, 1, 1, 0, 2,
1, 2, 1, 0, 0, 1, 1, 2, 3, 0, 3, 8, 0, 1, 1, 1, 1.

Answer the following questions. Hint: Include units.

- Find the average number of siblings.
- Find the median.
- Find the mode.

6.28 Hans visited a class of business students and asked for the number of cell phones each had ever owned.

4, 1, 2, 2, 3, 3, 2, 2, 4, 1, 3, 1, 3, 2,
1, 2, 2, 5, 5, 3, 1, 2, 3, 5, 3, 8, 6, 2, 3, 3, 2.

Answer the following questions. Hint: Include units.

- Find the average number of cell phones ever owned.
- Find the median.
- Find the mode.

[IM]

6.29 Florence visited a class of English majors and asked for the number of social media platforms each had. Pew Research Center published [Social Media Use in 2021](#).

10, 11, 4, 2, 3, 7, 5, 5, 5, 7, 6, 2, 3, 2,
4, 2, 4, 4, 4, 4, 10, 2, 3, 6, 7, 8, 8, 10, 9, 3, 3.

Answer the following questions. Hint: Include units.

- Find the average number of social media platforms.
- Find the median.
- Find the mode.

6.30 Hans showed Florence a frequency bar graph for the scores of a class of political science students after taking a pop-quiz on Gapminder. Answer the following question according to the bar graph of Figure 6.10.

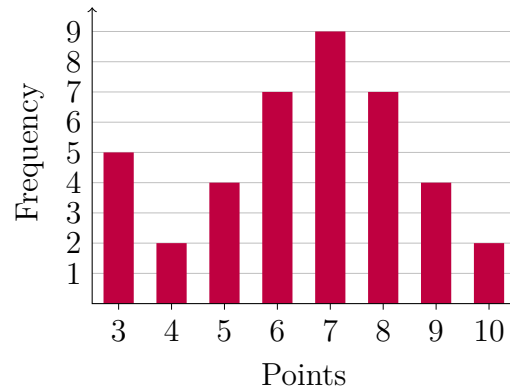


Figure 6.10: Quiz score distribution

- How many students took the quiz?

- b. Find the mean score.
- c. Find the median.
- d. Find the mode.
- e. Directly from the bar graph, is it possible to order the mean, median, and mode in ascending order?

6.31 Florence conducted a survey of biology classes regarding class sizes. She found the following expressed as a frequency bar graph in Figure 6.11.

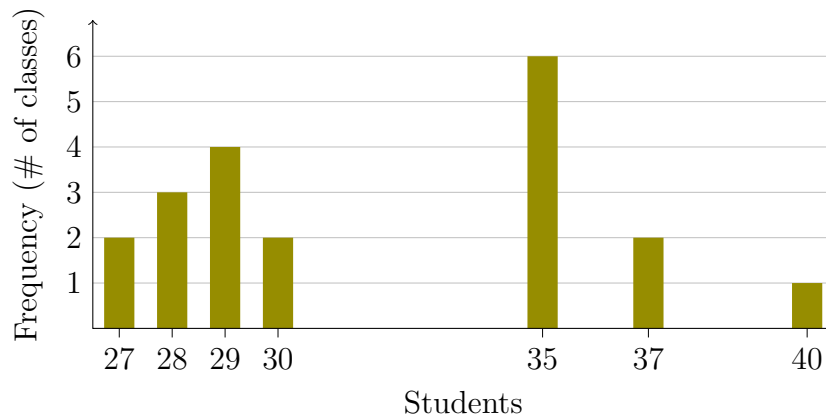


Figure 6.11: Class size distribution

- a. How many biology classes participated in the survey?
- b. Find the mean class size.
- c. Find the median.
- d. Find the mode.

- e. Directly from the bar graph, is it possible to order the mean, median, and mode in ascending order?

[IM]

6.32 Hans came upon a sample data of some members of the staff of **Lynn Valley Care Centre** tracking the number of contacts each had in the last week of March, 2020. See Table 6.12. Answer the following

Stem	Leaves
0	8 8
1	2 5 7 7 9
2	1 1 8 9
3	3 4 5 7 9
4	0 2
5	7
6	3

Figure 6.12: A stem-and-leaf diagram of staff contacts

questions based on the stem-and-leaf diagram.

- How many staff members were sampled?
- Find the mean number of contacts.
- Find the median.
- Find the mode.
- Construct a frequency table for the number of contacts grouped into 7 classes. Also include relative frequency and cumulative frequency columns.

A **cumulative frequency** of a certain event is the total frequency of all the events up to and including that certain event.

6.33 In VIV, Florence often gets visits from student avatars who are panicking near the end of a semester. Once, Avatar Alex visited and showed Florence that they scored 84, 69, 47, and 58 in World Politics with only one test left. If all five tests were out of 100 and weighed equally, and a B+ required at least a mean score of 76, was it possible for Alex to receive a B+?

6.34 After Florence showed Alex that they could not get a B+, Alex re-examined their scores of 84, 69, 47, and 58 in World Politics with only one test left. If all five tests were out of 100 and weighed equally, and a C− required at least a mean score of 60, was it possible for Alex to receive a C−? [IM]

6.35 Another time in VIV, Avatar Julian visited Florence about their final exam. Julian had all six test scores, each out of 100, 71, 88, 69, 57, 75, 82, split evenly for a total of 65 % of the course. Julian knew that the final exam which weighed 35 % of the course had a total of 100 also. At least how much should Julian get in the final to receive a mean of 80 for an A−?

Dispersion

6.36 Florence showed Hans some test scores from a class of her nursing students, see Table 6.6.

Score	90	80	70	60	50	40
Frequency	2	5	9	7	3	1

Table 6.6: Test scores

- Find the number of students who wrote that test.
- Find the range.

- c. Find the mean, median, and mode.
- d. Draw a box-and-whisker plot for the distribution of the number of points scored by nursing students in the test.

6.37 Hans showed Florence the following Gapminder scores from a group of journalism students, see Table 6.7.

Score (/18)	13	12	11	10	9	8	7	6	5
Frequency	2	1	4	3	5	2	1	1	2

Table 6.7: Quiz scores

- a. Find the number of journalism students who tried the quiz.
- b. Find the range.
- c. Find the mean, median, and mode.
- d. Draw a box-and-whisker plot for the distribution of the number of points scored by journalism students in the Gapminder quiz.

[IM]

6.38 Florence found some data from a sample of seniors living in Richmond Lions Manor regarding the number of visitors each had one week in May. Answer the questions using Figure 6.13 on the following page.

- a. How many seniors participated in the survey?
- b. Find the mean number of visitors in a week.
- c. Find the range.

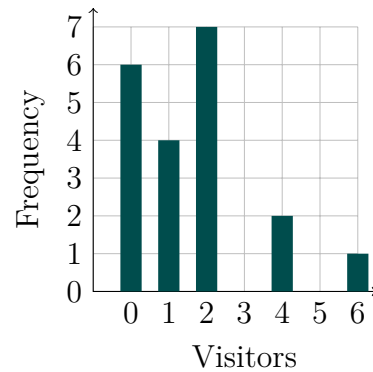


Figure 6.13: Visitors recorded for one week in May

- d. Find the sample variance.
- e. Find the sample standard deviation.
- f. Find the coefficient of variation.

6.39 Hans and Florence designed a dog park in VIV where visitors brought their virtual dogs for training tips. The visitors were asked the number of physical dogs they had ever owned. Answer the following questions.

1, 4, 3, 2, 3, 2, 2, 2, 4, 5, 4, 2, 2, 3, 0, 1, 1, 1, 2, 4, 6, 1, 2, 2

- a. How many visitors came to the dog park in VIV?
- b. Find the mean number of physical dogs ever owned by the visitors.
- c. Find the range.
- d. Find the sample variance.
- e. Find the sample standard deviation.

- f. Find the coefficient of variation.

[IM]

6.40 Florence was asked to improve service time at Capilano University's clinic. In order to do so, Florence gathered the waiting times (in minutes) for patients visiting the clinic over five consecutive days during the final exam period in April. Answer the following questions using Table 6.8.

Time (min)	0–9	10–19	20–29	30–39	40–49	50–59
Frequency	10	13	22	8	5	1

Table 6.8: Wait times at CapU's clinic

- How many patients' waiting times did Florence record?
- Find the mean waiting time.
- Find the range.
- Find the sample variance.
- Find the sample standard deviation.
- Find the coefficient of variation.

6.41 Florence and Hans as educators both understood the significance of digital literacy. After studying OECD's PISA report *21st-Century Readers—Developing literacy skills in a digital world*, they focused on Figure 1.2 on page 21 of the report which showed a 3-number summary. Answer the following questions.

When given a frequency distribution of classes, use the **class midpoint** (also called **class mark**) in calculations of mean, sample variance, and sample standard deviation. For example, in the class 0–9 minutes, the **class midpoint** is $(0 + 9)/2 = 4.5$ minutes.

The Organization for Economic Co-operation and Development, **OECD** in short, provides data, policy advice, and research for many countries in the world. One of the most well-known assessments from the OECD is the Programme for International Student Assessment, or **PISA**.

- a. Which country surveyed had the biggest change between 2012 and 2018 in the number of hours per week spent by students using the Internet?
- b. Which country had no change between 2015 and 2018 in the number of hours per week spent by students using the Internet?
- c. Which country surveyed had the biggest change between 2015 and 2018 in the number of hours per week spent by students using the Internet?

6.42 Florence and Hans continued studying OECD’s PISA report [21st-Century Readers—Developing literacy skills in a digital world](#). On page 40 of the report, an up-to-4-number summary chart was shown in Figure 2.1. Answer the following questions.

- a. Which country surveyed ranked first in economies and the percentage of students who reported having access to the Internet and a computer to use for schoolwork at home in 2018?
- b. How many countries were in 2018 still below the 2003 average?
- c. Which country improved most between 2009 and 2018?

[IM]

6.43 Florence found three box plots in Statistics Canada’s [Data Exploration’s Visualizing the box and whisker plot](#). Answer the following questions using Chart 4.5.2.1. The small circles indicate **outliers**.

- a. Find the first quartile, the median, and the third quartile of box A.
- b. Including outliers, which of A, B, or C has the largest range?

The **arithmetic mean**, simply called the **mean** or **average** in this chapter, is one of the **Pythagorean means**. The other two means are the **geometric mean** and the **harmonic mean**. With respect to *machine learning*, each of the means has a different purpose. Please read Brownlee’s blog: [Arithmetic, Geometric, and Harmonic Means for Machine Learning](#) for more detail.

- c. The **inter quartile range** (IQR) is the difference between Q_3 and Q_1 . Which has the largest IQR?
- d. Which one has the greatest sample variance?

[IM]

6.44 Florence showed Hans a poster from Canadian Perinatal Surveillance System: *Birth Weight for Gestational Age*. Answer the following questions.

- a. If a Canadian baby boy is born after 38 weeks in the womb and weighs 2.8 kg, which percentile does he fit on the graph?
- b. If a Canadian baby girl is born after 41 weeks in the womb and weighs 4.3 kg, which percentile does she fit on the graph?
- c. Home birth assistants feel safer to have newborns weigh at least 3.0 kg. For both boys and girls, find their gestational ages when their median weight reaches 3.0 kg.

[IM]

6.45 After Hans read that the data source for the *Birth Weight for Gestational Age* poster was based on a paper from 2001, Hans looked up a more recent data for Canadian baby girls born at 39 weeks of gestational age. Here is a sample of 10 data points for baby girls' weights in grams.

2973, 3722, 3026, 4105, 3125, 3590, 3220, 2800, 2550, 4305

Draw a box plot for the sample. Compare to the table from *Birth Weight for Gestational Age*, what can one conclude about the change in gestational weight?

An **outlier** is an extreme value in a data set, either extremely small or extremely large. There are many different methods in computing outliers for different distributions, but the simplest one relative to the box plot is **Tukey's fences** suggested by **John Tukey**: Let Q_1 and Q_3 denote the lower and upper quartiles respectively; an outlier lies outside the interval,

$$[Q_1 - 1.5(Q_3 - Q_1), Q_3 + 1.5(Q_3 - Q_1)]$$

6.46 After Hans read that the data source for the *Birth Weight for Gestational Age* poster was based on a paper from 2001, Hans looked up a more recent data for Canadian baby boys born at 35 weeks of gestational age. Here is a sample of 10 data points for baby boys' weights in grams.

3330, 3505, 2040, 2205, 3105, 3090, 3230, 1800, 1555, 2356

Draw a box plot for the sample. Compare to the table from *Birth Weight for Gestational Age*, what can one conclude about the change in gestational weight? [IM]

6.47 Hans found a report from [IMFBlog](#) on *Africa Faces New Shock as War Raises Food and Fuel Costs* due to the war in Ukraine. According to the charts under *Rising food insecurity*, answer the following questions.

- How much wheat did Russia export to Sub-Saharan Africa?
- Which Sub-Saharan African country imported the greatest percentage of wheat from Russia/Ukraine?
- In the bar graph with three bars labelled AE, EDME, and SSA, explain what 60.3 means for SSA.

[IM]

6.48 Hans and Florence were both educators and would like to know how much different governments spent on education. They found a report from [Fraser Institute: Education Spending in Public Schools in Canada: 2019 Edition](#) and the US Census Bureau's poster: *US School System Current Spending Per Pupil by Region: Fiscal Year 2020*. Answer the following questions.

- According to the table on page 11 of the report from Fraser Institute, how much did BC spend per student in public schools for 2015/16?

- b. Which province in Canada had the second lowest spending per student in public school education for 2015/16?
- c. With an appropriate currency converter, find the state spending a similar amount per pupil to BC. Remember to compensate for the fact that the bar graphs from the Census Bureau are based on 2020 data.

6.49 Project suggestion: Explain the phrase “within 3.2 percentage points 19 times out of 20” in the following quote

Abacus surveyed 1005 Canadians selected randomly from its online panel of 75000. The pool was taken between April 11 and 15 and the pollster says the results have an accuracy comparable to random poll of within 3.2 percentage points 19 times out of 20.

Some key words to search are **confidence interval**, **inferential statistics**, **sampling parameters**, or visit the [source](#) on WordReference.com.

Fraser Institute describes itself as a non-partisan think-tank with a mission to improve the quality of life for Canadians. One way of determining whether a charity organization is non-partisan is to see the donor base, especially large donors.

Solutions to Selected Exercises

1.1 If you treat each region at a time, then from Katharina's ham slicing discussion, there are infinitely many lines splitting the region equally. However, such a line may not necessarily split the opposite region equally.

Let us use the symmetry of the circle to help us. When a line is drawn through the centre of the circle, it makes an angle θ from the positive x -axis (the horizontal axis pointing towards the right). Let $B(\theta)$ be the difference in areas between the black regions on the right of the line and the left of the line. This area changes from one sign (say positive) to the opposite. By the IVT, there exists an angle that splits the regions equally. By rotational symmetry, this line also splits the white region equally because a similarly defined function $W(\theta)$ for white areas has opposite signs from $B(\theta)$, and two functions agree exactly when $B(\theta) = 0 = W(\theta)$. See Figure [A.1 on the following page](#).

1.2 Use the same reasoning as shown in the text margin to find that May 18th is Harmony's birthday.

1.3 September

1.4 Howard's birthday is October 14th.

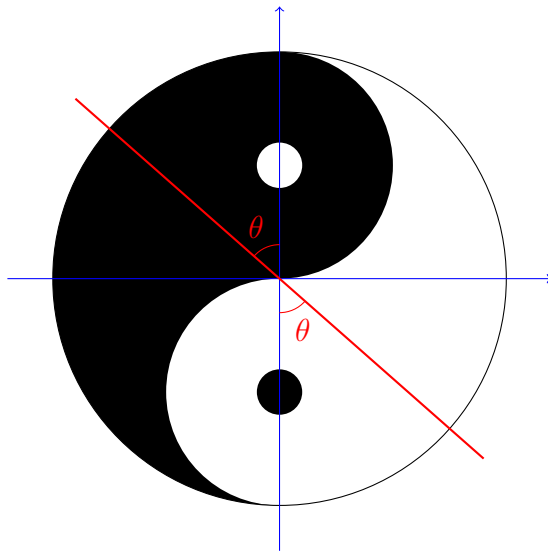


Figure A.1: Evenly divided Yin-Yang symbol

1.6 October

1.9

3^- 1	$2\div$ 4	2	4^+ 3
4	2 2	4^+ 3	1
$6\times$ 2	3	1	$2\div$ 4
4^+ 3	1	4 4	2

1.10 This problem is similar to the puzzle “Zara’s pass code” on [Ted-Ed](#). After the triple products of 36 are listed, one sees two equal sums: $1 + 6 + 6 = 2 + 2 + 9 = 13$, so knowing that there is an oldest daughter determines that the ages must have been 2, 2, and 9. Please go to the link for details.

1.11 Another way of generating triangular numbers. Like the handshaking problem, each new line needs to make an intersection with all previous lines, so $1 + 2 + 3 + \cdots + 9 = 45$.

1.12 This pattern is related to the triangular numbers. Beginning with 0 as the 0th triangular number, if you add one to each entry, you get 1, 2, 4, 7, 11, 16, finally, 22 regions with 6 lines.

1.13 The product of $11111 \times 11111 = 123454321$ because of the multiplication algorithm where the sum $11111 + 111110 + 1111100 + 11111000 + 111110000 + 1111100000$ shows that the ten-thousand’s place has five 1’s added together. This becomes a problem with ten 1’s except that if we remember to bundle the ten 1’s to the next place, then the answer is reachable without the calculator. Assume that **10** in the following number occupies the same billion’s place.

$$123456789\mathbf{10}987654321 = 1234567900987654321,$$

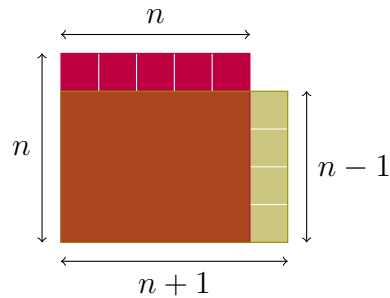
after rebundling tens to the next bigger place, that is, ten-billion’s place and hundred-billion’s place.

1.14 The next row, according to the pattern of adding two adjacent numbers from the row above, gives 1, 5, 10, 10, 5, 1. However, as a number, we must obey places, thus 161051 is the answer. One can check that $11^5 = 161051$.

1.17 If n is a number, then $(n-1)(n+1) = n^2 - 1$ as seen (in the case of $n \geq 1$) in [Figure A.2 on the next page](#). Therefore $4 \times 6 = (5-1)(5+1) =$

In geometry, switching lines and points gives dual results. The graph representing handshakes and the points of intersecting lines are duals of each other. For example, in [projective geometry](#), “two distinct lines on a plane meet at a point” is dual to “two distinct points on a plane determine a line.”

The pattern observed in [Solution 1.14](#) for the powers of 11 is described in [Pascal's Triangle](#)



The area of the region outside the rectangle is one unit larger than the area of the region outside the square.

Figure A.2: $(n - 1)(n + 1) = n^2 - 1$

$5^2 - 1 = 25 - 1 = 24$. The other examples follow the same pattern, in particular $899\,999\,999 \times 900\,000\,001 = 810\,000\,000\,000\,000\,000 - 1 = 809\,999\,999\,999\,999\,999$.

1.18 With Maple, one gets

0.004 115 226 337 448 559 670 781 893 004 115 226 337 448 559 670 781 893

a repeating decimal of 27 places. Long division confirms the repeating decimal.

1.19 $333\,333\,331 \div 17 = 19\,607\,843$, so it is not prime.

1.22 You may guess the answer right away, with 10 cm for Mei and 15 cm for Katharina. However, if we let Mei's piece be x cm long, then Katharina would receive $x + 5$ cm of the fruit roll up. All together, Mei's and Katharina's pieces add up to the original length. Therefore $x + x + 5 = 25$, or $2x = 20$, so $x = 10$. Mei's piece is 10 cm long.

1.23 Let Inderpreet's piece be x cm long, then Anjali who got 50 cm more would have $x + 50$ cm. Since the roll contains 6 m, or equivalently, 600 cm, we get $x + x + 50 = 600$, or

$$2x + 50 = 600, \quad 2x = 550,$$

```

0.004 115 226 337 448 559 670 781 893
243 ) 1.000 000 000 000 000 000 000 000 000
      972
      280
      243
      370
      243
      1270
      1215
      550
      486
      640
      486
      1540
      1458
      820
      729
      910
      729
      1810
      1701
      1090
      972
      1180
      972
      2080
      1944
      1360
      1215
      1450
      1215
      2350
      2187
      1630
      1458
      1720
      1701
      1900
      1701
      1990
      1944
      460
      243
      2170
      1944
      2260
      2187
      730
      729
      1
    
```

so $x = 275$. Therefore, Inderpreet had 275 cm and Anjali, 325 cm of ribbon.

1.24 Let Katharina's leftover share contain g gummy bears, then Anjali and Mei would each have $2g$ gummy bears. $2g + 2g + g = 5g = 30$, so $g = 6$. After sharing, Katharina had 6 gummy bears while Anjali and Mei each had 12.

1.25 Let Mei receive x pieces, then Anjali would get $2x$ pieces, and Katharina would get $3x$ pieces. All together,

$$x + 2x + 3x = 72, \quad \text{or} \quad 6x = 72,$$

so $x = 72/6 = 12$. Mei would get 12 pieces, Anjali, 24 pieces, and Katharina 36 pieces.

1.26 Let Anjali run x km. Then Katharina ran $x - 1$ km, and Mei ran $x - 23$ km. Therefore their total is $x + x - 1 + x - 23 = 3x - 24 = 72$, so $3x = 72 + 24 = 96$, so $x = \frac{96}{3} = 32$. Anjali ran 32 km for the fundraising campaign.

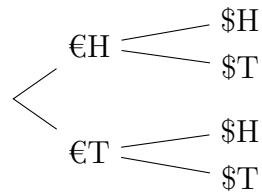
1.27 Let Simran's piece be x cm long. Then Anjali's must have been $3x$ cm long, and Gurpreet's, $2x$ cm long. Total $x + 3x + 2x = 6x$, which equals to 60 cm, so $6x = 60$ cm, so $x = 10$ cm.

Therefore, Simran had $x = 10$ cm, Gurpreet had $2x = 20$ cm, and Anjali had $3x = 30$ cm.

1.28 Let Mei's mom have x Skittles, then she would have $2x$ M&M's, and $2x + 30$ Coffee Crisps. All together she had $x + 2x + 2x + 30 = 5x + 30$ treats, so $5x + 30 = 200$, so $5x = 170$, so $x = 34$.

Mei's mom, therefore, had 34 Skittles, 68 M&M's, and 98 Coffee Crisps.

1.29 The possible outcomes are



1.30 Since each coin could show a head or a tail, 2 choices, Mei can expect $2^5 = 32$ outcomes.

1.31 Since there are six choices for each die, we get $6^2 = 36$ outcomes.

1.32 a. Let us name the seats: driver, passenger, behind driver, behind passenger. With 4 people to choose from for the driver's seat, then 3 for the passenger seat, 2 for the seat behind the driver, and 1 left for the seat behind the passenger, we have $4 \times 3 \times 2 \times 1 = 24$ seating arrangements.

b. With Mei on the driver's seat, only three people are left, so $3 \times 2 \times 1 = 6$ ways for the passengers to arrange themselves for the remaining seats.

1.33 There are 25 different topics for the first choice, 24 for the second, 23 for the third, so a total of $25 \times 24 \times 23 = 13\,800$ different project proposals possible.

1.34 We have a total of 8 different spices, so $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$ different orders in which Anjali could lay the spices out on a roasting tray.

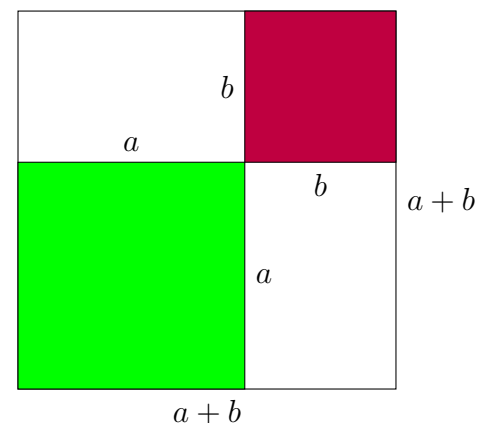
1.35 There are 6 choices of vegetables, and any vegetable could be chosen at every stage, so $6 \times 6 \times 6 \times 6 \times 6 = 6^5 = 7\,776$ different skewers possible. Remember that the top of the skewer is different from the bottom which has more bare stick.

1.36 When you use a tree diagram, the selection is ordered, as in the first coin followed by the second coin. Since it does not matter whether a nickel comes before a dime, each pair of coins, say a nickel and a dime in this case, is counted twice by the tree. To remove double counting, we need to divide the answer by 2, namely, $5 \times 4 \div 2 = 10$ ways of getting a pair of coins from Katharina's purse.

1.37 We can start building the tree by choosing which of the 4 formulas to repeat (4 choices), and which 2 picture proofs to get the repeated formulas (10 pairs of picture proofs). This leaves 3 picture proofs for 3 remaining formulas ($3 \times 2 \times 1 = 6$ choices). Therefore, a total of $4 \times 10 \times 6 = 240$ ways.

- 1.38**
- a. $x + 3 > x + 2$ True, just subtract x on both sides.
 - b. $x + 3 \geq x + 2$ True. Same reasoning as above.
 - c. $x + 3 \leq x + 2$ False, because 3 is not less than or equal to 2.
 - d. $x + 3 < x + 2$ False.
 - e. $x + 3 = x + 2$ False. Subtract x on both sides.
 - f. $x < x + 2$ True, the right hand side is 2 more than the left.
 - g. $x \leq x + 2$ True because it is less than (though not equal to, but this is an OR).
 - h. $a \geq -300$, for all whole numbers a . True (since the whole numbers are $0, 1, 2, 3, \dots$).
 - i. $a^2 + b^2 = (a + b)^2$, for some real numbers a and b . This is true when one of a and b is 0. If $ab \neq 0$, this is never true because $(a + b)^2 = a^2 + 2ab + b^2$.

Picture proof of square sum expansion



j. $\sqrt{a^2 + b^2} = a + b$, for some real numbers a and b . This is true for $ab = 0$ and a, b non-negative; otherwise, it is false simply by squaring both sides of the equation.

k. $\frac{a + b}{2} \geq \sqrt{ab}$, for non-negative real numbers a and b . This is the famous inequality of arithmetic mean (average of two numbers a and b) and geometric mean (\sqrt{ab}). We need to use the fact that the square of any number is at least 0. The particular square of interest is $(a - b)^2$ which is equal to 0 only when $a = b$. When $(a - b)^2$ is expanded, we get $a^2 - 2ab + b^2 \geq 0$ for all real numbers a and b . By adding $4ab$ on both sides of the inequality, we get

$$a^2 + 2ab + b^2 \geq 4ab.$$

Since both sides of the inequality are non-negative for non-negative real numbers a and b , we can take square roots on both sides to get $a + b \geq 2\sqrt{ab}$. Dividing by 2 on both sides, we get the desired inequality.

l. I know we were taught not to add fractions this way:

$$\frac{s}{t} + \frac{u}{v} = \frac{s + u}{t + v}$$

The correct way of adding two fractions is first to find a common denominator, in this case tv .

$$\frac{sv}{tv} + \frac{tu}{tv} = \frac{sv + tu}{tv}$$

We are trying to find integers for s, t, u , and v such that

$$\frac{s + u}{t + v} = \frac{sv + tu}{tv}.$$

An easy solution is to choose $s = u = 0$, then $s + u = 0$, and the addition of 0 is still 0. Another way is to choose $t = v$, but

The square root of 16, denoted by $\sqrt{16}$, is 4 because $4^2 = 16$.

non-zero, of course (or the fraction $a/0$ has no meaning). Then choose $s = -u$. This gives us zero as the sum. There may be other ways, but these are the simplest to explain.

- m. Assuming that a and b are non-negative, this equation is true when $ab = 0$. You can see this by squaring both sides of the equation to get

$$\text{LHS} = (\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b,$$

and

$$\text{RHS} = (\sqrt{a+b})^2 = a + b.$$

- n. For some integers n and m , $n^3 + m^3 \leq (n+m)^3$ when $3m^2n + 3mn^2 \geq 0$, by expanding $(n+m)^3$. This is equivalent to finding integers m and n such that $mn(m+n) \geq 0$. This is true for non-negative m and n 's. When $mn < 0$, that is, m and n are of opposite signs, then $m+n \leq 0$ guarantees the result because we will have two negative integers multiplied by a positive integer, or a $m+n = 0$ which gives a product of 0.

1.39 a. The picture of square numbers diagonally decomposed is described by $1+2+3+\cdots+(n-1)+n+(n-1)+\cdots+2+1 = n^2$.

- b. The 1×1 square subdivided is decomposed into fractional areas adding up to a total area of 1, described by

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

- c. The sum of 2 triangular numbers forms a rectangle of $n \times (n+1)$. For just one triangular number, we need to halve the area of the rectangle. Therefore, the equation is $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$.

- d. Let the area of the big triangle be 1. The largest white triangle is $1/4$ of the big triangle; the second largest white triangle is $1/4$ of $1/4$, namely area $1/16$. This yields the total area of white triangles described by

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}.$$

- e. Two interlacing consecutive square numbers is described by $1 + 3 + 5 + \cdots + (2n-1) + (2n+1) + (2n-1) + \cdots + 5 + 3 + 1 = n^2 + (n+1)^2$.
- f. This is similar to the shaded triangles, where the area of the green squares is $1/3$ of the total square area. The matching equation is

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}.$$

- g. The top blue part is the sum of n consecutive odd numbers. The bottom red part is the sum of the next n consecutive odd numbers, cleverly subdivided into three equal parts matching the shape of the blue part. The formula is

$$\frac{1 + 3 + \cdots + (2n-1)}{(2n+1) + (2n+3) + \cdots + (4n-1)} = \frac{1}{3}.$$

- h. A square number expressed as the sum of consecutive odd numbers has this formula, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.
- i. The sum on the right hand side of

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{\left(\frac{n(n+1)}{2}\right)} + \cdots = 2$$

can be written as


$$1 + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \dots \\ + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \dots$$

Since neighbouring terms across parentheses match with opposite signs, they cancel leaving $1 + \frac{2}{2} = 1 + 1 = 2$. When a long sum of infinitely many terms expanded to almost twice as many terms then cancel infinitely many terms to leave only two surviving terms, we call this a **telescoping sum**.

2.2 =||||⊥Π

2.4 324

2.5 1 214 000

2.7 $1\,214\,000 - 312\,300 = 901\,700 =$ 

2.9 The film was made in 1959 and earned nearly 147 million dollars.

2.11 2010

2.15 a. This is a three-digit number with digits (from the bottom) 1, 17, and 7. The value is therefore $1 \times 1 + 17 \times 20 + 7 \times 360 = 2861$.

b. This is a two-digit number with digits (again from the bottom) 10 and 14, so the value is $10 \times 1 + 14 \times 20 = 290$.

c. Here the digits are 0, 15, and 18, so the value is $0 \times 1 + 15 \times 20 + 18 \times 360 = 6780$.

2.16 The digits are (from the bottom, 8 and 2, so Lady Chimalmat brought $8 \times 1 + 2 = 48$ cobs of Mayan corn.

2.17 The digits are (from the bottom) 19 and 13, so Lady Chimalmat put $19 \times 1 + 13 \times 20 = 279$ beans in her salad.

2.19 If the top three dots were in the same place, they would be placed on the same level. Having the top two dots at the same level, and above the single dot means that the top two dots are in the 360s place, while we have $6 = 1 + 5$ in the 20s place, and 12 ones; total $12 \times 1 + 6 \times 20 + 2 \times 360 = 852$.

2.22 $0 \times 1 + 17 \times 20 + 7 \times 360 = 2860$

2.24 a. In Hindu-Arabic numerals, we get $121 + 345 = 466$, $\begin{array}{r} \bullet \\ \hline \bullet \end{array}$ in Mayan numerals. However, when you add directly using Mayan numerals, the 1s place is clear, $1 + 5 = 6$, putting the dot directly on the bar. In the 20s place, 18 20s yields 1 dot in the 360s place, leaving only a bar (that is, a 5) left in the 20s place. Remember to leave a gap between the top dot and the top line to indicate changing of places, so in summary, $1 \times 360 + 5 \times 20 + (1 + 5) = 466$.


b. First, translate the numbers $423 + 397 = 820$, then express 820

as $\begin{array}{r} \bullet \quad \bullet \\ \hline \text{⊖} \end{array}$, or carry out addition directly using Mayan numerals, and arrive at the same answer.

c. Translating to Hindu-Arabic numerals, we have $728 + 424 = 1152$,

or $\begin{array}{r} \bullet \quad \bullet \quad \bullet \\ \bullet \quad \bullet \\ \hline \hline \end{array}$ in Mayan numerals.

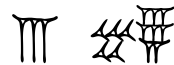
2.25 a. $379 - 288 = 91$ or $\begin{array}{r} \bullet \quad \bullet \quad \bullet \\ \bullet \\ \hline \hline \end{array}$

b. $732 - 415 = 317$ or 

2.27 35 000 litres.

2.30 Lady Shala harvested 609 bunches of grapes.

2.32 Lady Shala harvested $576 - 123 - 215 = 238$ pears.



2.34 a. $\text{𑀓𑀢𑀣} \llcorner \text{𑀓} + \text{𑀢} \llcorner \text{𑀓} = \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $814 + 397 = 1211$.

b. $\llcorner \text{𑀓} \text{𑀢} + \llcorner \text{𑀓} \llcorner \text{𑀓} = \llcorner \text{𑀓} \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $1985 + 1339 = 3324$.

c. $\text{𑀓} \text{𑀢} \llcorner \text{𑀓} + \text{𑀢} \llcorner \text{𑀓} \llcorner \text{𑀓} = \text{𑀓} \llcorner \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $3914 + 9923 = 13837$.

2.35 a. $\llcorner \text{𑀓} \llcorner \text{𑀓} - \text{𑀢} \llcorner \text{𑀓} = \llcorner \text{𑀓} \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $924 - 417 = 507$.

b. $\llcorner \text{𑀓} \llcorner \text{𑀓} - \llcorner \text{𑀓} \llcorner \text{𑀓} = \llcorner \text{𑀓} \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $2386 - 1589 = 797$.

c. $\text{𑀢} \llcorner \text{𑀓} \llcorner \text{𑀓} - \text{𑀢} \llcorner \text{𑀓} \llcorner \text{𑀓} = \llcorner \text{𑀓} \llcorner \text{𑀓}$, or equivalently in Hindu-Arabic numerals: $9732 - 8371 = 1361$.

2.36 TI-83 plus gives $\sqrt{2} \approx 1.414213562$.
 $1 + 24/60 + 51/3600 + 10/216000 \approx 1.414212963$.

2.37 a. $\text{𑀓} \text{𑀢}$ is $15 \times 20 + 7 = 307$.

b. $\text{𑀓} \text{𑀢} \text{𑀓}$ is $12 \times 400 + 4 \times 20 + 6 = 4886$.

c. $\overline{5}\overline{4}\overline{3}$ is $17 \times 400 + 8 \times 20 + 13 = 6973$.

d. $\overline{5}\overline{3}\overline{2}$ is $18 \times 400 + 11 \times 20 + 7 = 7427$.

2.39 $\overline{5}\overline{4} + \overline{3}\overline{2}$ is $\overline{5}\overline{8}\overline{1}$, or $58 + 23 = 81$ books on these two types of bears; that is $\overline{5}\overline{8}\overline{1}$ such books.

2.40 2500 is $\overline{2}\overline{5}\overline{0}\overline{0}$; 3000 is $\overline{3}\overline{0}\overline{0}\overline{0}$.

2.42 a. $\overline{1}\overline{5}\overline{1}\overline{7} + \overline{2}\overline{6}\overline{2}\overline{9} = \overline{4}\overline{1}\overline{4}\overline{6}$, or equivalently in Hindu-Arabic numerals: $1517 + 2629 = 4146$.

b. $\overline{2}\overline{8}\overline{9}\overline{2} + \overline{3}\overline{4}\overline{7}\overline{6} = \overline{6}\overline{3}\overline{6}\overline{8}$, or equivalently in Hindu-Arabic numerals: $2892 + 3476 = 6368$.

c. $\overline{5}\overline{7}\overline{0}\overline{2} - \overline{3}\overline{5}\overline{3}\overline{7} = \overline{2}\overline{1}\overline{6}\overline{5}$, or equivalently in Hindu-Arabic numerals: $5702 - 3537 = 2165$.

d. $\overline{7}\overline{4}\overline{2}\overline{7} - \overline{5}\overline{1}\overline{9}\overline{3} = \overline{2}\overline{2}\overline{3}\overline{4}$, or equivalently in Hindu-Arabic numerals: $7427 - 5193 = 2234$.

2.43 a. $303_5, 304_5, 310_5, 311_5, 312_5$

b. $252_6, 253_6, 254_6, 255_6, 300_6$

c. $30_4, 31_4, 32_4, 33_4, 100_4$

d. $1022_3, 1100_3, 1101_3, 1102_3, 1110_3$

e. $A39D_{16}, A39E_{16}, A39F_{16}, A3A0_{16}, A3A1_{16}$

f. $1001_2, 1010_2, 1011_2, 1100_2, 1101_2$

g. $675_8, 676_8, 677_8, 700_8, 701_8$

2.44 Write the place values for each number according to its base.

a. $372_9 = 3 \times 81 + 7 \times 9 + 2 \times 1 = 308$

b. $253_7 = 2 \times 49 + 5 \times 7 + 3 \times 1 = 136$

c. $425_6 = 4 \times 36 + 2 \times 6 + 5 \times 1 = 161$

d. $1\ 012\ 102_3 = 1 \times 729 + 0 \times 243 + 1 \times 81 + 2 \times 27 + 1 \times 9 + 0 \times 3 + 2 \times 1 = 875$

e. $AB9_{16} = 10 \times 256 + 11 \times 16 + 9 \times 1 = 2745$

f. $1\ 010\ 010\ 101_2 = 512 + 128 + 16 + 4 + 1 = 661$

2.45 Either write the place values for each base to determine the largest place or use successive division.

a. $34 = 100\ 010_2$ because $34 = 32 + 2 = 2^5 + 2^1$.

b. $77 = 1\ 001\ 101_2$

c. $123 = 1\ 111\ 011_2$

d. $394 = 1102_7$

e. $572 = 2352_6$

f. $2592 = 5040_8$

g. $297 = 129_{16}$

h. $2938 = B7A_{16}$

i. $9848 = 2678_{16}$

j. $12\ 192 = 2FA0_{16}$

2.46 a. $314_5 + 233_5 = 1102_5$

b. $354_6 + 445_6 = 1243_6$

c. $323_4 + 222_4 = 1211_4$

d. $1121_3 + 2121_3 = 11\ 012_3$

e. $A39F_{16} + 9387_{16} = 13\ 726_{16}$

f. $110\ 011_2 + 101\ 101_2 = 1\ 100\ 000_2$

g. $676_8 + 354_8 = 1252_8$

2.47 a. $421_5 - 232_5 = 134_5$

b. $534_7 - 246_7 = 255_7$

c. $423_5 - 134_5 = 234_5$

d. $110\ 101_2 - 10\ 011_2 = 100\ 010_2$

e. $A30E_{16} - 938F_{16} = F7F_{16}$

f. $112\ 211_3 - 101\ 122_3 = 11\ 012_3$

g. $751_8 - 364_8 = 365_8$

2.48 To octal, group each binary from the right, three digits in a group, and find the value for each group using binary expansion; for example, $011_2 = 2 + 1 = 3$.

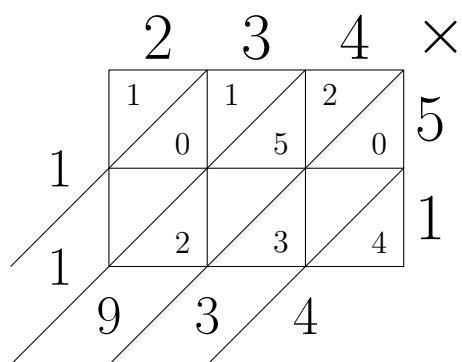
a. 5763_8 , $BF3_{16}$. b. 6571_8 , $D79_{16}$. c. 7425_8 , $F15_{16}$. d. 4357_8 , $8EF_{16}$.

To hexadecimal, group each binary from the right, four digits in a group and expand.

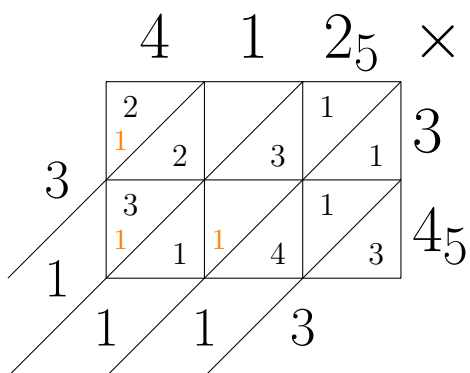
2.52 a. MMMCDXCVIII is 3498. b. $\overline{V}\overline{V}\overline{V}$ is 3476. c. DA_{16} is 3488.
 d. 110110100011_2 is 3491. e. $\overline{IV}\overline{III}$ is 3483. Therefore,

$$\overline{V}\overline{V}\overline{V} < \overline{IV}\overline{III} < DA_{16} < 110110100011_2 < \text{MMMCDXCVIII}.$$

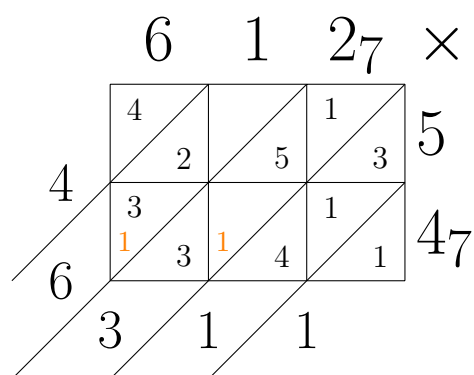
2.54 Fill in the cells for 234×5 then add the numbers along each diagonal, then use the calculator to check that $234 \times 51 = 11934$.



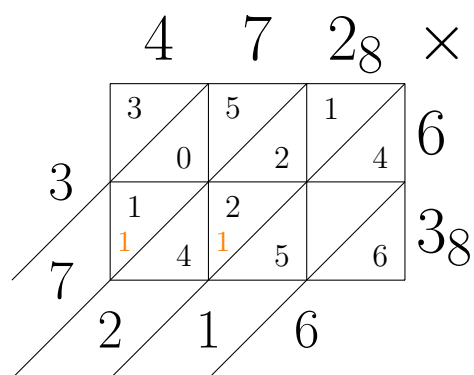
2.57 Using the multiplication table for base-5, one fills the cells.



2.58 Using the multiplication table for base-7, one fills the cells.



2.59 Using the multiplication table for base-8, one fills the cells.



2.60 Start with two known cells in a row or in a column to find one-digit common factors; for example, $\gcd(42, 18) = 6$, so 6 must be at the right end of the bottom row with 42 and 18. Similarly, $\gcd(12, 20) = 4$, so 4 is above 6 at the right end of the galley. Note that 2 is also a common factor of 12 and 20, but $20 \div 2 = 10$, a two-digit quotient, not allowed on top of a column in the galley.

$$\begin{array}{r}
 735 \times \\
 \hline
 280 \\
 1460 \\
 \hline
 8100
 \end{array}$$

A grid diagram for the multiplication 735×46 . The grid is 3 columns wide and 3 rows high. The top row contains the digits 7, 3, 5, and a multiplication symbol \times . The right side of the grid shows the partial products 4, 6, and 0. The grid cells contain the following numbers:

- Row 1: (1,1)=2, (1,2)=1, (1,3)=2
- Row 2: (2,1)=4, (2,2)=1, (2,3)=3
- Row 3: (3,1)=8, (3,2)=1, (3,3)=0

 Diagonal lines separate the grid into three sections. The top-left section (top-left to bottom-right) contains the numbers 2, 8, 1, 2, 4, 3, 8, 1, 0. The top-right section (top-right to bottom-left) contains the numbers 1, 2, 1, 3, 1, 8, 1, 0. The bottom section (bottom-left to top-right) contains the numbers 8, 1, 0.

2.63 Fill the galley:

$$\begin{array}{r}
 101_2 \times \\
 \hline
 101 \\
 101 \\
 \hline
 110_2
 \end{array}$$

A grid diagram for the multiplication $101_2 \times 101_2$. The grid is 3 columns wide and 3 rows high. The top row contains the digits 1, 0, 1, and a multiplication symbol \times . The right side of the grid shows the partial products 1, 1, and 0. The grid cells contain the following numbers:

- Row 1: (1,1)=1, (1,2)=0, (1,3)=1
- Row 2: (2,1)=1, (2,2)=0, (2,3)=1
- Row 3: (3,1)=0, (3,2)=0, (3,3)=0

 Diagonal lines separate the grid into three sections. The top-left section (top-left to bottom-right) contains the numbers 1, 0, 1, 1, 1, 0, 0, 0. The top-right section (top-right to bottom-left) contains the numbers 0, 1, 0, 1, 0, 1, 0, 0. The bottom section (bottom-left to top-right) contains the numbers 0, 0, 0.

2.64 Fill the galley:

		E	8	7	$A_{16} \times$	
	B	6	5	8		D 2_{16}
B	6	8	B	2		
E	1 1	1 1	0 1	E	4	
B	4	1	4			

2.65 Fill the galley using Table 2.3:

		\overline{w}		-	\times
		\overline{w}			
		\overline{w}			
		\overline{w}			
		\overline{w}			

2.67 There are two approaches: to determine separately whether $25 \pmod{8}$ equals $47 \pmod{8}$, or to examine the difference $47 - 25$ is a multiple of 8. In this case, it is not, so False.

- a. $25 \equiv 47 \pmod{8}$ False b. $15 \equiv 38 \pmod{7}$ False c. $43 \equiv 79 \pmod{12}$ True
 d. $32 \equiv 96 \pmod{6}$ False e. $93 \equiv 18 \pmod{15}$ True
 f. $82 \equiv 26 \pmod{4}$ True

2.69 a. $2 + 17 \pmod{9} \equiv 1$

- b. $15 - 8 \pmod{3} \equiv 1$
- c. $4 \times 19 \pmod{7} \equiv 6$
- d. $22 + 46 \times 3 \pmod{16} \equiv 0$
- e. $193 - 38 \times 2 \pmod{25} \equiv 17$
- f. $11 \times 26 - 7 \pmod{8} \equiv 7$

2.71 A table of values for the unknown from 1 is a good start.

- a. $7x \equiv 3 \pmod{4}$

When $x = 1$, $7 \equiv 3 \pmod{4}$, so $x = 1$ is a solution. To find all solutions, one finds that $x = 2, 3, 4$ are not solutions. Since the $7 \times 4 \equiv 0 \pmod{4}$, adding multiples of 4 to the first solution $x = 1$ will yield all the solutions. Thus, $x = 1 + 4k$ for any integer k .

- b. $6 - a \equiv 5 \pmod{8}$

Starting at $a = 1$, one finds that $6 - 1 = 5 \equiv 5 \pmod{8}$, so $a = 1 + 8k$ for any integer k . Note that in a table of values, as $a = 2, 3, 4, \dots$, the difference $6 - a$ will 4, 3, 2, ...

- c. $3 \times (2 + n) \equiv 4 \pmod{6}$

Using the distributive law to expand the left hand side, we get

$$6 + 3n \equiv 4 \pmod{6}, \quad \text{or} \quad 3n \equiv 4 \pmod{6},$$

since $6 \equiv 0 \pmod{6}$. As a table of values shows from $n = 1$, $3n \equiv 3$ or $0 \pmod{6}$, never $4 \pmod{6}$. This has to do with $\gcd(3, 6) = 3$. More theory on modular arithmetic can be learned in a course on number theory. It suffices to show the table of values for the first six n 's for the conclusion that this equation has no solution.

d. $52 - 2y \equiv 3 \pmod{21}$

Note first that $52 \equiv 10 \pmod{21}$, so the equation is equivalent to

$$10 - 2y \equiv 3 \pmod{21}, \quad \text{or} \quad 7 \equiv 2y \pmod{21}.$$

A table of values yields $2 \times 11 \equiv 1 \pmod{21}$. Multiplying both sides by 7, one gets

$$2 \times 77 \equiv 7 \pmod{21}.$$

Now, 77 is certainly not the smallest positive integer solution, but $y = 77 + 21k$ for any integer k gives all solutions. In particular, $k = -3$ gives $y = 77 - 63 = 14$, the smallest positive solution.

This solution uses the multiplicative inverse of 2 (mod 21), (i.e. 11) to solve for y in $7 \equiv 2y \pmod{21}$. The same technique could be used to solve for any integer b in $b \equiv 2y \pmod{21}$ by letting $y = b \times 11$.

e. $10b \equiv 9 \pmod{13}$

The table shows that $10 \times 4 \equiv 1 \pmod{13}$, or 4 is the multiplicative inverse of 10 (mod 13). Therefore, $b = 4 \times 9 + 13k$ for integers k . Again, 36 is not the smallest positive integer solution for b , but $36 - 26 = 10$ is.

f. $12u + 2 \equiv 7 \pmod{9}$ is equivalent to solving $12u \equiv 5 \pmod{9}$. According to the table of values for u , one concludes that $12u \equiv 3, 6, \text{ or } 0 \pmod{9}$, never 5. Thus, this equation has no solution for u because $\gcd(12, 9) = 3$.

2.72 a. Find the smallest positive integer x such that both equations are true: $x \equiv 7 \pmod{9}$ and $x \equiv 2 \pmod{5}$.

Begin by listing all positive solutions of $x \equiv 2 \pmod{5}$ from $x = 2, 7, 12, 17, 22, 27, \dots$

Check each to see which one satisfies $x \equiv 7 \pmod{9}$. In this case, 7 is the smallest solution. Since $\text{lcm}(9, 5) = 45$, $x = 7 + 45k$ for any integer k covers all solutions.

- b. Find the second smallest positive integer y such that both equations are true: $y \equiv 1 \pmod{3}$ and $y \equiv 4 \pmod{7}$.

By listing all positive solutions for $y \equiv 1 \pmod{3}$, one finds that 4 also satisfies $y \equiv 4 \pmod{7}$. However, the question asks for the second smallest positive integer, so we need to use $\text{lcm}(3, 7) = 21$ in the general solution $y = 4 + 21k$ and substitute $k = 1$ to get $y = 25$.

- c. Find a positive solution nearest 100 such that all three congruences are true: $z \equiv 2 \pmod{6}$, $z \equiv 4 \pmod{10}$, and $z \equiv 0 \pmod{8}$.

The easiest congruence to use is $z \equiv 4 \pmod{10}$ where $z = 4, 14, 24, 34, 44, \dots$

Among these, one picks out those that also satisfy $z \equiv 0 \pmod{8}$, namely, 24, 64, 104, or in general, $24 + 40k$. Out of these candidates, one checks for the last congruence: $z \equiv 2 \pmod{6}$, and finds $104 \equiv 2 \pmod{6}$, so $z = 104 + k \text{lcm}(6, 8, 10) = 104 + 120k$. The solution closest to 100 is 104.

- d. Find the positive solution nearest 200 such that all three congruences are true: $u \equiv 2 \pmod{7}$, $u \equiv 3 \pmod{11}$, and $u \equiv 4 \pmod{9}$.

Starting with $u \equiv 3 \pmod{11}$ gives the fewest possible values to check: 3, 14, 25, 36, 47, 58, or in general, $u = 3 + 11k$. Since $58 \equiv 4 \pmod{9}$, the last congruence, $u \equiv 2 \pmod{7}$, must come from solutions of the form $u = 58 + 99k$. But, $58 \equiv 2 \pmod{7}$, so choosing $k \equiv 0 \pmod{7}$ will do. To summarize, $u = 58 + 11 \times 9 \times 7k$, or $u = 58$, or 751, The positive solution nearest 200 is $u = 58$.

2.73 Since $59 \equiv 3 \pmod{7}$, we need to go forwards by three days from Tuesday, thus Friday.

2.75 Since $366 \equiv 2 \pmod{7}$, we need to go forwards by two days from Tuesday. Therefore, her birthday happened on Thursday the following year.

2.77 Two modular arithmetic operations happen in time: mod 60 for minutes and mod 24 for a 24-hour clock. When we work on minutes, $35 + 27 = 62 \equiv 2 \pmod{60}$, giving us 2 minutes and a bundle of sixty minutes as one hour carried to the hour place.

For the hours, $2 + 33 + 1 = 36 \equiv 12 \pmod{24}$, which means one and a half day forward from Wednesday, so Thursday at 2:02 pm.

2.80 We need to use the formula

$$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12} \equiv 0 \pmod{10},$$

to see that $0 + 6 + 6 + 8 + 15 + 8 + 3 + 0 + 9 + 2 + 0 + 3 = 60 \equiv 0 \pmod{10}$. Therefore this UPC is valid.

- 3.1**
- a. $\angle 4$ is corresponding to the 75° angle.
 - b. $\angle 6$ is vertically opposite to the 75° angle.
 - c. $\angle 2$ forms an alternate interior angle pair with the 75° angle.
 - d. $\angle 3$ forms an *interior angle on the same side of the transversal* pair with the 75° angle.
 - e. $\angle 6$ and $\angle 4$ form a pair of alternate exterior angles congruent to the 75° angle.

3.3 Since $\angle c$ and 118° lie on the same transversal, $\angle c \cong 180^\circ - 118^\circ = 62^\circ$ because a straight angle is 180° .

Since $\angle b$ forms an alternate interior angle pair with the given 118° when one considers the horizontal pair of parallel lines, $\angle b \cong 118^\circ$.

Considering the up-down pair of parallel lines with a horizontal transversal where angles $\angle a$ and $\angle b$ lie, one sees that they form an alternate interior pair of angles, so $\angle a \cong \angle b \cong 118^\circ$.

3.5 Since the top and bottom sides are parallel, using the property of alternate interior angles, $2y = 3y - 15$, so $y = 15$.

Consider the triangle containing three marked angles. Using the angle sum of a triangle, one obtains $4x + 10 + 2x + 2y = 180$, or $6x + 10 + 2 \times 15 = 180$, where one substitutes 15 for y . Simplify the equation involving x only, one gets $6x = 140$, or $x = 70/3$.

3.7 Consider line t_1 where $\angle a$ and 100° form an alternate exterior angle pair, so $\angle a \cong 100^\circ$.

Also on line t_1 , $\angle b$ and 100° form a straight angle, so $\angle b \cong 80^\circ$.

On line t_3 , the corresponding angle to 80° forms a straight angle with $\angle c$, so $\angle c \cong 100^\circ$.

Lines t_1 and t_3 are parallel because the angles $\angle b \cong 80^\circ$ are both supplementary to the vertical angle of $\angle a$.

Line t_2 and the angle 25° are irrelevant to this question.

3.8 The top triangle is a right triangle because $\angle 5 \cong 90^\circ$. This yields $\angle 17 \cong 50^\circ$.

Angle $\angle 12$ forms an alternate exterior angle pair with 40° , so $\angle 12 \cong 40^\circ$.

Angle $\angle 13$ is supplementary to a right angle, so $\angle 13 \cong 90^\circ$.

3.11 Since a full turn in a circle is 360° , the hour hand sweeps $360^\circ/12 = 30^\circ$ in one hour. Furthermore, one hour has sixty minutes, so $30^\circ/60 = 0.5^\circ$, or half a degree in one minute.

3.13 Since the minute hand sweeps a full turn, i.e. 360° , in one hour, or sixty minutes, the minute hand sweeps $360^\circ/60 = 6^\circ$ in one minute.

3.14 Let the minute hand pointing North (12) be at 0° position. Then the hour hand pointing at 2 would be at $2 \times 30^\circ$ or 60° position. Therefore, the acute angle formed by the hands is 60° .

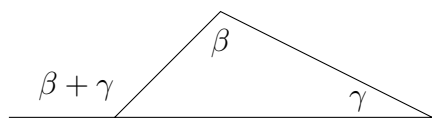
3.16 Let North (12) be 0° position. Then the hour hand at 2 : 27 would be at $2 \times 30^\circ + 27^\circ/2 = 60^\circ + 13.5^\circ = 73.5^\circ$ position because the hour hand sweeps 0.5° per minute. The minute hand at 27 minutes is at $27 \times 6^\circ = 162^\circ$ because it sweeps 6° per minute. Therefore, the acute angle formed by the hands is $162^\circ - 73.5^\circ = 88.5^\circ$.

3.18 *First solution:* Choose a triangle containing one of the pointy ends. In this seven-point star, there are at least two choices, the triangle containing E or F . Suppose we choose the latter triangle, and label the higher interior angle $\angle 1$ and the lower one, $\angle 2$. Notice that $\angle 1$ as an exterior angle has the same size as the sum of the non-adjacent interior angles $\angle C$ and $\angle G$. To see that $\angle 2$ is the sum of the remaining four pointy angles, begin with the triangle containing D with $\angle 2$ as its exterior angle. By the exterior angle property of a triangle, $\angle 2$ is the same size as $\angle D$ and $\angle 3$. But $\angle 3$ is congruent to $\angle 4 + \angle A$, and $\angle 4 \cong \angle E + \angle B$, so to summarize,

$$\angle 2 \cong \angle D + \angle 3 \cong \angle D + \angle 4 + \angle A \cong \angle D + \angle E + \angle B + \angle A.$$

Combined with $\angle 1$ and $\angle F$, one gets $\angle F + \angle 1 + \angle 2 = 180^\circ$ from the angle sum in a triangle. Replacing angles $\angle 1$ and $\angle 2$ by 6 remaining pointy angles, one shows that the angle sum of all seven pointy angles is 180° .

Second solution: If the (measures of the) angles of a triangle are α , β , and γ , then $\alpha + \beta + \gamma = 180^\circ$, so $180^\circ - \alpha = \beta + \gamma$. This is the Exterior Angle Property.



Consider the triangle spanned by C , G , and point at $\angle 1$ (the triangle with blue vertices). By the Exterior Angle Property,

$$\angle 1 = \angle C + \angle G.$$

Now consider the triangle spanned by D and the points at $\angle 2$ and $\angle 3$ (the triangle with red vertices). Using the Exterior Angle Property again,

$$\angle 2 = \angle D + \angle 3.$$

Similarly, considering the triangle spanned by A and the points at $\angle 3$ and $\angle 4$,

$$\angle 3 = \angle A + \angle 4,$$

and considering the triangles spanned by B , E , and the point at $\angle 4$,

$$\angle 4 = \angle B + \angle E.$$

Therefore

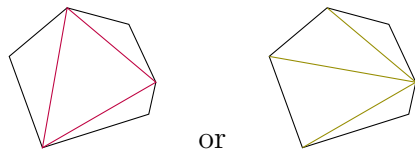
$$\begin{aligned} \angle 2 &= \angle D + \angle 3 \\ &= \angle D + \angle A + \angle 4 \\ &= \angle D + \angle A + \angle B + \angle E. \end{aligned}$$

Since the angle sum of the triangle spanned by F and the points at $\angle 1$ and $\angle 2$ is 180° , it follows that

$$\begin{aligned} 180^\circ &= \angle F + \angle 1 + \angle 2 \\ &= \angle F + (\angle C + \angle G) + (\angle D + \angle A + \angle B + \angle E) \\ &= \angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G, \end{aligned}$$

so the desired angle sum is 180° .

3.20 Join non-adjacent vertices of the hexagon to partition the hexagon into four non-overlapping triangles. For example,



The interior angle sum is $4 \times 180^\circ = 720^\circ$.

3.22 To show whether two triangles are similar, one needs to establish either corresponding angles or a fixed ratio for corresponding side lengths. With a protractor, one can measure all the angles to find that corresponding angles are congruent pairwise. With a compass, one can compare lengths to see that the corresponding side length in the bigger triangle is double that of the smaller.

In conclusion, the triangles are similar, i.e. $\triangle ABC \sim \triangle DEF$.

3.25 Since $\triangle ABC \sim \triangle DEF$, $6 : 8 = 2 : x$, so $x = 8/3$.

3.28 One requires at least three equilateral triangles at one point to form a solid. Since each angle in an equilateral triangle is 60° , $3 \times 60^\circ = 180^\circ < 360^\circ$. This makes a tetrahedron using 4 triangles for its 4 faces.

When one puts 4 triangles together at a corner, $4 \times 60^\circ = 240^\circ < 360^\circ$. This makes an octahedron, using 8 triangles in total.

When one puts 5 triangles together at a corner, $5 \times 60^\circ = 300^\circ < 360^\circ$. This makes an icosahedron, using 20 triangles in total.

These are three of the five Platonic solids.

If you put 6 triangles together at a corner, $6 \times 60^\circ = 360^\circ$, so this fails to make a solid. Therefore only the tetrahedron, octahedron, and icosahedron are possible.

3.33 $x = 13$, $y = 9$, $z = 8$, $a = 16$, $b = 24$, $c = 25$, $d = 26$, $e = 20$.

3.34 Triangles with sides y , a , and c are similar to the 3–4–5 triangle.

Triangles with sides x and d are similar to the 5–12–13 triangle.

The remaining three triangles are not similar because each contains a prime and the others do not have a multiple of that prime to establish similarity.

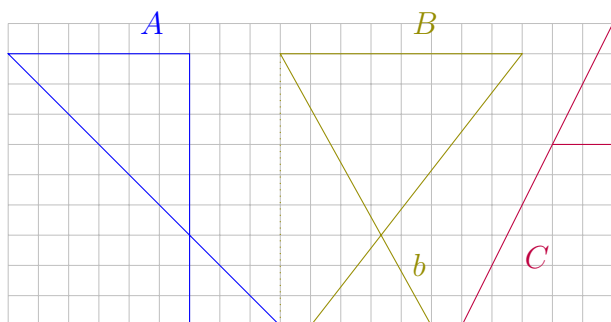
3.35 Let h denote the length of the hypotenuse. Using the Pythagorean Theorem on $\triangle A$, $h^2 = 8^2 + 15^2 = 64 + 225 = 289$, so $h = 17$.

Similarly, for $\triangle B$, $h = 10$ because we have a 3–4–5 triangle with each side doubled.

The last triangle, $\triangle C$, is an isosceles right triangle: $h^2 = 5^2 + 5^2 = 2 \times 5^2$, so $h = 5\sqrt{2}$.

3.37 In A , the top and bottom sides are parallel because both are horizontal. Therefore, using the property of alternate interior angles between parallel lines and its transversal, we have congruent acute angles at the top and bottom of A . Vertically opposite angles give a second pair of congruent angles. The last pair is the right angles. Since all three angles in each triangle are pairwise congruent, the triangles in A are similar. The smaller triangle in A is an isosceles right triangle with a leg length of 3, so its perimeter is $3 + 3 + 3\sqrt{2} = 6 + 3\sqrt{2}$.

A similar angle argument applies to B to establish similarity of the two triangles. Since the top side is 8 long and the bottom is 4, the ratio of big triangle to small triangle is 2 : 1. This fact is used to get the slant sides of the smaller triangle. Add the dotted lines in the figure below to see one of the slant sides as the hypotenuse of a right triangle of leg lengths 5 at the bottom and 9 up. Then the slant side b in the small triangle is $1/3$ of the hypotenuse h where $h^2 = 5^2 + 9^2 = 106$. So $b = \sqrt{106}/3$. Similarly, the other slant side is $\sqrt{9^2 + 7^2}/3 = \sqrt{130}/3$. In conclusion, the perimeter of the smaller triangle in B is $4 + \sqrt{106}/3 + \sqrt{130}/3$.



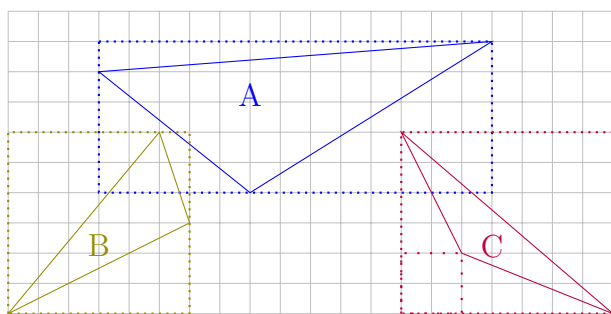
The triangles in C are both right triangles sharing the top acute angle, so clearly similar (right angle, the top angle, and the third angle being $180^\circ - 90^\circ - \text{the top angle}$) with a side ratio of $2 : 5$. The perimeter of the smaller triangle is $2 + 4 + \sqrt{2^2 + 4^2} = 6 + \sqrt{20} = 6 + 2\sqrt{5}$.

3.38 Two similar triangles are formed by the light beam: $\triangle ABC \sim \triangle EDC$, so $x : 7 = 5 : 3$. Therefore, $x = 35/3$.

3.41 A similar triangle has the same side length ratio, so assume that the lengths were $3x$, $4x$, and $5x$ long, and its sum was $3x + 4x + 5x = 60$, i.e. $12x = 60$, so $x = 5$.

Therefore, the lengths were 15, 20, and 25 units long.

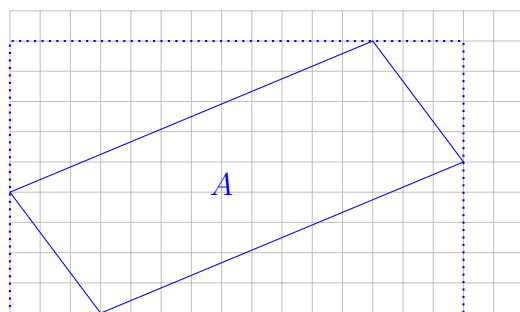
3.45 Since none of the triangles has an integer base with a corresponding height, to find exact areas in this case, one needs an enclosing rectangle first; see the figure below. Once the rectangle is drawn, all one needs to do is to subtract the right angles outside the triangle in question. For example, for $\triangle A$, the blue dotted lines show the enclosing rectangle. The little sliver of a right triangle above $\triangle A$ has area $13/2$, the lower left right triangle to $\triangle A$ has area 20, and the left right triangle to $\triangle A$ has area 10. Since the enclosing rectangle has area $13 \times 5 = 65$, $\triangle A$ has area $65 - 13/2 - 20 - 10 = 57/2 = 28.5$ square units.



A similar approach can be used for $\triangle B$ to obtain an area of 10.5 square units.

For $\triangle C$, the purple enclosing rectangle only contains two vertices of $\triangle C$, so one needs to remove a 2×2 square in the lower left hand corner to get an area of $42 - 21 - 4 - 5 - 4 = 8$ square units for the triangle.

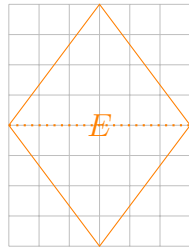
3.47 Enclosing quadrilateral A in a rectangle, one first sees many right triangles with leg lengths as part of a Pythagorean triple: 3–4–5 and 5–12–13. This makes it easier to find the perimeter: $(5 + 13) \times 2 = 36$. Since opposite sides are congruent in length, the quadrilateral is a parallelogram.



Without a given height to any base, it is easier to find the area of this parallelogram by subtracting right triangles from the enclosing rectangle of 15×9 . Therefore, the area of parallelogram A is $135 - 2 \times (3 \times 4/2) - 2 \times (5 \times 12/2) = 135 - 12 - 60 = 63$ square units.

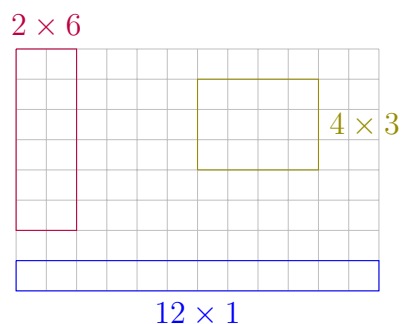
3.49 The slant sides have length 5 using the Pythagorean Theorem, so all four sides are congruent, thus a **rhombus**. The perimeter of the rhombus is $4 \times 5 = 20$ units. Since the rhombus is lying on one of its sides, it can be viewed as a parallelogram with base 5 and height 4 so an area of $5 \times 4 = 20$ square units.

3.51 The slant sides have length 5 using the Pythagorean Theorem, so all four sides are congruent, thus a **rhombus**. The perimeter of the rhombus is $4 \times 5 = 20$ units. Since the rhombus is standing on one of its vertices, it can be bisected by its horizontal diagonal to get two congruent isosceles triangles with base 6 and height 4 so an area of $2 \times (6 \times 4/2) = 24$ square units.



3.54 All five quadrilaterals are rhombi (plural of rhombus) of side length 5, so all five have the same perimeter of 20 units. However, the further the vertices are from each other, the smaller the area. The square, or quadrilateral **G** has congruent diagonal, and has the largest area of 25 square units. Quadrilateral **D** is the longest diagonal and has the smallest area of 15 square units. In the extreme, one can obtain a rhombus of perimeter 20 and 0 area by squashing quadrilateral **D** down to a height of 0.

3.55 Archimedes studied three up to a rotation of 90° : a 1×12 rectangle, a 2×6 rectangle, and a 3×4 rectangle. Of the three, the 1×12 or 12×1 rectangle had the longest perimeter of 26 units.



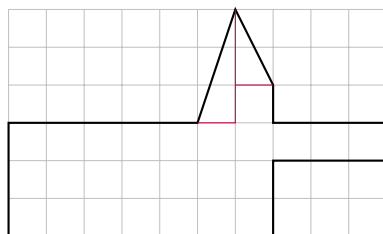
3.58 There are only five triangles, two of which are right triangles: 6–8–10 and 5–12–13. The other three are harder to find: 6–25–29, 7–15–20, and 9–10–17.

3.61 Subdivide into rectangles to get an area of 17 square units and a perimeter of 20 units.

3.63 Subdivide into rectangles to get an area of 18 square units and a perimeter of 24 units.

3.65 Subdivide into rectangles to get an area of 28 square units. The perimeter is equal to the perimeter of the enclosing rectangle, so 28 units. This is an equable shape.

3.67 Subdivide into rectangles and triangles or other shapes to get an area of $25 + \frac{1 \cdot 3}{2} + \frac{1 \cdot 2}{2} = 27\frac{1}{2}$ square units.



Polygon G

The perimeter is $25 + \sqrt{1^2 + 3^2} + \sqrt{1^2 + 2^2} = 25 + \sqrt{10} + \sqrt{5} \approx 30.40$ units. (The shortcut of Exercise 3.65 on page 123 couldn't apply here since the sides of Polygon G are not all parallel to the sides of the enclosing rectangle.)

3.69 Trapezoid A can be viewed as a 3×10 rectangle with two right triangles removed, so $30 - 3 - 6 = 21$ square units for its area. Using the Pythagorean Theorem, the slant sides are $\sqrt{13}$ and 5 units long, so the perimeter of the trapezoid is $4 + 10 + 5 + \sqrt{13} = 19 + \sqrt{13}$ units long.

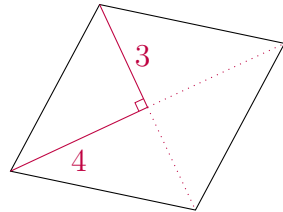
Trapezoid B is an isosceles trapezoid because the slant sides are congruent in length. Its area is $(6 + 2) \times 4/2 = 16$ square units. One slant side is $\sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$, so its perimeter is $6 + 2 + 2 \times 2\sqrt{5} = 8 + 4\sqrt{5}$.

Trapezoid C is an 8×9 rectangle with a right triangle removed, so an area of $72 - 16 = 56$ square units. The right triangle has legs of 4 and 8, a $1 : 2$ ratio, the same ratio as the slant height of B , just twice as big. Therefore, the slant side of C is $4\sqrt{5}$ long, and the perimeter of C is $8 + 9 + 5 + 4\sqrt{5} = 22 + 4\sqrt{5}$ units long.

3.70 First, one needs to find the length, $l = 95/5 = 19$. With the length, the perimeter is then $2 \times (19 + 5) = 48$ units long.

3.72 First, one needs to find the length, $l = (42/2) - 5 = 16$. Using the length in the area formula of a rectangle, the area is then $16 \times 5 = 80$ square units.

3.74 The diagonals of a rhombus bisect each other at 90° ,



The side length of the rhombus is therefore $\sqrt{3^2 + 4^2} = 5$, so the perimeter is $4 \times 5 = 20$ units. Moreover, the area is $4 \times \frac{3 \times 4}{2} = 24$ square units.

3.75 The smaller circle with a radius of 4 yields an area of $\pi r^2 = \pi 4^2 = 16\pi$.

The larger circle with a circumference of $40 = 2\pi r$ yields a radius of $r = 40/(2\pi) = 20/\pi$. Taking this radius, the larger circle will have an area of $\pi(20/\pi)^2 = 400/\pi$ square units.

3.77 The area of the circle is $\pi r^2 = \pi(24/(2\pi))^2 = 144/\pi \approx 45.8366$ square units.

One needs the side length of the square of perimeter 24, i.e. $s = 6$. With that, the square's area is $6^2 = 36$ square units.

One side of the equilateral is 8; using the Pythagorean Theorem, considering half an equilateral triangle with one leg of 4 and a hypotenuse of 8, one gets the other leg of $4\sqrt{3}$. The area is then $8 \times 4\sqrt{3}/2 = 16\sqrt{3} \approx 27.7128$.

The circle clearly has the largest area, by approximately 9.8366 square units.

3.78 Using the area formula for circles, if $r = 3$, then the area is $\pi 3^2 = 9\pi$ square units. Since 80° is $80/360$ of the whole circle, the sector area is $80/360$ of 9π or $2/9 \times 9\pi = 2\pi$ square units.

Similar steps are applied to the other two sectors:

$$\frac{130}{360} \times \pi 2^2 = \frac{13}{36} \times 4\pi = \frac{13\pi}{9},$$

and

$$\frac{210}{360} \times \pi 1^2 = \frac{7}{12} \times \pi = \frac{7\pi}{12},$$

3.79 Using the circumference formula $c = 2\pi r$ for a circle of radius r , the perimeter is $2r + 2\pi r \times \frac{60}{360} = 30$ units. Solve for r in the equation

to get $r = 15/(1 + \pi/6) = \frac{90}{\pi+6} \approx 9.8451$. The sector area is then $\frac{1}{6}\pi r^2$ with the r just found, or $a = \frac{1350\pi}{(\pi+6)^2} \approx 50.7505$.

Similar steps are applied to the other two sectors:

$$r = \frac{13}{1 + \frac{11\pi}{36}} = \frac{468}{11\pi + 36} \approx 6.6329,$$

$$a = \frac{110}{360} \times \pi r^2 = \frac{66924\pi}{(11\pi + 36)^2} \approx 42.2323,$$

and

$$r = \frac{10}{1 + \frac{7\pi}{9}} = \frac{90}{7\pi + 9} \approx 2.9041,$$

$$a = \frac{280}{360} \times \pi r^2 = \frac{6300\pi}{(7\pi + 9)^2} \approx 20.6070.$$

3.80 A cone.

3.82 Solid *A* has a volume of $14 \times 17 \times (16 - 11) + 10 \times 10 \times 11 = 2290$ cubic units and a surface area of $4 \times 10 \times 11 + 10 \times 10 + 2 \times (14 \times 17 + 14 \times 5 + 17 \times 5) - 10 \times 10 = 1226$ square units.

3.84 Solid *C* has a volume of $24 \times 27 \times 31 = 20\,088$ cubic units and a surface area of $2 \times 24 \times 27 + 2 \times 24 \times 31 + 2 \times 35 \times 27 = 4674$ square units.

3.86 Solid *E* on page 129 has a volume of $34 \times 46 \times 47/2 = 36\,754$ cubic units and a surface area of $46 \times 34 + 47 \times 34 + \sqrt{46^2 + 47^2} \times 34 + 2 \times \frac{46 \times 47}{2} = 5324 + 170\sqrt{173} \approx 7560.0$ square units.

3.88 The cylinder's volume is $\pi r^2 h = \pi 6^2 \times 12 = 432\pi$ cubic units. The surface area is $2\pi r^2 + 2\pi r h = 2\pi \times 6(6 + 12) = 216\pi$ square units.

3.92 The surface area is that of a rectangular prism with its 6 faces with two circles removed, but the curved surface of the cylinder added, so $10\,822 + \frac{1545\pi}{2} \approx 13\,249$ square units.

The volume of a rectangular prism with a cylinder removed is $lwh - \pi r^2 h = 58 \times 17 \times 59 - \pi \frac{15^2}{2} \times 59 = 58\,174 - \frac{13\,275\pi}{4} \approx 47\,748$ cubic units.

3.93 The surface area of the cylinder with the top missing is $\pi r^2 + 2\pi r h$ where $h = 11 - 5 = 6$ because $r = 10/2 = 5$. The hemisphere has a surface area of $4\pi r^2/2 = 2\pi 5^2$. All together, we get $25\pi + 60\pi + 50\pi = 135\pi \approx 424.1$ square units.

The volume of solid *L* is $\pi 5^2 \times 6 + \frac{4}{3}\pi 5^3/2 = \frac{700\pi}{3} \approx 733.0$ cubic units.

3.94 The surface area is that of half a cylinder of $r = 13/2$, $h = 13$ plus the square base of side $s = 13$, so $\frac{507\pi}{4} + 169 \approx 567.2$ square units.

The volume of a half cylinder is simply $\pi r^2 h/2 = \frac{2197\pi}{8} \approx 862.8$ cubic units.

3.95 The surface area of the rectangular box is $2 \times (12 \times 17 + 12 \times 24 + 17 \times 24)$, but a strip of 8×17 at the base and two semicircles $2 \times \pi 4^2/2$ on two faces are gone. Instead, half a curved surface of a cylinder $4\pi \times 17$ is on the under side. Therefore, the surface area is $1664 + 52\pi \approx 1827$

The volume of solid *N* is $4896 - 136\pi \approx 4469$ cubic units.

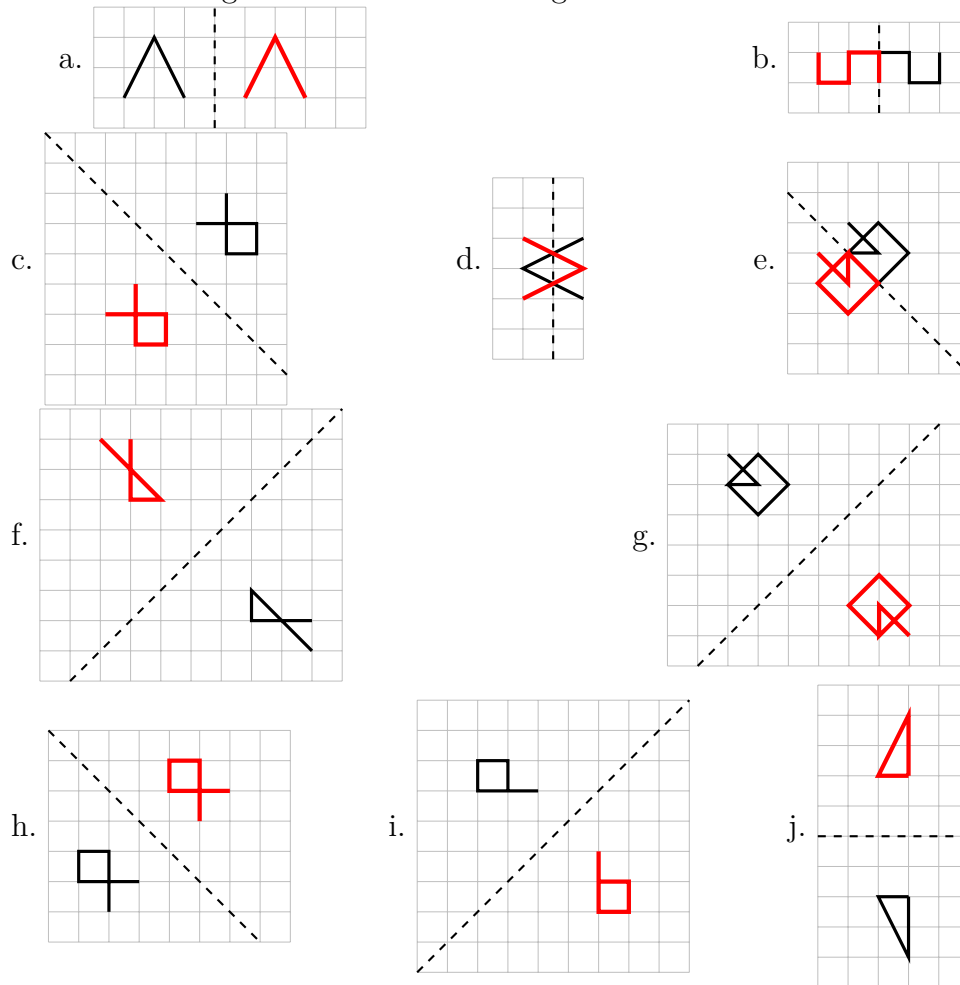
3.96 The surface area is $1666 + \frac{1377\pi}{2} \approx 3829$ square units. The volume is $17^2 \times 32 + \pi \frac{17^2}{2} \times 32 = 9248 + 2312\pi \approx 16\,511$ cubic units.

3.97 The surface area consists of five rectangles in the inner hole $5 \times 23 \times 48$. These five rectangles also appear on the outer face of the nut. Partitioning the nut into rectangular prisms and sector prisms, we get $5 \times 40 \times 23 \times 2$ for the front and back surfaces and $2 \times \pi 23^2$, i.e.

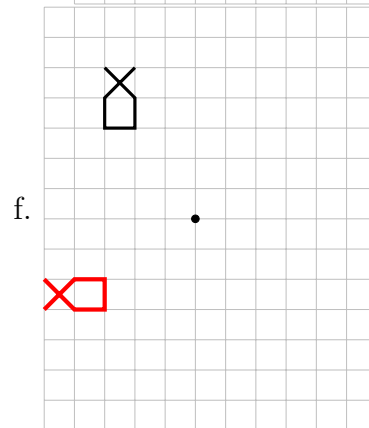
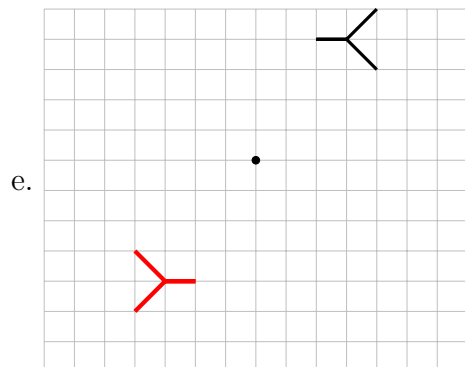
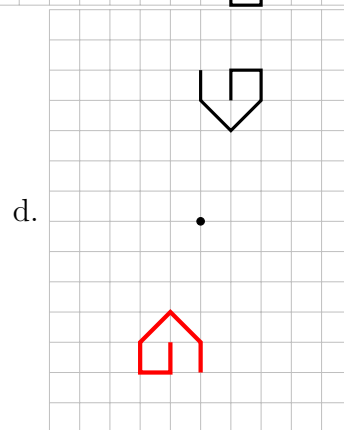
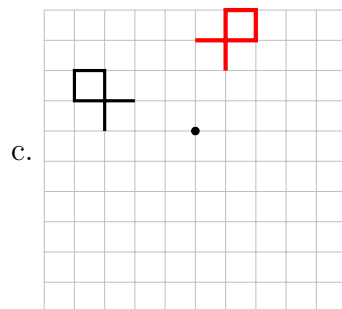
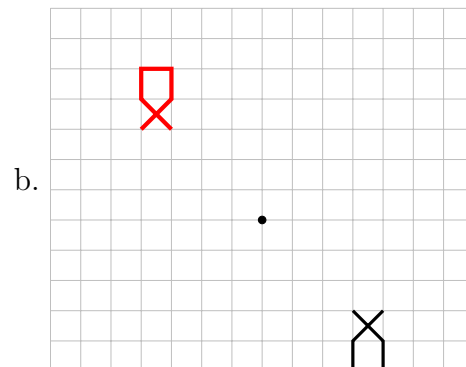
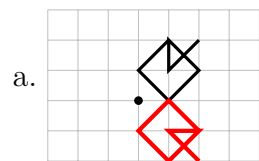
two circles of radius 23. The curved part on the outer surface is the curved surface of a cylinder, $2\pi 23 \times 48$. Together, the surface area is $28\,400 + 3266\pi \approx 38\,660$ square units.

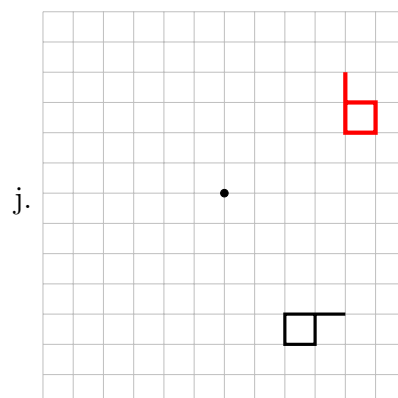
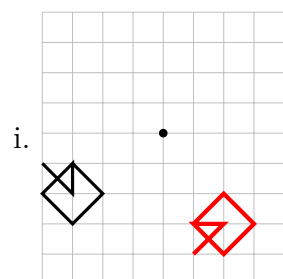
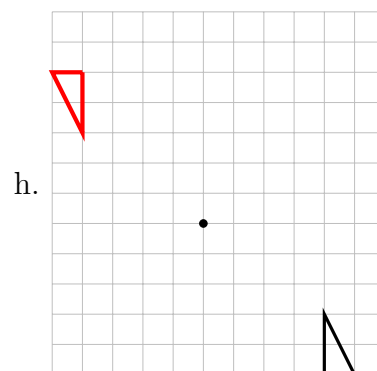
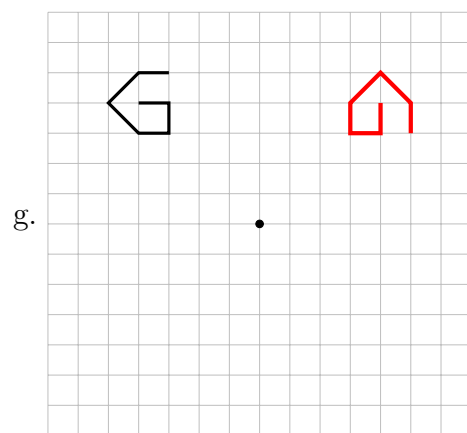
The volume is $5 \times 40 \times 23 \times 48 + \pi 23^2 \times 48 = 220\,800 + 25\,392\pi \approx 300\,571$ cubic units.

3.98 The red figure is the reflected image.



3.100 The red figure is the image after rotation.





4.1 Answers are shown in blue.

a. $\frac{1}{2} = \frac{5}{10} = 0.5$

b. $\frac{4}{5} = \frac{8}{10} = 0.8$

c. $\frac{13}{52} = \frac{1}{4} = \frac{25}{100} = 0.25$

d. $\frac{3}{8} = \frac{375}{1000} = 0.375$

e. $\frac{4}{50} = \frac{8}{100} = 0.08$

f. $\frac{63}{84} = \frac{3}{4} = \frac{75}{100} = 0.75$

g. $\frac{1}{20} = \frac{5}{100} = 0.05$

h. $\frac{1}{40} = \frac{25}{1000} = 0.025$

i. $\frac{3}{50} = \frac{6}{100} = 0.06$

j. $\frac{7}{25} = \frac{28}{100} = 0.28$

k. $\frac{1}{80} = \frac{125}{10\,000} = 0.0125$

l. $\frac{1}{200} = \frac{5}{1000} = 0.005$

m. $\frac{6}{500} = \frac{12}{1000} = 0.012$

n. $\frac{8}{250} = \frac{32}{1000} = 0.032$

o. $\frac{35}{400} = \frac{875}{10\,000} = 0.0875$

4.2 Answers are shown in blue.

a. 50 % of 350 is 175.

b. 45 % of 1000 is 450.

c. 36 % of 50 is 18.

- d. 30 % of 150 is 45.
 e. 30 % of 15 is 45.
 f. 20 % of 150 is 30.
 g. 2 % of 250 is 5.
 h. 60 % of 50 is 30.
 i. 6 % of 500 is 30.
 j. 200 % of 15 is 30.
 k. 20 % of 150 is 30.
 l. 20 % of 15 is 3.
 m. 2 % of 1500 is 30.
 n. 200 % of 15 is 30.
 o. 12 % of 100 is 12.
 p. 80 % of 40 is 32.

4.3 a. $29\% = 0.29$; b. $58.2\% = 0.582$; c. $7.5\% = 0.075$;
 d. $30\% = 0.30 = 0.3$; e. $125.5\% = 1.255$; f. $0.25\% = 0.0025$;
 g. $0.007\% = 0.00007$

4.4 a. $0.635 = 63.5\%$; b. $0.02 = 2\%$; c. $0.45 = 45\%$;
 d. $3.7 = 370\%$; e. $43 = 4300\%$; f. $2.015 = 201.5\%$;
 g. $0.0089 = 0.89\%$; h. $0.33333\dots = 33\frac{1}{3}\%$

4.5 See Table A.1 on the following page

The first repeating decimal question uses an overline on 3 to indicate that the digit 3 is repeated. All the other repeating decimal questions are similarly denoted. When only one digit is repeated, ... may be used.

Fraction	Decimal	Percent
$\frac{2}{5}$	0.4	40 %
$\frac{6}{100} = \frac{3}{50}$	0.06	6 %
$\frac{2}{5}$	0.4	40 %
$\frac{375}{1000} = \frac{3}{8}$	0.375	37.5 %
$\frac{3}{200}$	0.015	1.5 %
$\frac{3}{10}$	0.3	30 %
$\frac{5}{100} = \frac{1}{20}$	0.05	5 %
$\frac{8}{100} = \frac{2}{25}$	0.08	8 %
$\frac{25}{100} = \frac{1}{4}$	0.25	25 %
$\frac{5}{8}$	0.625	62.5 %
$\frac{3125}{10^5} = \frac{1}{32}$	0.031 25	$3\frac{1}{8}$ %
$\frac{25}{10} = 2\frac{1}{2}$	2.5	250 %
$1\frac{1}{3}$	$1.\bar{3}$	133.333... %
$\frac{1}{6}$	$0.1\bar{6}$	16.666... %
$2\frac{5}{9}$	$2.\bar{5}$	255.555... %
$\frac{2}{7}$	$0.\overline{285714}$	$28.\overline{571428}$ %
$\frac{3}{11}$	$0.\overline{27}$	$27.\overline{27}$ %
$4\frac{5}{6}$	$4.8\bar{3}$	483.333... %
$1\frac{4}{7}$	$1.\overline{571428}$	$157.\overline{142857}$ %
$\frac{1}{99}$	$0.\overline{01}$	$1.\overline{01}$ %

Table A.1: Fractions, decimals, and percent

4.6 a. $1.75 \times 139 = 243.25$, so 243 is the nearest whole number.

b. $\frac{20}{15} \times 40\% = \frac{8}{15} = 0.5\bar{3}$

c. $\frac{3}{275} = 0.01\overline{09}$

d. $\frac{145}{8} = 18.125$

e. $\frac{159}{7}$

f. 334.53

g. -24.7

h. 146.08

i. $\frac{4100}{107} \approx 38.318$

j. $\frac{550}{3} = 183.\bar{3}$

k. $\frac{1200}{31} \approx 38.710$

l. 103.49%

m. 10.34%

n. 12%

4.7 The passing rate in percent is $112/(112 + 81) \approx 58.03\%$ because the total who wrote the test is the base amount.

4.9 The total tax rate is 12%, so the total cost would be $30.94 \times 1.12 \approx 34.65$. Round to 2 decimal places for Canadian currency because the smallest unit is a cent.

4.10 A 15% discount leaves 85%, so $61.44 \times 0.85 = 52.224$. Keeping all digits helps retain accuracy. The new price reduced by 50% results in $0.5 \times 52.224 = 26.112$. So the final sale price was \$26.11.

4.11 Tax rate was $12.22/188 = 0.065 = 6.5\%$.

4.13 Total including taxes would be $1.12 \times 435 = 487.20$ dollars.

4.15 Let the original price of the sofa be x , then $1.125x = 597.38$, so $x = 597.38/1.125 = 531.00\bar{4}$. Round the answer to \$531, so $597.38 - 531 = 66.38$ dollars in taxes.

4.19 Say Tuesday's value was x . Then Wednesday's value was $(1 - 0.039)x$, so Friday's value was $1.021(1 - 0.039)x = 14\,721.53$, so $x = \$15\,003.89$.

4.21 Janette skipped one question out of 12, so $1/12 \approx 8.3\%$.

4.24 $5580 \div 0.93 = 6000$. The stadium could host 6000 people.

4.25 $1.016^4 = 1.065\,552\,45$, or an increase of 6.56% over four years.

4.26 The area of a circle is $A = \pi r^2$. Let r be the radius of the original circular patio. The enlarged patio would have radius $1.089r$, so the new area would be $\pi(1.089r)^2 = 1.185\,921\pi r^2$. The percentage increase was $(1.185\,921 - 1)\pi r^2 / (\pi r^2) = 0.185\,921$ or 18.59% .

4.28 $\pounds 1 = \$46.84$. Dividing both sides of the equation by 46.84, one gets $\pounds \frac{1}{46.84} = \1 , so $\pounds 0.021\,35 = \$1$.

4.31 $30\,000 \div 1500 = 20$, so the Lebanese pound lost $(20 - 1)/20 = 95\%$ of its value.

4.32 Assume something was worth 1 bolivar in the beginning of 2016. By the end of 2016, it would be worth 8 bolivar, by the end of 2017, $8 \times 40 = 320$ bolivar; in the end of 2018, $17\,000 \times 320 = 5\,440\,000$. The buying power of a one-million bolivar note from the beginning of 2016 would be worth $10^6 / (544 \times 10^4) \approx 0.1838$, not even a fifth of one bolivar.

4.34 $I = Prt = 6000 \times 0.02 \times 1 = 120$ dollars.

4.36 $376/4700 = 0.08$, so 8%.

4.39 See Table A.2. Round dollar answers to two decimal places.

Future Value	Principal	rate	t (years))
a) \$8243.63	\$7800	$1\frac{5}{8}\%$	3.5
b) \$11 077.50	\$10 500	2.75%	2
\$500 500	c) \$314 903.66	$5\frac{3}{4}\%$	10.25
\$250 000	\$80 000	d) 14.17%	15
\$45 000	\$9000	$8\frac{3}{8}\%$	e) 47.76

Table A.2: Future value, principal, rate, and time

4.41 Since the principal of \$5000 remains the same, we focus on the factor $(1 + r/n)^n$ for the effect after one year. If the GIC was invested for more than a year, then $(1 + r/n)^{nt} = ((1 + r/n)^n)^t$. Because the exponential function is one-to-one, the larger $(1 + r/n)^n$ is, the larger $(1 + r/n)^{nt}$ will be.

- 2% compounded semi-annually: $(1 + 0.02/2)^2 = 1.0201$
- 1.8% compounded monthly: $(1 + 0.018/12)^{12} \approx 1.018149245$
- 1.78% compounded weekly: $(1 + 0.0178/52)^{52} \approx 1.017956264$
- 1.75% compounded daily: $(1 + 0.0175/360)^{360} \approx 1.017653589$
(using 360 days a year)

Regardless of the dollar amount for the GIC or the length of time to keep the GIC, the highest yield came from 2% compounded semi-annually.

4.43 Let the principal be P . Then $P + 5237 = P(1 + 0.0225/12)^{12 \times 5} \approx P \times 1.118954383$. Subtracting P on both sides, $5237 \approx 0.118954383P$ so $P \approx 5237/0.118954383 \approx 44\,025.28$.

Brian's grandparents bought a \$44 025.28 GIC.

4.45 $1100 = 1000(1 + r/365)^{3650}$, or $1.1 = (1 + r/365)^{3650}$. Raise both sides to the fractional power of $1/3650$ to get $(1.1)^{1/3650} = 1 + r/365$. Subtracting 1 on both sides,

$$\frac{r}{365} = (1.1)^{1/3650} - 1, \quad \text{so} \quad r = 365 \times ((1.1)^{1/3650} - 1) \approx 0.0095311424,$$

less than 1%.

4.47 According to the compound interest future value formula, let t be the number of years David needed to wait. Then

$$50\,000 = 45\,000(1 + 0.01/12)^{12t}, \quad \text{or} \quad \frac{50}{45} = (1 + 0.01/12)^{12t}.$$

To solve for t , one needs to take log on both sides:

$$\log\left(\frac{50}{45}\right) = 12t \log(1 + 0.01/12),$$

so

$$t = \frac{\log(\frac{50}{45})}{12 \log(1 + 0.01/12)} \approx 10.54044098,$$

or 10 years and 198 days.

4.50 Patrick lost 15% of the value, or the rate was -15% .

4.51 Applying the compound interest formula for 4 years with $A = 50\,000$, $P = 10$, we get

$$50\,000 = 10(1 + r/1)^4, \quad \text{so} \quad r = \sqrt[4]{5000} - 1 \approx 7.40896,$$

or 740.90%.

4.52 $942\,500/38.01 \approx 24\,796.11$. Divided equally, each Canadian owed \$24 796.11 national debt.

4.54 $I = Prt = 800 \times 0.18 \times \frac{40}{365} \approx 15.78$ dollars.

4.56 $888 \times 12 \times 5 = 53\,280$, so finance charge is $53\,280 - 28\,888 = 24\,392$. Using simple interest, the annual rate is $24\,392/28\,888/5 \approx 0.16887$, or 16.89%. It would be cheaper to rent a car.

4.57 Let x be the original taxi ride cost before GST and the tip, then

$$1.17x = 74.52, \quad \text{so} \quad x = 74.52/1.17 \approx 63.69,$$

or the taxi driver charged \$63.69 before adding a 5% GST and a 12% tip (17% in total).

4.58 $1137.28 \times 12 \times 7 = 95\,531.52$, so interest paid would be $95\,531.52 - 85\,000 = 10\,531.52$. At first, the interest may seem a large percentage of the loan, $10\,531.52/85\,000 \approx 0.1239$; however, after dividing the rate by 7, the annual rate only came to 1.77%, a super deal.

4.60 a. The monthly instalment would be $30\,000 \times (1 + 0.04 \times 5) / (12 \times 5) = 600$, using add-on interest method.

b. The total finance charge on the loan would be $30\,000 \times 0.04 \times 5 = 6000$ dollars.

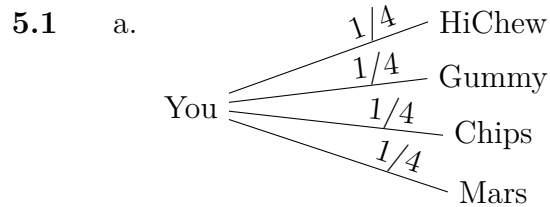
c. The balance on the loan after the first instalment can be found after finding the interest portion of the first instalment of \$600. Since \$6000 in interest was spread over 60 months, the first month's interest was \$100, so \$500 went to reducing the loan from \$30 000 to \$29 500, the new balance on the loan.

4.63 a. Over 36 months, David's monthly payments would be $1800 \times (1 + 0.015 \times 3) / 36 = 52.25$ dollars.

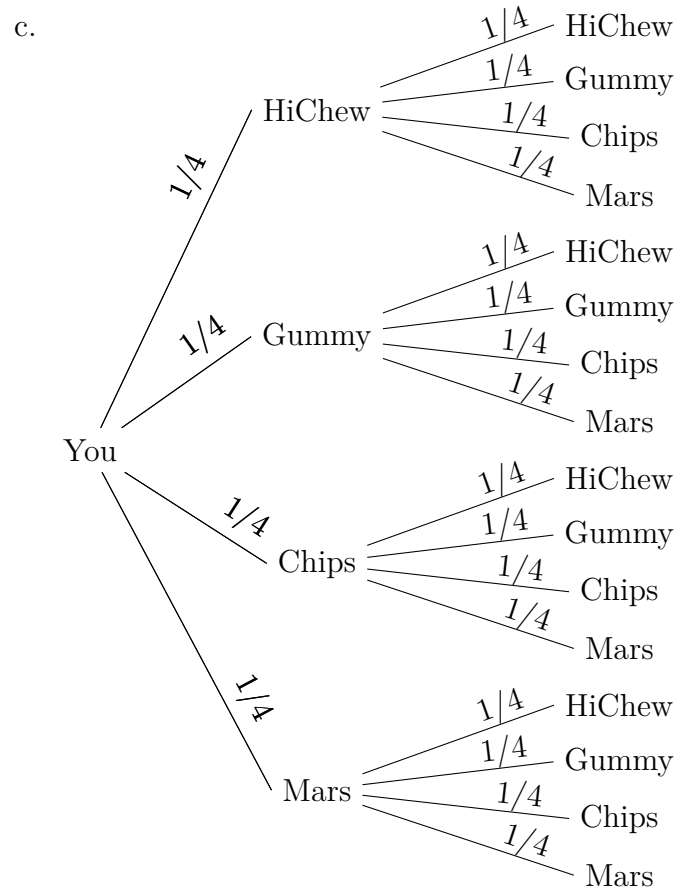
- b. If David put down \$300 of his own money, he only needed to borrow \$1500. Therefore, the monthly payments would be $1500 \times (1 + 0.015 \times 3)/36 \approx 43.54166$, or \$43.55, a reduction of \$8.70 every month.
- c. If David wanted monthly payments of \$40 instead, one needs algebra to solve for the principal before deciding what the down payment should be. Let the amount borrowed (the principal) be P .

$$40 = \frac{P \times (1 + 0.015 \times 3)}{36}, \quad \text{so} \quad P = 40 \times 36/1.045 \approx 1377.99.$$

The down payment was $1800 - 1377.99 = 422.01$ dollars.



- b. {Mars, HiChew, Gummy, Chips}



d. One can use a table of 16 ordered pairs as outcomes for the sample space. One can also list the ordered pairs as a set. Each item will be denoted by its initial.

$$\{(M, M), (M, H), (M, G), (M, C), (H, M), \\ (H, H), (H, G), (H, C), (G, M), (G, H), (G, G), \\ (G, C), (C, M), (C, H), (C, G), (C, C)\}$$

e. $\{(M, C), (H, C), (G, C), (C, M), (C, H), (C, G)\}$

- f. Note that it is the entire sample space for spinning the spinner twice except for (G, G) .

$$\begin{aligned} &\{(M, M), (M, H), (M, G), (M, C), (H, M), \\ &\quad (H, H), (H, G), (H, C), (G, M), (G, H), \\ &\quad (G, C), (C, M), (C, H), (C, G), (C, C)\} \end{aligned}$$

- g. With three kinds of candy, nine pairs are possible:

$$\begin{aligned} &\{(M, M), (M, G), (M, C), (G, M), \\ &\quad (G, G), (G, C), (C, M), (C, G), (C, C)\} \end{aligned}$$

- h. The non-Mars can come last, in the middle, or first.

$$\begin{aligned} &\{(M, M, C), (M, M, G), (M, M, H), (M, C, M), \\ &\quad (M, G, M), (M, H, M), (C, M, M), (G, M, M), (H, M, M)\} \end{aligned}$$

- 5.3** a. After one spin, $P(\text{HiChew}) = \frac{1}{6}$.
- b. After one spin, $P(\text{Mars} \cup \text{Chips}) = \frac{1}{3}$.
- c. After two spins, $P(\text{Mars} \cap \text{Chips}) = \frac{1}{18}$ from 2 outcomes, i.e. (M, C) and (C, M) out of 36 ordered pairs.
- d. After two spins, $P(\text{Mars} \cup \text{Chips}) = \frac{5}{9}$. With the help of a probability tree, one sees that if the first spin gives Mars, the second spin can be anything; similarly for Chips in the first spin. Together they contribute a probability of $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. If the first spin is neither Mars nor Chips, a probability of $\frac{4}{6} = \frac{2}{3}$, the second spin must give either a Mars or a Chips, so $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$. Combining both cases, one obtains

$$\frac{1}{3} + \frac{2}{9} = \frac{5}{9}.$$

- e. After two spins, $P(\text{no Gummy}) = (\frac{5}{6})^2 = \frac{25}{36}$.
- f. After two spins, $P(\text{at least one Mars}) = 1 - (\frac{5}{6})^2 = \frac{11}{36}$. Here one uses the property of complements by finding first the probability of getting no Mars after two spins, then subtracts the answer from 1.
- g. After two spins, $P(\text{exactly one Chips}) = \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{10}{36} = \frac{5}{18}$.

5.6 Using the property of the complement, one finds the probability of getting no tail to be $(1/2)^4$, so $1 - \frac{1}{16} = \frac{15}{16}$.

5.8 A regular tetrahedron is the solid for a four-sided die. Each face is an equilateral triangle. The sample space for rolling two D4's has 16 outcomes.

- a. $P(X_1 + X_2 = 3) = \frac{2}{16} = \frac{1}{8}$ from $(\blacktriangle 1, \blacktriangle 2)$ and $(\blacktriangle 2, \blacktriangle 1)$.
- b. $P(X_1 + X_2 \geq 5) = \frac{4+3+2+1}{16} = \frac{5}{8}$
- c. $P(X_1 + X_2 < 4) = \frac{3}{16}$
- d. $P(|X_1 - X_2| > 2) = \frac{1}{8}$ since the difference between the dice has to be 3 because the die's minimum is 1 and maximum is 4. Therefore the only possibilities are $(\blacktriangle 1, \blacktriangle 4)$ and $(\blacktriangle 4, \blacktriangle 1)$.
- e. $P(X_1 + X_2 = 2n) = 1/2$ because of the 16 outcomes in the sample space, half are even.
- f. $P(X_1 + X_2 = 2n - 1) = 1/2$ because of the 16 outcomes in the sample space, half are odd.
- g. $P(X_1 + X_2 = 2) + P(X_1 + X_2 = 5) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$.

5.10 Using Table 5.1 on page 181, one sees $1+2+3+4+5 = 15$ outcomes satisfying the condition. Thus $P(X_g + X_v < 7) = 15/36 = 5/12$.

5.13 a. $P(4\clubsuit) = \frac{1}{52}$

b. $P(4) = \frac{4}{52} = \frac{1}{13}$

c. $P(\clubsuit) = \frac{13}{52} = \frac{1}{4}$

d. $P(5 \cup \text{queen}) = \frac{4+4}{52} = \frac{2}{13}$

e. $P(8 \cup \text{royal}) = \frac{4+12}{52} = \frac{4}{13}$

f. $P(\heartsuit \cup \text{king}) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

5.14 There are 4 hands of Royal flush and 36 hands of Straight flush.

5.15 a. $P(\text{one pair}) = 1\,098\,240/2\,598\,960 = 352/833 \approx 0.4226$

b. $P(\text{royal flush}) = 4/2\,598\,960 = 1/649\,740 \approx 1.539 \times 10^{-6}$

c. $P(\text{full house}) = 3744/2\,598\,960 = 6/4165 \approx 1.657 \times 10^{-4}$

d. $P(\text{straight flush}) = 40/2\,598\,960 = 1/64\,974 \approx 1.539 \times 10^{-5}$

e. $P(\text{three of a kind}) = 54\,912/2\,598\,960 = 88/4165 \approx 0.021\,13$

f. $P(\text{flush}) = 5108/2\,598\,960 = 1277/649\,740 \approx 0.001\,965$

g. $P(\text{two pair}) = 123\,552/2\,598\,960 = 198/4165 \approx 0.047\,54$

5.18 Drawing the probability tree in Figure A.3 on the next page takes a bit of effort, but once you have it, answering the questions is easy.

a. $P(\text{red in the first draw}) = \frac{1}{5} + \frac{1}{12} = \frac{17}{60}$.

b. $P(\text{teal in the second draw}) = \frac{3}{40} + \frac{1}{12} + \frac{3}{40} + \frac{1}{8} + \frac{1}{40} + \frac{1}{24} + \frac{1}{8} + \frac{1}{6} = \frac{43}{60}$.
 Note that this the same as $P(\text{teal in the first draw}) = \frac{3}{10} + \frac{5}{12} = \frac{43}{60}$
 since we don't know anything about the other draw in either case.

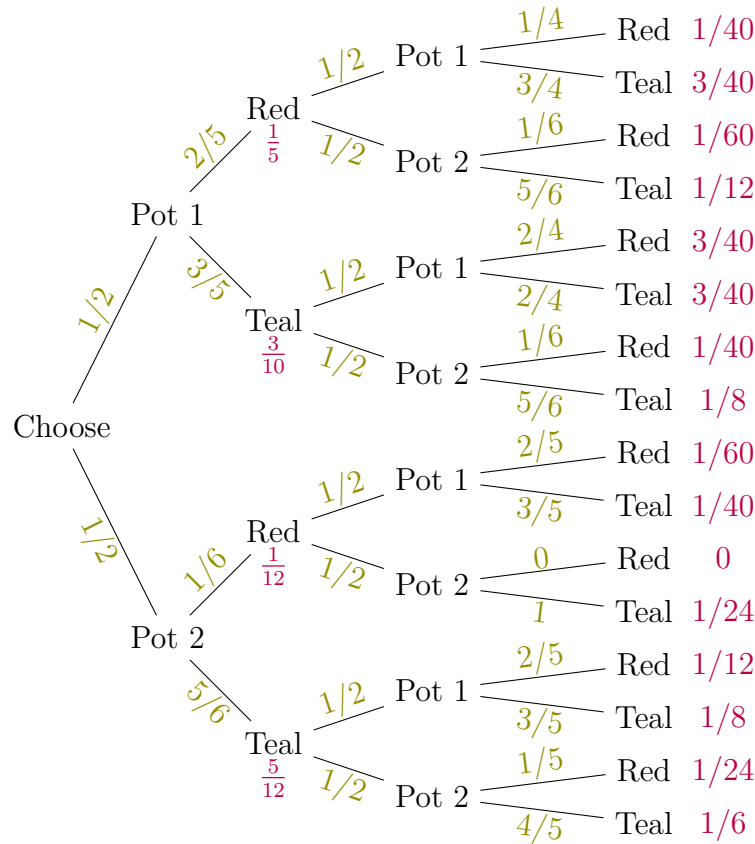


Figure A.3: The probability tree for Ham's game of red and teal balls

- c. $P(\text{two red balls}) = \frac{1}{40} + \frac{1}{60} + \frac{1}{60} + 0 = \frac{7}{120}$ and $P(\text{two teal balls}) = \frac{3}{40} + \frac{1}{8} + \frac{1}{8} + \frac{1}{6} = \frac{59}{120}$. Therefore $P(\text{two balls of the same colour}) = \frac{7}{120} + \frac{59}{120} = \frac{11}{20}$.
- d. In the previous part we found that the probability of two reds is $\frac{7}{120}$. The probability of *not* getting two reds is then $1 - \frac{7}{120} = \frac{113}{120}$,

so the odds in favour of getting two red balls is

$$\frac{7/120}{113/120} = \frac{7}{113} \quad \text{or } 7 : 113.$$

5.22 $P(X = 8) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$

5.23 A sample space of 8×6 shows where a sum of 15 is. Thus $P(X_1 + X_2 = 15) = 4/48 = \frac{1}{12}$.

5.27 Using permutations, Hamlet could form $9 \cdot 8 \cdot 7$ 3-digit numbers without repeating digits. Of those, $5 \cdot 8 \cdot 7$ would be odd, so $5/9$ is the probability.

5.30 22% chance of winning implies 78% of losing, so the odds against the team would be $78 : 22 = 39 : 11$.

5.32 The probability of winning the World Cup soccer was $\frac{5}{8+5} = \frac{5}{13} \approx 0.3846 = 38.46\%$.

5.33 The probability of winning the World Series was $\frac{4}{4+9} = \frac{4}{13} \approx 0.3077 = 30.77\%$.

5.34 Notices that $1 : 1\,000\,000$ is very close to $1/1\,000\,001 = 9.999\,99 \times 10^{-7}$. For small probabilities, the scientific notation helps one focus on the differences.

5.35 Adding the totals,

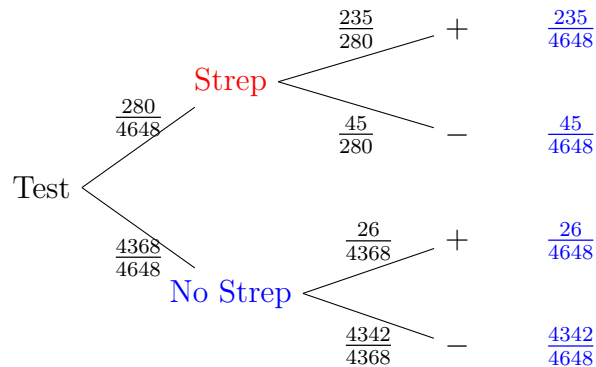
	Issue in Survey		
	Keep	Free	Total
Live in Vancouver	4235	8275	12 510
Live in suburbs	7829	6098	13 927
Total	12 064	14 373	26 437

- a. The probability of a randomly chosen person supporting keeping the cetaceans is $\frac{12\,064}{26\,437} \approx 0.4563$.
- b. The probability of a randomly chosen person not being a resident of Vancouver is $\frac{13\,927}{26\,437} \approx 0.5268$.
- c. The probability of a randomly chosen person from Vancouver who supported the keeping of cetaceans is $\frac{4235}{12\,510} \approx 0.3385$.
- d. The probability of a randomly chosen person from the suburbs opposing the keeping of cetaceans is $\frac{6098}{13\,927} \approx 0.4379$.

5.37 Add the totals to the table and read that $P(\text{Strep} \mid +) = 235/261 \approx 90.04\%$.

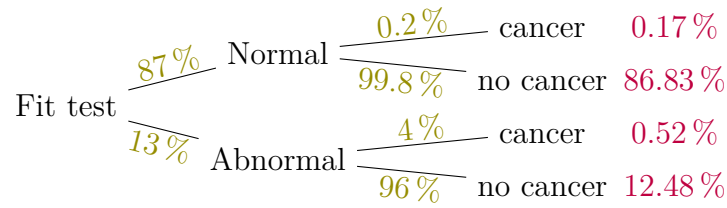
		Streptococcus		
		Have Strep A	No Strep A	Total
Test	+	235	26	261
Results	-	45	4342	4387
Total		280	4368	4648

Alternatively, from a probability tree below, $P(+)=\frac{235}{4648}+\frac{26}{4648}=\frac{261}{4648}$, and $P(\text{Strep} \cap +)=\frac{235}{4648}$, so $P(\text{Strep} \mid +)=\frac{235}{261}$, the same as the answer computed directly from the table.



5.39 Of the 10% to 15% of those taking FIT that had abnormal results, less than 4% had cancer: between $0.1 \cdot 0.04 = 0.004$ and $0.15 \cdot 0.04 = 0.006$. Less than 0.6% of the population had colon cancer.

5.40 The probability tree:



Therefore

$$P(- | C) = \frac{P(- \cap C)}{P(- \cap C) + P(+ \cap C)} = \frac{0.87 \cdot 0.002}{0.87 \cdot 0.002 + 0.13 \cdot 0.04} \approx 0.2507 = 25.07\%.$$

This is alarmingly high, over a quarter of the time. This result shows the importance of paying attention to symptoms.

5.41 In the table below the given numbers are black. Deduce the blue numbers by subtraction. Of the 1003 healthy people 0.30% say that they suffer from hypochondria, so 3 people think they are hypochondriacs even though they are healthy. The purple numbers and, finally, the red number now follow by subtraction.

	hyp.	not hyp.	Total
Yes	1	3	4
No	9	1000	1009
Total	10	1003	1013

The chance of a false negative (saying no when you actually are a hypochondriac) is $\frac{9}{10} = 90\%$. Self-assessment for **hypochondria** is a pretty bad test.

- 5.45 a. $P(\bar{B}) = \frac{10}{19} \approx 52.63\%$
- b. $P(A \text{ and } B) = \frac{4}{19} \approx 21.05\%$
- c. $P(B | \bar{A}) = \frac{5}{9} \approx 55.55\%$
- d. $P(A | B) = \frac{4}{9} \approx 44.44\%$
- e. $P(A \text{ and } \bar{B}) = \frac{6}{19} \approx 31.58\%$
- f. By definition of **independent events**, one needs to check whether $P(A \cap B) = P(A) \cdot P(B)$, or, equivalently, whether $P(B) = P(B | A)$.
In this case, $P(B) = \frac{9}{19}$ while $P(B | A) = \frac{4}{10} = \frac{2}{5}$. Therefore $P(B) \neq P(B | A)$, so A and B are NOT INDEPENDENT.

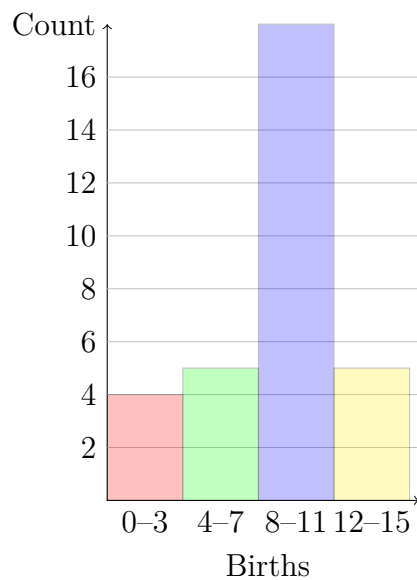
- 5.47 a. The chance of not getting both rain and fog to occur on Samlette's birthday is $1 - 0.3 = 0.7$.
- b. The probability of not getting rain on Samlette's birthday is $1 - 0.7 = 0.3$
- c. The probability of having either rain or fog (or both) on Samlette's birthday is $0.7 + 0.4 - 0.3 = 0.8$.

5.48 $0.7^3 = 0.343$.

- 5.50 a. $P(A \cup B) = 0.9 + 0.7 - 0.65 = 0.95$.
- b. $P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.05$.
- c. $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.65}{0.70} = 0.9286$.

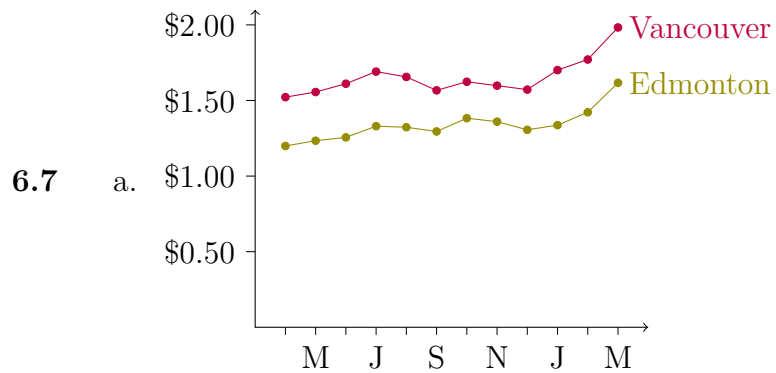
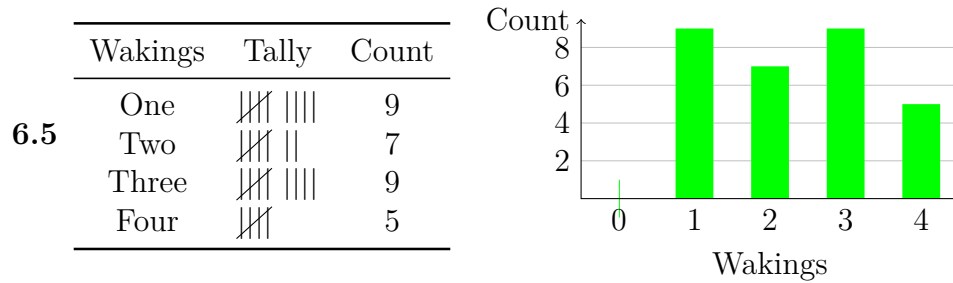
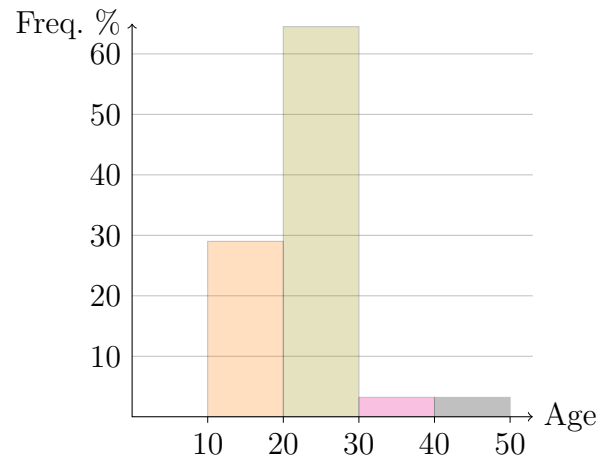
6.1

Births	Tally	Count
0-3		4
4-7		5
8-11		18
12-15		5



6.3

Stem	Leaves	Age	Freq.	Rel. freq. (%)
1	888999999	10-19	9	29.0
2	13031123453010227348	20-29	20	64.5
3	7	30-39	1	3.2
4	8	40-49	1	3.2



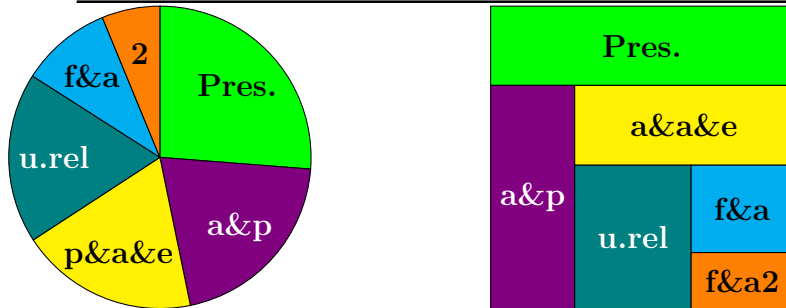
b. The price jump was likely caused by Russia's invasion of Ukraine.

c. Vancouver: $\frac{198.3}{177.1} - 1 = 11.97\%$. Edmonton: $\frac{161.7}{142.2} - 1 = 13.71\%$.

6.9 The purple line graph from the tab: Energy Highlights has the greatest positive slope, indicating the largest percentage increase.

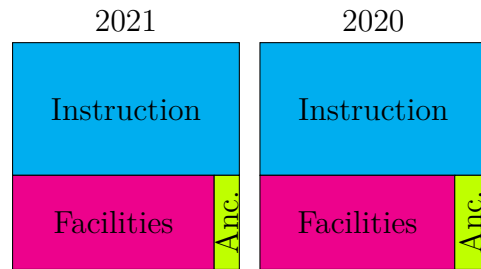
6.10

Title	Compensation	Percentage	Angle (°)
President	266 450	26.2	94.5
VP, a&p	208 461	20.5	73.9
VP, p&a&e	193 341	19.0	68.5
VP, f&a	98 934	9.7	35.1
VP, f&a 2	63 095	6.2	22.4
VP, u.rel	185 240	18.2	65.7
Total	1 015 521	99.8	360.1



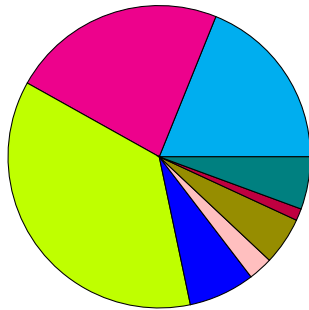
6.12 From the report:

Expense	2021	2020
Instruction	69 744 057	70 989 665
Facilities	44 059 884	43 283 217
Ancillary	5 480 966	7 092 677



6.14

Asset	Percentage	Angle
Public Equity	18.9	68.0
Private Equity	23.0	82.8
Hedge Funds	36.4	131.0
Real Estate	7.1	25.6
Natural Resources	2.6	9.4
Bonds/TIPS	5.1	18.4
Other Real Assets	1.3	4.7
Cash & Other	5.6	20.2



6.16 $2849 / (43 + 429 + 261 + 182 + 2849) \cdot 100\% = 75.69\%$.

6.17 a. July and December

b. March, 2021 export exceeded that of 2020's by almost one billion dollars.

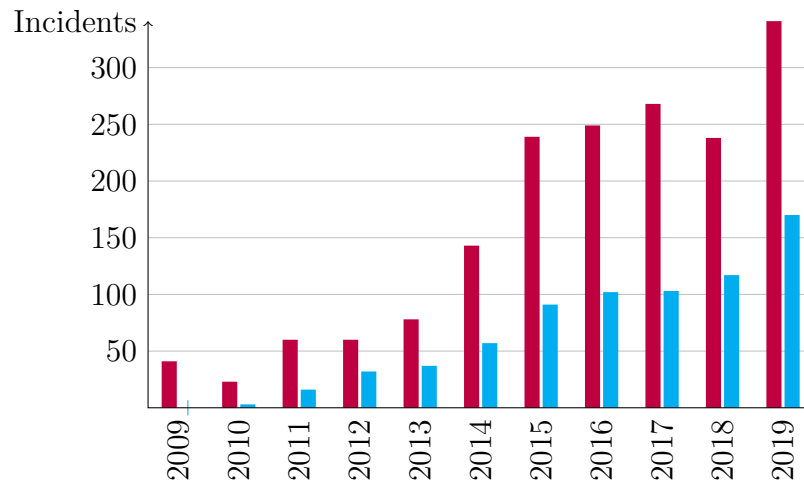
- c. July looks smaller than August but the difference according to the numbers shown below the bars is the same.
- d. Both used 2020 as the base amount, so 2.1 B in 2020 is as large a base as 2.0 B in 2021. $(2.0 - 2.1)/2.1 \approx -4.76\%$ compared to $(2.1 - 2.0)/2.0 \approx 5.00\%$. Note that the percentages differ from the graph's because of rounding from **trade data** of Statistics Canada: July \$2.050 417 billion in 2020 to \$1.973 157 billion in 2021; August \$2.025 984 billion in 2020 to \$2.119 417 billion in 2021.

- 6.19**
- a. For both boys and girls, the percentages according to gender add up to a number greater than 100. If each category indicates percentage of the whole population of girls or boys, then there must be some who are involved in more than one activity.
 - b. Wages gaps are separated into Full-time jobs vs. Full-time informal work with family, friends, or neighbours. Under full-time, girls earn on average 15.26 dollars per hour while boys earn 18.01 dollars per hour. For the other category, girls earn 8.67 dollars per hour while boys earn 14.98 dollars per hour. The more relevant to me is the disparity in the full-time informal work with family, friends, or neighbours where these people seem to value girls' work much less.
 - c. In a stacked bar graph, the focus shifts to the total dollars earned per hour, more difficult to see the difference in wage gap between the genders.

- 6.22**
- a. 0.045 %, five times more likely.
 - b. 0.085 %, 25 times more likely.
 - c. 0.053 %, provincial infraction.

- d. 0.083 %, criminal offence.
- e. No, his BAC would be around 0.093 %, so driving would be a criminal offence.

- 6.23** a. In 2014 the rate was 0.6 per 100 000 population while in 2015 it was 0.9, so an increase of $\frac{0.9-0.6}{0.6} = 50\%$. Similarly, from 2018 to 2019 the increase was $\frac{1.4-1.0}{1.0} = 40\%$, so the rate of increase went down.
- b. The original stacked bar graph is better at showing the development of the overall number of cases regardless of how they happen to be classified in law.



- c. In the original graph, the labels on the left pertain to the stacked bars while the labels on the right pertain to the line graph.

- 6.24** Search for the population of the US in 2017 to find 325.1 million. Adding the death rates per 100 000 population accounting for all three types of deaths of despair, one gets $6.8 + 14.5 + 21.6 = 42.9$ deaths per 100 000.

Finally, $42.9 \times 325.1 \times 10^6 \div 10^5 \approx 139\,468$ deaths of despair in the US in 2017.

6.25 a. $\bar{x} = 45.25$ years; b. $\tilde{x} = 45.5$ years; c. Mode: 42 years.

6.27 a. $\bar{x} = 1.26$ siblings; b. $\tilde{x} = 1$ sibling; c. Mode: 1 sibling.

6.29 a. $\bar{x} = 5.26$ platforms; b. $\tilde{x} = 4$ platforms; c. Mode: 4 platforms.

6.30 a. $5 + 2 + 4 + 7 + 9 + 7 + 4 + 2 = 40$ students.

b. $\bar{x} = \frac{3 \times 5 + 4 \times 2 + 5 \times 4 + 6 \times 7 + 7 \times 9 + 8 \times 7 + 9 \times 4 + 10 \times 2}{40} = \frac{260}{40} = 6.5.$

c. $5 + 2 + 4 + 7 = 18$ students score 3, 4, 5, or 6 points. The 20th and 21st students therefore both score 7 points, so $\tilde{x} = 7$.

d. The most common score is 7 points.

e. Not obvious in this case.

6.32 a. 20 staff members.

b. $\bar{x} = \frac{8+8+12+15+17+17+19+21+21+28+29+33+34+35+37+39+40+42+57+63}{20} = \frac{575}{20} = 28.75$ contacts

c. $\tilde{x} = \frac{28+29}{2} = 28.5$ contacts

d. Three modes: $\{8, 17, 21\}$.

e. See Table [A.3 on the following page](#).

6.33 Let x be the score of the fifth test. One needs to solve for x in the following inequality.

$$\frac{84 + 69 + 47 + 58 + x}{5} \geq 76, \quad \text{so} \quad 84 + 69 + 47 + 58 + x \geq 76 \cdot 5,$$

so $x \geq 380 - 258 = 122$. Since the maximum score was 100, it was impossible for Alex to receive a B+.

Number of Contacts	Frequency (staff)	Relative frequency	Cumulative frequency
0–9	2	$2/20 = 0.1$	2
10–19	5	$5/20 = 0.25$	7
20–29	4	$4/20 = 0.2$	11
30–39	5	$5/20 = 0.25$	16
40–49	2	$2/20 = 0.1$	18
50–59	1	$1/20 = 0.05$	19
60–69	1	$1/20 = 0.05$	20

Table A.3: A frequency table for contacts

6.35 Let Julian's final score be x . One needs to solve for x in the following inequality.

$$0.65 \frac{71 + 88 + 69 + 57 + 75 + 82}{6} + 0.35x \geq 80, \quad \text{so } x \geq 91.7619,$$

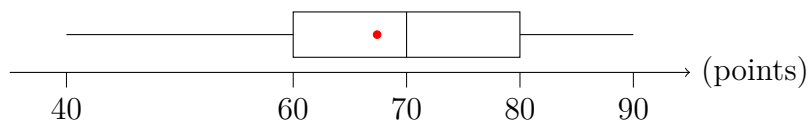
or Julian needed at least 91.77 in the final exam to get an A–.

6.36 a. $2 + 5 + 9 + 7 + 3 + 1 = 27$ students.

b. The range is 40–90 points.

c. $\bar{x} = \frac{2 \times 90 + 5 \times 80 + 9 \times 70 + 7 \times 60 + 3 \times 50 + 40}{27} = \frac{1820}{27} \approx 67.4$. Since 1 + 3 + 7 scored 40, 50, or 60 points, the 14th student scored $\tilde{x} = 70$ points. The most common score was 70 points.

d. Marking the mean with a red dot,



6.38 a. $6 + 4 + 7 + 2 + 1 = 20$ seniors.

b. $\bar{x} = \frac{0 \times 6 + 1 \times 4 + 2 \times 7 + 4 \times 2 + 6 \times 1}{20} = \frac{32}{20} = 1.6$ visits per senior.

c. The range is 0–6.

d. The problem states that this is a *sample*, so you must use the sample version of the formula for variance:

$$s^2 = \frac{1}{20 - 1} \left(6(0 - 1.6)^2 + 4(1 - 1.6)^2 + 7(2 - 1.6)^2 + 2(4 - 1.6)^2 + (6 - 1.6)^2 \right) = \frac{48.80}{19} \approx 2.568$$

e. The (sample) standard deviation is then $s = \sqrt{s^2} = \sqrt{2.568} = 1.603$ visits per senior.

f. The coefficient of variation is $\frac{s}{\bar{x}} = \frac{1.603}{1.6} \approx 1.002 = 100.2\%$.

6.40 a. $10 + 13 + 22 + 8 + 5 + 1 = 59$.

b. The table does not provide the actual data but only a summary. The best you can do, then, is to use the midpoint of each interval: $\bar{x} \approx \frac{10 \times 4.5 + 13 \times 14.5 + 22 \times 24.5 + 8 \times 34.5 + 5 \times 44.5 + 54.5}{59} = \frac{1325.5}{59} \approx 22.47$ minutes.

c. As far as the table shows, the patients waited 0–59 minutes.

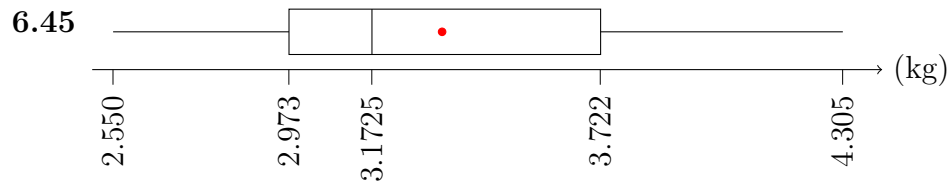
d. Again, the best estimate is

$$s^2 \approx \frac{1}{59 - 1} \left(10(4.5 - \bar{x})^2 + 13(14.5 - \bar{x})^2 + 22(24.5 - \bar{x})^2 + 8(34.5 - \bar{x})^2 + 5(44.5 - \bar{x})^2 + (54.5 - \bar{x})^2 \right) = \frac{8755.9}{58} \approx 151.0$$

e. $s = \sqrt{s^2} \approx 12.29$.

f. The coefficient of variation is $\frac{s}{\bar{x}} \approx 0.5469$.

- 6.41** a. Costa Rica had the biggest change, from below 20 hours to over 40 hours per week spent by students using the Internet.
- b. The grey circle overlaps the blue triangle for Czech Republic.
- c. Denmark.



The red dot indicates the mean at 3.34 kg. The small sample is not enough for drawing a conclusion regarding a trend.

- 6.48** a. According to the table on page 11 of the report from Fraser Institute, BC spent \$11 656 per student in 2015/16.
- b. BC had the second lowest spending per student in public school education.
- c. Back in 2016, BC's expense of C\$11 656 **would have been** about US\$9019. With **inflation** that's about US\$9726 in 2020. Tennessee and Mississippi come closest to that with US\$9896 and US\$9653, respectively.

Bibliography

- [1] Angus Deaton Anne Case. *Deaths of Despair and the Future of Capitalism*. Princeton University Press, 2020. ISBN: 9780691190785. URL: <https://press.princeton.edu/books/hardcover/9780691190785/deaths-of-despair-and-the-future-of-capitalism> (cit. on p. 250).
- [2] Joel Bakan. *The Corporation: The Pathological Pursuit of Profit and Power*. Penguin Random House Canada, 2004. ISBN: 978-0-140-290042 (cit. on p. 239).
- [3] Rachel Carson. *Silent Spring*. Houghton Mifflin Company, 1962. ISBN: 978-0618249060 (cit. on p. 151).
- [4] David Chalmers. “The Virtual and the Real”. In: *Disputatio* 9 (2017), pp. 309–352. URL: <https://www.sciendo.com/article/10.1515/disp-2017-0009> (cit. on p. 223).
- [5] David Chilton. *The Wealthy Barber: The Common Sense Guide to Successful Financial Planning*. Stoddart, 1989. ISBN: 978-0-7737-53181 (cit. on pp. 141, 147).
- [6] Stillman Drake. *Galileo Studies: Personality, Tradition, and Revolution*. University of Michigan Press, 1970. ISBN: 978-0472082834 (cit. on p. 18).
- [7] Euclid. *The Thirteen Books of The Elements*. Trans. Greek by Sir Thomas L. Heath. Dover Publications, Inc. ISBN: 0-486-60088-2 (cit. on p. 84).
- [8] Kitty Ferguson. *The Music of Pythagoras: How an Ancient Brotherhood Cracked the Code of the Universe and Lit the Path from Antiquity to Outer Space*. Walker Publishing Company, Inc. ISBN: 978-0-8027-16316 (cit. on p. 93).

- [9] Stephen Few. *Show Me the Numbers: Designing Tables and Graphs to Enlighten*. Analytic Press, 2004. ISBN: 978-0970601995 (cit. on p. 223).
- [10] Richard P. Feynman. “*Surely You’re Joking, Mr. Feynman!*”: *Adventures of a Curious Character*. W. W. Norton. ISBN: 978-0-393-35562-8 (cit. on p. 30).
- [11] Martin Gardner. “Mathematical Games”. In: *Scientific American* 229 (Oct. 1973), pp. 114–119. URL: <https://www.jstor.org/stable/24923225> (cit. on p. 19).
- [12] Martin Gardner. *My Best Mathematical and Logic Puzzles*. Dover Publications. ISBN: 978-0486281520. URL: <https://www.amazon.ca/My-Best-Mathematical-Logic-Puzzles/dp/0486281523> (cit. on p. 2).
- [13] Martin Gardner. *Penrose Tiles to Trapdoor Ciphers*. The Mathematical Association of America, 1988. ISBN: 978-0716719878. URL: <https://www.amazon.com/Penrose-Trapdoor-Ciphers-Martin-Gardner/dp/0716719878> (cit. on p. 31).
- [14] Nicomachus of Gerasa. *Introduction to Arithmetic*. URL: <https://archive.org/details/NicomachusIntroToArithmetic> (cit. on p. 21).
- [15] Daniel Gilbert. *Stumbling on Happiness*. Vintage Canada. ISBN: 978-0-676-97858-2 (cit. on p. 146).
- [16] Catherine Gildiner. *Good Morning, Monster: Five Heroic Journeys to Recovery*. Penguin Random House Canada. ISBN: 978-0-735-23696-7 (cit. on p. 228).
- [17] Kazuo Ishiguro. *Never Let Me Go*. Penguin Random House Canada, 1990. ISBN: 978-0-307-40099-4 (cit. on p. 161).
- [18] Harold R. Jacobs. *Mathematics A Human Endeavor*. 3rd ed. W. H. Freeman and Company, 1994. ISBN: 978-0-716-72426-1 (cit. on p. 31).
- [19] Rebecca Rockefeller Liesl Clark. *The Buy Nothing Get Everything Plan: Discover the Joy of Spending Less, Sharing More, and Living Generously*. Atria Books, 2020. ISBN: 978-1-9821-13797 (cit. on p. 141).
- [20] Katherine May. *Wintering: The Power of Rest and Retreat in Difficult Times*. Riverhead Books. ISBN: 978-0593189481. URL: <https://katherine-may.co.uk/> (cit. on p. 60).
- [21] Cynthia D. Moe-Lobeda. *Resisting Structural Evil: Love as Ecological-Economic Vocation*. 1517 Media, Fortress Press. ISBN: 978-1451462678. URL: <https://resistingstructuralevil.com/> (cit. on p. 45).
- [22] Roger B. Nelsen. *Proofs without Words: Exercises in visual thinking*. The Mathematical Association of America, 1993. ISBN: 9-780883-857007. URL: <https://bookstore.ams.org/clrm-pww-set/> (cit. on p. 18).
- [23] George Orwell. *1984*. Penguin UK, 2000. ISBN: 9-780140817744 (cit. on p. 162).
- [24] Thomas Pirnot. *Mathematics All Around*. 6th ed. Pearson, 2018. ISBN: 978-0134434681 (cit. on p. xxiii).

- [25] Greta Podleski. *Yum & yummer: Ridiculously tasty recipes that'll blow your mind, but not your diet!* One Spoon Media, 2017. ISBN: 9-781775-047001 (cit. on p. 149).
- [26] Jonathan Hunter Robert Maunder. *Damaged: Childhood Trauma, Adult Illness, and the Need for a Health Care Revolution.* University of Toronto Press. ISBN: 9781487528348. URL: <https://utorontopress.com/9781487528348/damaged/> (cit. on p. 228).
- [27] Michael J. Sandel. *The Tyranny of Merit: What's Become of the Common Good?* Macmillan, 2020. ISBN: 978-0-374-28998-0 (cit. on p. 235).
- [28] Michael J. Sandel. *What money can't buy: The moral limits of markets.* Farrar, Straus and Giroux, 2012. ISBN: 978-0-374-53365-6 (cit. on pp. 151, 176).
- [29] E. Seneta. "Lewis Carroll as a Probabilist and Mathematician". In: *Math. Scientist* 9 (1984), pp. 79–94 (cit. on p. 193).
- [30] Karl J. Smith. *The Nature of Mathematics.* 9th ed. Brooks/Cole, 2001. ISBN: 978-0534368906 (cit. on p. xxiii).
- [31] Krista Stelkia. "Police Brutality in Canada: A Symptom of Structural Racism and Colonial Violence". In: (July 2020). URL: <https://yellowheadinstitute.org/2020/07/15/police-brutality-in-canada-a-symptom-of-structural-racism-and-colonial-violence/> (cit. on p. 234).
- [32] Bryan Stevenson. *Just Mercy: A Story of Justice and Redemption.* Random House, 2015. ISBN: 978-0812984965 (cit. on p. 234).
- [33] Edward R. Tufte. *The Visual Display of Quantitative Information.* Graphics Press, 1983. ISBN: 9780961392109 (cit. on p. 223).
- [34] Yanis Varoufakis. *Talking to my daughter about the economy or How capitalism works—and how it fails.* Farrar, Straus and Giroux, 2017. ISBN: 978-0374718435 (cit. on p. 153).

Index

- $>$, greater than, 15
 \approx , approximately equal, 16
 \cong , congruence, 16
 \equiv , equivalence, 16
 \geq , greater or equal, 15
 \sim , similar, 16
 $\binom{m}{n}$, binomial coefficient, 188
 ${}_m C_n$, choose, 188
 $n!$, factorial, 188
 \neg , negation, 16
 \vee , or, 16
 \wedge , and, 16
 $[x]$, floor function, 51
 μ , average, 16
%, percent, 141
 $P(E)$, probability, 179
 \bar{A} , set complement, 187
 $\{ \dots \}$, set, 178
 \cap , set intersection, 186
 \cup , set union, 16, 186
 $\emptyset = \{ \}$, the empty set, 16
 \in , belongs to, 16
 \subset , proper subset, 16
 \subseteq , subset, 16
 \bar{x} , mean, 233
 σ , standard deviation, 237
 \tilde{x} , median, 236
- a posteriori, 195
a priori, 181
absolute increase, 144
add-on interest method, 158
algorithm, 31
alternate exterior angles, 89
alternate interior angles, 89
altitude, 112
amortized loan, 160
and, 186
angles, 85

- alternate exterior, 89
 - alternate interior, 89
 - complementary, 89
 - corresponding, 87–89, 100
 - supplementary, 89
 - vertically opposite, 86, 88
- annuity, 158, 160
- arc, 95
- Archimedean solids, 114
- arithmetic mean, 259
- ASCII, 58
- average, 233, 234, 259
- average daily balance method, 155, 156
- axis of symmetry, 99, 100
- Babylonian numerals, 45
- bar, 226
- Bayes factor, 196
- bi-weekly, 152
- biased, 223
- binary, 52
- binomial coefficient, 188
- binomial experiment, 182
- box breathing, 94
- box plot, 238
- box-and-whisker plot, 238
- Caesar cipher, 63
- categories, 224
- causation, 242
- central tendency, 233
- centre, 233
- certain event, 179
- certainty, 179
- chance, 179
- channel decoder, 61
- channel encoder, 61
- check digit, 67
- circle, 95
- circumference, 95
- class mark, 258
- class midpoint, 258
- classes, 224
- clock-arithmetic, 65
- closed figure, 91
- clustered bar chart, 231
- coefficient of variation, 239
- combination, 188
- complement, 184, 185, 187
- complementary angles, 89
- composite, 31, 32
- compound interest, 151
 - bi-weekly, 152
 - continuously, 170
 - formula, 149
 - monthly, 150
- compounding periods, 150
- concave polygon, 92
- conditional probability, 191
- confidence interval, 262
- congruence
 - angles, 86
 - arithmetic, 65
- consumer price index, 169
- continuity, 3
- continuous, 3

- continuous probability distribution, 179
- continuously compounding, 170
- convex polygon, 91, 92
- correlation, 242
- corresponding angles, 87–89, 100
- cost of living, 169
- cube, 96
- cubic unit, 96

- data, 223
- decimal, 75
- deductive reasoning, 3
- denominator, 141
- depreciation, 148
- deviation, 237
- directed graph, 200
- discrete, 179
- disjunction, 16
- dispersion, 237
- distribution, 225, 226, 236, 252, 253
 - symmetric, 237
- double blind, 242

- edges, 7
- Egyptian numerals, 39
- empirical probability, 195
- empty set, 16, 179
- equable, 122
- equations, 8
- equilateral triangle, 92
- error correcting code, 62
- event, 178
 - certain, 179
 - complement, 184, 185
 - impossible, 179
 - independent, 183, 191
 - mutually exclusive, 186
 - simple, 179
- exact interest, 173
- experiment, 178
 - binomial, 182
- experimental, 242
- exponent, 152
- exterior, 88

- factorial, 188
- failure, 182
- fair, 181
- false positive rate, 197
- figure-eight knot, 42
- finance charge, 155, 156, 160
- first quartile, 238
- five-number summary, 238
- floor function, 51
- fraction, 141–142
 - expansion, 142
 - reduction, 142
- frequency, 224, 225
- future amount, *see* future value
- future value, 148, 150, 158

- galley method, 55
- gambler's fallacy, 183
- geometric mean, 259
- GIC, 147
- glide, 100

- reflection, 99
- graph, 7, 200
 - directed, 200
- grouped bar chart, 231
- groups, 224
- half line, 85
- harmonic mean, 259
- hash mark, 38
- hexadecimal, 55
- histogram, 226
- Horner's method, 53
- hypotenuse, 92, 114
- impossible event, 179
- Incan quipu, 42
- independence, 183, 191, 196, 218, 320
- inductive reasoning, 9
- inferential statistics, 262
- inflation, 143, 170
- instalment loan, 158, 160
- inter quartile range, 260
- interest
 - add-on, 158
 - average daily balance, 155, 156
 - compounded, 149
 - bi-weekly, 152
 - continuously, 170
 - monthly, 150
 - exact, 173
 - ordinary, 173
 - simple, 147
 - unpaid balance, 155
- interior angles, 88
 - on the same side of the transversal, 89
- Intermediate Value Theorem, 3
- intersection, 186
- inverse, 66
- isosceles, 92, 93
- IVT, *see* Intermediate Value Theorem
- Kaktovik numerals, 47
- lattice method, *see* galley method
- least common multiple, 66, 67
- legs, 92
- lemma, 88
- line, 84
- line graph, 229
- line segment, 85
- long knot, 42
- Mayan numerals, 43
- mean, 233–235, 237, 259
- median, 233, 234, 236
- mod, *see* modular arithmetic
- mode, 233, 234, 236
- modular arithmetic, 64
- modular function, 51
- modulus, 65
- monthly, 150
- multibar graph, 231
- mutually exclusive, 186
- non-adjacent, 89
- non-fungible token, 174
- nonpositional, 39

- number, 39
- numerals, 39
 - Babylonian, 45
 - binary, 52
 - counting rods, 38
 - Egyptian, 39
 - hexadecimal, 55
 - Incan quipu, 42
 - Kaktovik, 47
 - Mayan, 43
 - octal, 51, 55
 - other bases, 50
 - penta-vigesimal, 48
 - Roman, 40
 - tally marks, 38
- numerator, 141
- observational, 242
- octal, 51, 55
- odds, 185, 196
- or, 186
- ordinary interest, 173
- outcome, 178
- outlier, 238, 259
- parallel, 87
- penta-vigesimal, 48
- percent, 141
 - increase, 144
- percentile, 238
- perimeter, 115
- permutation, 13, 188
- perpendicular, 85
- pie chart, 230
- plane, 84
- Platonic solids, 114
- point, 84
- polygon
 - concave, 92
 - convex, 92
 - regular, 92
- population, 223
 - deviation, 237
 - standard deviation, 237
 - variance, 237
- prime, 31
- probability, 179
 - conditional, 191
 - empirical, 195
- probability tree, 180
- proof, 88
- Pythagorean Theorem, 94
- Pythagorean triples, 94
- QR code, 59
- quartile, 238
- radius, 95
- random variable, 181
- randomness, 181, 182
- range, 237, 238
- ray, 85
- rectangular prism, 96, 97
- reduced to the lowest terms, 142
- reflection, 99
- reflection symmetry, 99

- regular, 92
- regular polygon, 92
- regular solids, 114
- relative comparison, 144
- relative frequency, 227
- relative increase, 144
- replacement
 - with, 190
 - without, 190
- RESP, 142
- rhombus, 294
- right angle, 85, 86
- right cone, 131
- right pyramid, 131
- right triangle, 92
- rigid motion, 99
- Roman numerals, 40
- rotation, 99, 100
- rotational symmetry, 100
- rounding, 234
- RRSP, 148
- sample, 223
 - standard deviation, 237
 - variance, 237
- sample space, 178, 179
- sampling parameters, 262
- second quartile, 238
- sector, 95
- set, 178
 - braces, 178
 - complement, 187
 - empty, 16, 179
 - intersection, 186
 - union, 16, 186
- similar, 92, 93
- simple event, 179
- simple figure, 91
- simple interest, 147
 - formula, 147
- single knot, 42
- slide, 100
- slide or glide symmetry, 87
- source, 61
- source decoder, 61
- source encoder, 61
- sphere, 101
- spread, 237
- spreadsheet, 22–23, 51, 150, 234, 238
- square numbers, 11, 12
- square pie chart, 230, 231
- square unit, 95
- stacked bar graph, 232
- standard deviation, 237
 - population, 237
 - sample, 237
- stem-and-leaf plot, 225
- straight angle, 85, 86
- subset, 178
- success, 182
- success rate, 169
- successive division, 52
- successive multiplication, 53
- supplementary angles, 89
- symmetric distribution, 237
- symmetry, 99

tally, 224, 225
telescoping sum, 273
tetrahedron, 114
TFSA, 148
theorem, 85, 88
theoretical probability, 181
third quartile, 238
translation, 99, 100
translational symmetry, 87, 100
transversal, 87–89
tree diagram, 14
triangular numbers, 11
Tukey's fences, 260

unary, 38
unbiased, 223
union, 186
Universal Product Code, 59
unpaid balance method, 155

variables, 8
variance, 237
 population, 237
 sample, 237
Venn diagram, 186
vertical angles, 86
vertically opposite angles, 86, 88
vertices, 7

waffle chart, 230
with replacement, 190
without replacement, 190