CHAPTER IV

# AN ALGORITHM FOR CERTIFYING $\sum_{\mathbf{k}} F(n,\mathbf{k}) = f_n$

In this chapter, we will develop the $r$-variable analogues of the algorithm in Chapter 2. We will take the values of $I_1^*, I_2^*, \ldots, I_r^*$ and $J^*$ from Chapter 3, and input them into the algorithm to obtain directly $a_0(n), a_1(n), \ldots, a_J(n)$, not all zero, and rational functions $R_1(n,\mathbf{k}), R_2(n,\mathbf{k}), \ldots, R_r(n,\mathbf{k})$.

Here Theorem 4.2A, the analogue of Theorem 3.2A in [WZ3] is what we need to construct the algorithm.

**Theorem 4.1.** [WZ3, Theorem 4.2A] *Let $F$ be a proper-hypergeometric term. Then there are a positive integer $J$, polynomials $a_0(n), a_1(n), \ldots, a_J(n)$ and hypergeometric functions $G_1, \ldots, G_r$ such that for every $(n,\mathbf{k}) \in \mathbb{N}^{r+1}$ at which $F \neq 0$ and $F$ is well-defined at all of the arguments that appear in*

$$\sum_{\mathbf{i}=0}^{\mathbf{I}} \sum_{j=0}^{J} \alpha(\mathbf{i}, j, n) F(n-j, \mathbf{k}-\mathbf{i}) = 0 \tag{4.1}$$

*we have*

$$\sum_{j=0}^{J} a_j(n) F(n-j,\mathbf{k}) = \sum_{i=1}^{r} \big( G_i(n, k_1, \ldots, k_i, \ldots, k_r) - G_i(n, k_1, \ldots, k_i - 1, \ldots, k_r) \big). \tag{4.2}$$

*Moreover this recurrence is non-trivial, and each $G_i(n,\mathbf{k})$ is of the form $R_i(n,\mathbf{k})F(n,\mathbf{k})$, where the $R$'s are rational functions of their arguments.*

In the proof of Theorem 4.1 in [WZ3], we let $N$ be the operator that shifts (down) the variable $n$, that is $Nf(n) = f(n-1)$. Further, for each $i = 1, \ldots, r$ we let $K_i$ be the operator that shifts the variable $k_i$, that is $K_i f(\mathbf{k}) = f(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$. Then (4.1) is equivalent to an assertion

$$H(N, n, K_1, \ldots, K_r)F(n, \mathbf{k}) = 0,$$

where $H$ is a polynomial in its arguments and does not involve $\mathbf{k}$. We can expand $H$ in a Taylor's series about $\mathbf{K} = \mathbf{1}$, to obtain

$$H(N, n, \mathbf{K}) = H(N, n, \mathbf{1}) + \sum_{i=1}^{r}(K_i - 1)V_i(N, n, \mathbf{K})$$

in which the $V_i$'s are polynomials in their arguments. We apply the right hand side of the last equality to $F(n, \mathbf{k})$, and (4.2) follows.

We generalize the idea from Chapter 2 to the multivariable case. From Chapter 3, we can compute the upper bounds for $\mathbf{I}^*$ and $J^*$ which are used to find the degree in $\mathbf{k}$ of the numerator polynomial of $R_i(n, \mathbf{k})$, $i \in [r]$. Into (4.2), we substitute

$$\sum_{\substack{\mathbf{0} \le \mathbf{e} \le (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e} \cdot \mathbf{1} \le \mathcal{N}_i}} \frac{c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n, \mathbf{k})}$$

for $R_i(n, \mathbf{k})$ where the $c_i(\mathbf{e}, n)$ are unknown polynomials in $n$. The procedure that follows yields a homogeneous linear system with the $c_i$'s and $a_j$'s as the unknowns for which a solution is guaranteed by [WZ3, Theorem 4.1].

## Algorithm for the Certificate

Step 1. Rename $k_i$'s so that $I_1 \ge I_2 \ge \cdots \ge I_r$ for the given $I_i$'s.

Step 2. Obtain $\sum_{i=1}^{r}(K_i - 1)V_i(N, n, \mathbf{K})$ in the following way. For the Taylor's series expansion of $H(N, n, \mathbf{K})$ about $\mathbf{K} = \mathbf{1}$, we first sum all terms that are divisible by $(K_1-1)$, factor $(K_1-1)$ to make the sum equal to $(K_1-1)V_1(N,n,\mathbf{K})$. The remaining terms of $H(N,n,\mathbf{K})$ are no longer divisible by $(K_1-1)$. We then proceed to sum all the remaining terms that are divisible by $(K_2-1)$, and get $(K_2-1)V_2(N,n,K_2,\ldots,K_r)$ as the sum. Successively we sum the terms until we reach the last sum, namely, $(K_r-1)V_r(N,n,K_r)$.

Step 3. Divide (4.2) by $F(n,\mathbf{k})$ on both sides to get

$$\sum_{j=0}^{J}\frac{a_j(n)F(n-j,\mathbf{k})}{F(n,\mathbf{k})}=\sum_{i=1}^{r}\Bigg(R_i(n,k_1,k_2,\ldots,k_r) - \frac{R_i(n,k_1,\ldots,k_i-1,\ldots,k_r)F(n,k_1,\ldots,k_i-1,\ldots,k_r)}{F(n,\mathbf{k})}\Bigg). \tag{4.3}$$

Step 4. Into (4.3), substitute for $R_i(n,\mathbf{k})$

$$\sum_{\substack{0\le j\le J\\ \mathbf{0}\le\mathbf{i}\le(0,\ldots,0,I_i-1,I_{i+1},\ldots,I_r)}}\frac{d(\mathbf{i},j,n)F(n-j,\mathbf{k}-\mathbf{i})}{F(n,\mathbf{k})},$$

where $d$'s are polynomials in $n$ only.

Step 5. Compute a common denominator for $R_i(n,\mathbf{k})$, i.e.,

$$D_{R_i}(n,\mathbf{k})=P(n,\mathbf{k})\prod_{s=1}^{p}(a_sn+\mathbf{b}_s\cdot\mathbf{k}+c_s)^{\underline{(a_s)^+J+(I_i-1)(b_{is})^++\sum_{i<t\le r}I_t(b_{ts})^+}} \times\prod_{s=1}^{q}(u_sn+\mathbf{v}_s\cdot\mathbf{k}+w_s+1)^{\overline{(-u_s)^+J+(I_i-1)(-v_{is})^++\sum_{i<t\le r}I_t(-v_{ts})^+}}.$$

Step 6. Calculate the degree in $\mathbf{k}$ of the numerator polynomial $N_{R_i}(n,\mathbf{k})$ over the denominator polynomial $D_{R_i}(n,\mathbf{k})$.

$$\mathcal{N}_i:=\deg_{\mathbf{k}}P(n,\mathbf{k})+(I_i-1)(\mathcal{B}_i+(V_i-B_i)^+)+J(\mathcal{A}+(U-A)^+)+\sum_{i<t\le r}I_t(\mathcal{B}_t+(V_t-B_t)^+),$$

where (as in Chapter 3)

$$U:=\sum_{\substack{s\\ \mathbf{v}_s\neq0}}u_s,\quad V_l:=\sum_s v_{ls},\quad A:=\sum_{\substack{s\\ \mathbf{b}_s\neq0}}a_s,\quad B_l:=\sum_s b_{ls},$$

and

$$\mathcal{A} := \sum_{\substack{s \\ \mathbf{b}_s \neq 0}} (a_s)^+ + \sum_{\substack{s \\ \mathbf{v}_s \neq 0}} (-u_s)^+, \quad \mathcal{B}_l := \sum_s (b_{ls})^+ + \sum_s (-v_{ls})^+.$$

Step 7. Conclude that $R_i(n, \mathbf{k})$ has the form

$$\sum_{\substack{\mathbf{0} \leq \mathbf{e} \leq (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e} \cdot \mathbf{1} \leq \mathcal{N}_i}} \frac{c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n, \mathbf{k})}.$$

Step 8. Substitute $R_i$'s into (4.3), and collect all terms to one side of the equal sign.

Step 9. Find a common denominator for the resulting expression and make the coefficients of each monomial in $\mathbf{k}$ zero.

Step 10. Solve the resulting system of homogeneous equations for the $a_j$'s and $c_i$'s. Again we are guaranteed that a solution exists from Theorem 4.1 in [WZ3].