CHAPTER I

THE ORDER OF THE RECURRENCE FOR $F(n,k)$

We show slightly better upper bounds for the order of the recurrence satisfied by a given proper-hypergeometric term $F(n,k)$. We follow the proof of Theorem 3.1 in [WZ3] and hold fast unto the estimates to obtain our bounds.

Definition 1.1. [WZ3] A *proper-hypergeometric term* is a function of the form

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k, \tag{1.1}$$

where P is a polynomial and ξ is a parameter. The a's, b's, u's and v's are assumed to be specific integers, i.e., they are integers and do not depend on any other parameters. The c's and the w's are also integers, but they may depend on parameters. We will say that F is *well-defined* at (n,k) if none of the numbers $\{a_s n + b_s k + c_s\}_1^p$ is a negative integer. We will say that $F(n,k) = 0$ if F is well-defined at (n,k) and at least one of the numbers $\{u_s n + v_s k + w_s\}_1^q$ is a negative integer, or $P(n,k) = 0$.

Definition 1.2. [WZ3] A proper-hypergeometric term F is said to satisfy a k-free recurrence at a point $(n_0,k_0) \in \mathbb{Z}^2$ if there are integers I, J and polynomials $\alpha_{i,j} = \alpha_{i,j}(n)$ that do not depend on k and are not all zero, such that the relation

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)F(n-j,k-i) = 0 \tag{1.2}$$

holds for all (n,k) in some $\mathbb{R}^2$ neighborhood of (n_0,k_0), in the sense that F is well-defined at all of the arguments that occur, and the relation (1.2) is true.

Theorem 1.3. [WZ3, Theorem 3.1] *Every proper-hypergeometric term F satisfies a non-trivial k-free recurrence relation. Indeed there exist I,J and polynomials $\alpha_{i,j}(n)$ ($i = 0,\dots,I; j = 0,\dots,J$) not all zero, such that* (1.2) *holds at every point $(n_0,k_0) \in \mathbb{Z}^2$ for which $F(n_0,k_0) \neq 0$ and all of the values $F(n_0-j,k_0-i)$ that occur in* (1.2) *are well-defined. Furthermore there exists such a recurrence with $(I,J) = (I^*,J^*)$, where*

$$J^* = \sum_s |b_s| + \sum_s |v_s|, \quad I^* = 1 + \deg(P) + J^*\Big(\Big(\sum_s |a_s| + \sum_s |u_s|\Big) - 1\Big). \tag{1.3}$$

1.1 Slightly better upper bounds

Notation. We let $x^+ := \max\{0,x\}$. The set $\{1,2,\dots,I\}$ is denoted by $[I]$, and $[I]_0$ means $[I]\cup\{0\}$. We let $x^{\underline{m}}$ denote $x(x-1)\cdots(x-m+1)$, and $x^{\overline{m}}$ denote $x(x+1)\cdots(x+m-1)$ for positive integers m. We define $x^{\underline{0}} = 1 = x^{\overline{0}}$.

We improve the bounds for I^* and J^* by

Theorem 1.4. *Let*

$$\begin{aligned} U &:= \sum_{\substack{s\\ v_s\neq 0}} u_s, \quad V := \sum_s v_s, \qquad A := \sum_{\substack{s\\ b_s\neq 0}} a_s, \quad B := \sum_s b_s,\\ \mathcal{A} &:= \sum_{\substack{s\\ b_s\neq 0}} (a_s)^+ + \sum_{\substack{s\\ v_s\neq 0}} (-u_s)^+, \quad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+, \end{aligned}$$

and $\delta = \deg_k P(n,k)$. Then J^* and I^* in (1.3) *of Theorem 1.3 can be replaced by*

$$J^* = \mathcal{B} + (V-B)^+, \quad \text{and} \quad I^* = 1 + \delta + J^*\left(\mathcal{A} + (U-A)^+ - 1\right).$$

Proof. Fix some I, $J > 0$, and suppose (n_0,k_0) is a point that satisfies the two conditions of the theorem. Since we assumed that all of the a_s, b_s, u_s, v_s in Definition 1.1 are integers, we have that for all (n,k) in some $\mathbb{R}^2$ neighborhood of (n_0,k_0), all of the ratios $F(n-j,k-i)/F(n,k)$ are well-defined *rational functions* of n and k. (See (1.1) for $F(n,k)$.) Hence we can form a linear combination

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)\frac{F(n-j,k-i)}{F(n,k)} \tag{1.4}$$

of these rational functions, in which the α's are to be determined, if possible, so as to make the sum vanish identically in the neighborhood.

As in [WZ3], the problem is to find a common denominator for the summand in (1.4). Instead we find a common denominator $D(n,k)$ for

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\frac{F(n-j,k-i)}{F(n,k)}.$$

Clearly, $D(n,k)$ is also a common denominator for the summand in (1.4).

Consider

$$\frac{F(n-j,k)}{F(n,k)}=\frac{P(n-j,k)}{P(n,k)}\prod_{s=1}^{p}\frac{(a_sn+b_sk+c_s-a_sj)!}{(a_sn+b_sk+c_s)!}\prod_{s=1}^{q}\frac{(u_sn+v_sk+w_s)!}{(u_sn+v_sk+w_s-u_sj)!} \tag{1.5}$$

which contributes to the denominator $D(n,k)$, if $a_s>0$, or $u_s<0$, or both.

In (1.5), if $a_s>0$ for some $s\in[p]$, then

$$\frac{(a_sn+b_sk+c_s-a_sj)!}{(a_sn+b_sk+c_s)!}=\frac{1}{(a_sn+b_sk+c_s)^{\underline{a_sj}}}.$$

Since $(a_sn+b_sk+c_s)^{\underline{a_sj}}$ divides $(a_sn+b_sk+c_s)^{\underline{a_sJ}}$ for $0<j\le J$ and $a_s>0$, a common denominator for $\sum_{j=0}^{J}\frac{F(n-j,k)}{F(n,k)}$ is

$$P(n,k)\prod_{\substack{s=1\\a_s>0}}^{p}(a_sn+b_sk+c_s)^{\underline{a_sJ}}\prod_{\substack{s=1\\u_s<0}}^{q}(u_sn+v_sk+w_s+1)^{\overline{-u_sJ}}. \tag{1.6}$$

Similarly, a common denominator for $\sum_{i=0}^{I} \frac{F(n,k-i)}{F(n,k)}$ is

$$P(n,k) \prod_{\substack{s=1 \\ b_s>0}}^{p} (a_s n + b_s k + c_s)^{\underline{b_s I}} \prod_{\substack{s=1 \\ v_s<0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-v_s I}}. \tag{1.7}$$

Putting (1.6) and (1.7) together, we have

$$D(n,k) = P(n,k) \prod_{s=1}^{p} (a_s n + b_s k + c_s)^{\underline{\max_{i,j}(a_s j + b_s i)^+}} \prod_{s=1}^{q} (u_s n + v_s k + w_s + 1)^{\overline{\max_{i,j}(-u_s j - v_s i)^+}}.$$

Clearly,

$$\max_{\substack{i\in[I]_0 \\ j\in[J]_0}} (a_s j + b_s i)^+ = (a_s)^+ J + (b_s)^+ I,$$

and

$$\max_{\substack{i\in[I]_0 \\ j\in[J]_0}} (-u_s j - v_s i)^+ = (-u_s)^+ J + (-v_s)^+ I.$$

If we let $\delta := \deg_k P(n,k)$, then the degree in k of $D(n,k)$ is

$$\begin{aligned} &\delta + J\Big(\sum_{\substack{s\in[p] \\ b_s\neq 0}} (a_s)^+\Big) + I\Big(\sum_{s\in[p]} (b_s)^+\Big) + J\Big(\sum_{\substack{s\in[q] \\ v_s\neq 0}} (-u_s)^+\Big) + I\Big(\sum_{s\in[q]} (-v_s)^+\Big) \\ &= \delta + J\Big(\sum_{\substack{s\in[p] \\ b_s\neq 0}} (a_s)^+ + \sum_{\substack{s\in[q] \\ v_s\neq 0}} (-u_s)^+\Big) + I\Big(\sum_{s\in[p]} (b_s)^+ + \sum_{s\in[q]} (-v_s)^+\Big). \end{aligned}$$

Next, we find the degree in k of the numerator polynomial $N(n,k)$ in (1.4) with $D(n,k)$ as the common denominator. Consider the (i,j)th term in

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \frac{F(n-j,k-i)}{F(n,k)}. \tag{1.8}$$

Since

$$\begin{aligned} \frac{F(n-j,k-i)}{F(n,k)} &= \frac{P(n-j,k-i)}{P(n,k)} \xi^{-i} \\ &\times \frac{\prod\limits_{\substack{s=1 \\ a_s j + b_s i<0}}^{p} (a_s n + b_s k + c_s + 1)^{\overline{-a_s j - b_s i}} \prod\limits_{\substack{s=1 \\ u_s j + v_s i>0}}^{q} (u_s n + v_s k + w_s)^{\underline{u_s j + v_s i}}}{\prod\limits_{\substack{s=1 \\ a_s j + b_s i>0}}^{p} (a_s n + b_s k + c_s)^{\underline{a_s j + b_s i}} \prod\limits_{\substack{s=1 \\ u_s j + v_s i<0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-u_s j - v_s i}}}, \end{aligned}$$

by letting

$$N_{i,j} := \prod_{\substack{s=1 \\ a_s j + b_s i < 0}}^{p} (a_s n + b_s k + c_s + 1)^{\overline{-a_s j - b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i > 0}}^{q} (u_s n + v_s k + w_s)^{\underline{u_s j + v_s i}},$$

and

$$D_{i,j} := \prod_{\substack{s=1 \\ a_s j + b_s i > 0}}^{p} (a_s n + b_s k + c_s)^{\underline{a_s j + b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i < 0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-u_s j - v_s i}},$$

we have

$$\frac{F(n-j,k-i)}{F(n,k)} = \frac{P(n-j,k-i)}{P(n,k)} \xi^{-i} \frac{N_{i,j} D(n,k)}{D_{i,j} D(n,k)}.$$

Hence, the degree in k of the numerator of the (i,j)th term in (1.8) with $D(n,k)$ as the denominator, i.e., $P(n-j,k-i)\xi^{-i} N_{i,j} D(n,k)/\big(D_{i,j} P(n,k)\big)$, is

$$\begin{aligned}(1.9)\quad \delta &+ \sum_{\substack{b_s \neq 0 \\ a_s j + b_s i < 0}} (-a_s j - b_s i) + \sum_{\substack{v_s \neq 0 \\ u_s j + v_s i > 0}} (u_s j + v_s i) \\ &+ \deg_k D(n,k) - \sum_{\substack{b_s \neq 0 \\ a_s j + b_s i > 0}} (a_s j + b_s i) - \sum_{\substack{v_s \neq 0 \\ u_s j + v_s i < 0}} (-u_s j - v_s i) - \delta \\ &= \deg_k D(n,k) + \sum_{v_s \neq 0} (u_s j + v_s i) - \sum_{b_s \neq 0} (a_s j + b_s i).\end{aligned}$$

Taking the maximum over i,j of the last line of (1.9) gives

$$\begin{aligned}\deg_k N(n,k) &= \max_{i,j} \Big(\deg_k D(n,k) + \sum_{v_s \neq 0} (u_s j + v_s i) - \sum_{b_s \neq 0} (a_s j + b_s i) \Big) \\ &= \deg_k D(n,k) + \max_{i,j} \Big(j \sum_{\substack{s \\ v_s \neq 0}} u_s + i \sum_s v_s - j \sum_{\substack{s \\ b_s \neq 0}} a_s - i \sum_s b_s \Big).\end{aligned}$$

Let

$$U := \sum_{\substack{s \\ v_s \neq 0}} u_s, \quad V := \sum_s v_s, \quad A := \sum_{\substack{s \\ b_s \neq 0}} a_s, \quad B := \sum_s b_s.$$

We can rewrite $\deg_k N(n,k)$ as

$$\begin{aligned}\deg_k N(n,k) &= \deg_k D(n,k) + \max_{i,j}\big(j(U-A)+i(V-B)\big)\\ &= \deg_k D(n,k) + J(U-A)^+ + I(V-B)^+.\end{aligned}$$

Knowing the degree in k of $N(n,k)$, we deduce that there are $1+\deg_k N(n,k)$ homogeneous linear equations to solve in $(I+1)(J+1)$ unknowns, namely, the $\alpha_{i,j}$'s. A system of solutions for the $\alpha_{i,j}$'s exists, if $(I+1)(J+1) \geq 2+\deg_k N(n,k)$. From the inequality, we will obtain an upper bound for J.

Let

$$\mathcal{A} := \sum_{\substack{s\\ b_s\neq 0}} (a_s)^+ + \sum_{\substack{s\\ v_s\neq 0}} (-u_s)^+, \quad \text{and} \quad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+.$$

Then,

$$\begin{aligned}\deg_k N(n,k) &= \deg_k D(n,k) + \max_{i,j}\Big(j(U-A)+i(V-B)\Big)\\ &= \delta + J\mathcal{A} + I\mathcal{B} + J(U-A)^+ + I(V-B)^+.\end{aligned}$$

If $\mathcal{B}+(V-B)^+ \neq 0$, we let $J^* = \mathcal{B}+(V-B)^+$, and solve for I^* in $(I+1)(J+1) \geq 2+\deg_k N(n,k)$ to get $I^* = 1+\delta+\big(\mathcal{A}+(U-A)^+-1\big)\big(\mathcal{B}+(V-B)^+\big)$ as an upper bound.

If $\mathcal{B}+(V-B)^+ = 0$, namely

$$\sum_s b_s^+ + \sum_s (-v_s)^+ + (\sum_s v_s - \sum_s b_s)^+ = 0,$$

then $b_s = 0$ for all $s \in [p]$, and $v_s = 0$ for all $s \in [q]$. In other words, the factorial part of $F(n,k)$ is independent of k. In this case,

$$\begin{aligned}\sum_k F(n,k) &= \frac{\prod_{s\in[p]}(a_s n + c_s)!}{\prod_{s\in[q]}(u_s n + w_s)!}\sum_k P(n,k)\xi^k\\ &= \frac{\prod_{s\in[p]}(a_s n + c_s)!}{\prod_{s\in[q]}(u_s n + w_s)!}P(n,\xi D)\frac{1}{1-\xi}.\end{aligned}$$

The sum above is summable but infinite. Since we are concerned with only terminating hypergeometric series, we can disregard the case $\mathcal{B} + (V - B)^+ = 0$. $\square$

Remark. If $P(n,k)$ in $F(n,k)$ is a constant, then $\delta = 0$. In this case, the I^* and J^* from Theorem 1.4 agree with the results in [W2] when $\mathcal{B} + (V - B)^+ \neq 0$.

1.2 Examples

Example 1.5. Take $F(n,k) = \binom{n}{k}^2$. We express $F(n,k)$ in the form of Definition 1.1 to get $n!^2/(k!^2(n-k)!^2)$. Then $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $u_1 = u_2 = 0$, $u_3 = u_4 = 1$, $v_1 = v_2 = 1$, $v_3 = v_4 = -1$, $U = 2$, $V = 0$, $A = 0$, $B = 0$, $\mathcal{A} = 0$, $\mathcal{B} = 2$. Since $U - A = 2$ and $V - B = 0$, we get $J^* = 2$ and $I^* = 3$.

The following two examples are from [W2, p. 4].

Example 1.6. [W2] Fix a positive integer m, and put

$$F(n,k) = \binom{n}{k}^m = \frac{n!^m}{k!^m (n-k)!^m}.$$

Then $a_i = 1$, $i \in [m]$; $b_i = 0$, $i \in [m]$; $u_i = 0$, $i \in [m]$; $u_i = 1$, $i \in [2m] \setminus [m]$; $v_i = 1$, $i \in [m]$; $v_i = -1$, $i \in [2m] \setminus [m]$. Thus $A = 0$, $B = 0$, $U = m$, $V = 0$, $\mathcal{A} = 0$, $\mathcal{B} = m$. Hence $J^* = m$, and $I^* = (m-1)m + 1$.

Example 1.7. [W2] If $F(n,k) = (n+k+\alpha+\beta)!/(k!\,(n-k)!\,(k+\alpha)!)$, then the f_n's where $f_n(x) = \sum_k F(n,k)x^k$ are the Jacobi polynomials. (See Formula (Jacobi) in Introduction for Jacobi polynomials.) A similar calculation as in the previous examples shows that $J^* = 2$ and $I^* = 1$. This is the best possible.