INTRODUCTION

The study of ordinary and partial differential equations led to the investigation of special functions, those bearing the names of Gauss, Hermite, Jacobi, Laguerre and Legendre. Therefore, Askey [As3] defined special functions as "functions that occur often enough to merit a name". Most special functions are expressible as hypergeometric series, i.e. a series $\sum_{k=0}^{\infty} a_k$ such that the ratio a_{k+1}/a_k of consecutive terms is a rational function of k. For example, the Hermite polynomials

$$H_n(x) := n! \sum_k \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!} \tag{Hermite}$$

has $a_{k+1}/a_k = -(n-2k)(n-2k-1)/(4x^2(k+1))$; the Laguerre polynomials

$$L_n^{\alpha}(x) := \sum_k \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{Laguerre}$$

has $a_{k+1}/a_k = (n-k)(-x)/((\alpha+k+1)(k+1))$; the Legendre polynomials

$$P_n(x) := \frac{1}{2^n} \sum_k \binom{n}{k}^2 (x-1)^k (x+1)^{n-k} \tag{Legendre}$$

has $a_{k+1}/a_k = (n-k)^2(x-1)/((x+1)(k+1)^2)$; and the general Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) := \frac{1}{2^n} \sum_k \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k} \tag{Jacobi}$$

has

$$\frac{a_{k+1}}{a_k} = \frac{(x+1)(n+\alpha+1)(n+\beta+1)}{(n-k+1)(n+\beta-k+1)}.$$

The first hypergeometric series that rose to fame and became *the* hypergeometric series of the 19th century was the ${}_2F_1$, often called the Gaussian hypergeometric, for Gauss in his doctoral dissertation of 1812 [Gau] presented a thorough investigation of the series. Prior to Gauss, Euler [E] and Pfaff [Pf] also discovered many remarkable properties of ${}_2F_1$. The study of hypergeometric series became so important that W. W. Sawyer once remarked [S] "There must be many universities today where 95 per cent, if not 100 per cent, of the functions studied by physics, engineering, and even mathematics students, are covered by this single symbol $F(a, b; c; x)$ [i. e. ${}_2F_1$]."

In 1870, ${}_2F_1$ was generalized to ${}_mF_n$.

Definition. [GKP, p. 205] The *general hypergeometric series* is a power series in z with $m + n$ parameters, and it is defined as follows in terms of rising factorial powers:

$$ {}_mF_n\left[\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix} \middle| z \right] = \sum_{k \geq 0} \frac{a_1^{\overline{k}} \ldots a_m^{\overline{k}}}{b_1^{\overline{k}} \ldots b_n^{\overline{k}}} \frac{z^k}{k!}, $$

where $a^{\overline{k}}$ (also denoted by $(a)_k$) $:= a(a+1)(a+2)\ldots(a+k-1)$. To avoid division by zero, none of the b's may be zero or a negative integer. Other than that, the a's and b's may be anything. The a's are said to be *upper parameters*, and the b's are *lower parameters*. The last quantity z is called the *argument*.

We should note that most literature about hypergeometric series uses the notation in the definition. Sometimes, a one-line notation '$F(a_1, \ldots, a_m; b_1, \ldots, b_n; z)$' is also used (as in Sawyer's remark). However, Graham, Knuth and Patashnik do not have subscripts m and n around F in [GKP] because it is clear how many parameters are upper and lower parameters.

We are now witnessing a fast comeback of special functions and their associated hypergeometric series. Moreover, the q-analogues of special functions and hypergeometric series, called q-series have proved to be very useful in number theory, combinatorics, physics, group theory, [An5] and other areas of science and mathematics.

Andrews in 1974 [An1] first pointed out the great relevance of hypergeometric series to binomial coefficient identities. Indeed, special functions and hypergeometric series satisfy many identities, most of which involve binomial coefficients. We quote the following paragraph from [WZ4, p. 148 ¶2].

> There are countless identities relating special functions (e.g., [PBM, R, An5, As1]. In addition to their intrinsic interest, some of them imply important properties of these special functions, which in turn sometimes imply deep theorems elsewhere in mathematics (e.g., [deB, Ap]). Just as important for mathematics are the extremely successful attempts to instill meaning and insight, both representation-theoretic (e.g., [Mi]) and combinatorial (e.g., [Fo2]), into these identities.

Special functions share an even more remarkable property recently pointed out in [Z2, Z4, WZ2]: Most special functions can be written in the form

$$P_n = \sum_{k=0}^{\infty} F(n,k)$$

where n is an auxiliary parameter, and one has that not only is $F(n,k+1)/F(n,k)$ a rational function of k, but is a rational function of (n,k), and in addition, so is $F(n+1,k)/F(n,k)$. It is easy to check that $F(n+1,k)/F(n,k)$ is indeed a rational function of (n,k)

in the examples given before. We will call such an F a *hypergeometric term*[1] as in [WZ3]. This observation led Zeilberger [Z4] to conclude that a hypergeometric term is [WZ3] "an entirely rational, finitary object," and "can be handled by finite methods and machines [Z4], [WZ1], [WZ2]." Thus was born Wilf and Zeilberger's algorithmic proof theory for hypergeometric identities [WZ3].

Sister Celine Fasenmyer working under the supervision of Rainville found an algorithm for obtaining recurrence relations satisfied by hypergeometric polynomials. She presented the method by examples in her Ph. D. thesis [F1] in 1945 and in two subsequent papers [F2, F3]. Before the 1940's, 'it seemed customary upon entering the study of a new set of polynomials to seek recurrence relations, pure or mixed by essentially a hit-and-miss process' [R, p. 233]. With Sister Celine's technique, it was possible to find pure recurrences for a certain class of hypergeometric polynomials. Verbaeten [V] in 1974 showed how to make her technique general in the one summation case.

Independent of Verbaeten's work, Zeilberger [Z2] showed how to apply Sister Celine's method systematically. Furthermore, Zeilberger realized that Sister Celine's technique implies *all binomial identities are trivial* in the sense that one only needs to check a finite number of special cases to establish the truth of the identity of interest. Indeed, Zeilberger is the first to realize that Sister Celine's technique opened the door to automatic proving of hypergeometric identities. Central to Zeilberger's discovery is the fact that given a proposed hypergeometric expression $\sum_k F(n,k) = \sum_k G(n,k)$, we can show that the equality holds for all n by showing that both $\sum_k F(n,k)$ and $\sum_k G(n,k)$ satisfy the same

[1]More generally, a function $F(k_1, k_2, \ldots, k_r)$ is said to be a *hypergeometric term* if, for all $i \in [r]$, the ratio $F(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_r)/F(k_1, \ldots, k_r)$ is a rational function in all the variables.

recurrence and agree for some initial values of n.

Zeilberger's development of the proof theory for hypergeometric multisum identities began in the late 70's. A decade later, Wilf and Zeilberger employed Gosper's algorithm [G] in the discovery of WZ-pairs for proving hypergeometric identities [WZ1, WZ2, Z5]. (Almost all known single-sum hypergeometric identities can be proved using WZ-pairs.) Recently [WZ3], Wilf and Zeilberger formalized, systematized, and generalized Sister Celine' technique to prove hypergeometric identities. They defined *proper*-hypergeometric terms [WZ3, p. 596] for which her method will always produce recurrence relations. For the first time, an explicit a priori upper bound for the order of the recurrence satisfied by the hypergeometric term $F(n, \mathbf{k})$ is known [WZ3, Theorem 3.1]. Further, they gave admissibility conditions [WZ3, p. 602] on $F(n, \mathbf{k})$ for $\sum_{\mathbf{k}} F(n, \mathbf{k})$ to satisfy the same recurrence as $F(n, \mathbf{k})$. In addition to the proof theory for (multisum) hypergeometric identities, they successfully applied Sister Celine's technique to q-hypergeometric identities to obtain an a priori upper bound for the order of the recurrence, and for the first time presented an algorithmic proof theory for the q-hypergeometric identities. Combining the notion of WZ-pairs and the proof theory for multisum ordinary/q hypergeometric identities, they showed how to prove ordinary/q hypergeometric identities using WZ-tuples. (Again, almost all known identities satisfy recurrence relations in the form of WZ-tuples.) The proof theory was also extended to identities involving multiple integrals. For this dissertation, we will consider Wilf–Zeilberger's algorithmic proof theory only for the discrete ordinary single/multisum identities.

Sister Celine Fasenmyer in her Ph. D. dissertation [F1] presented many examples of hypergeometric series $\sum_k F(n, k)x^k$ for which she found recurrence relations by first obtaining the recurrence for $F(n,k)x^k$. Her technique finds a recurrence relation for the hypergeometric term, $F(n,k)x^k$ with polynomial-in-(n,x) coefficients. Three decades later, Zeilberger applied Sister Celine's method for proving proposed hypergeometric identities [Z1, Z2] in the following way. Suppose we would like to show that $\sum_k F(n,k) = f(n)$, where $F(n,k)$ and $f(n)$ are hypergeometric terms. Then we consider the ordinary generating function of $F(n,k)$, namely, $\sum_k F(n,k)x^k$, and obtain a recurrence relation for $F(n,k)x^k$ using Sister Celine's method. After dividing the recurrence relation by the smallest common factor $(x-1)^l$, and setting $x=1$, we get a recurrence for $F(n,k)$. If we sum over k, we will, if lucky, get a recurrence for the sum $\sum_k F(n,k)$. Because the coefficients of the recurrence relation are polynomials in (n,x) by Sister Celine's technique, the coefficients of the recurrence for the sum $\sum_k F(n,k)$ are polynomials in n only. It is now trivial to check whether $f(n)$ satisfies this recurrence relation. If this is so, and if $f(n) = \sum_k F(n,k)$ for certain initial values of n, then it follows by induction, that $f(n) = \sum_k F(n,k)$ for all n. The necessary initial values to check are the numbers up to (and including) the sum of the order of the recurrence and the highest integer zero of the leading (polynomial-in-n) coefficient of the recurrence. In short, *we have reduced proving the identity into checking a few initial values of* n. Furthermore, Zeilberger expressed the view [Z2, p. 122] that given $\sum_k F(n,k) = f(n)$, where $F(n,k)$ and $f(n)$ are hypergeometric terms, there exists an n_1 such that the identity $\sum_k F(n,k) = f(n)$ is true for all n if (and only if) it is true for $n \le n_1$. We give an explicit, pre-computable n_1 in this paper. (See Theorem 5.1 and its proof in Chapter 5.)

In Chapter 1, we follow the proof of [WZ3, Theorem 3.1] and sharpen upper bounds for the order of the recurrence satisfied by the summand in the case of just one summation index.

Chapter 2 contains an algorithm for finding the certificate $R(n,k)$, a rational function in n and k, needed to prove identities in the WZ-pair fashion. The algorithm is similar to the one described in [WZ3, pp. 592–593] for finding the certificate $R(n,k)$ directly. It uses the sharper upper bounds from Chapter 1.

Chapter 3 is a multivariable version of Chapter 1. To accomplish this generalization, we need to solve a certain minimization problem, estimate the number of positive zeros of a particular polynomial, and find an upper bound for the zeros of that polynomial.

Chapter 4 is the multivariable analogue of Chapter 2. We present an algorithm for finding the certificates $R_i(n,\mathbf{k})$ for $i \in [r]$ that are needed in proving identities using WZ-tuples. As in Chapter 2 which used bounds from Chapter 1, the bounds from Chapter 3 are used in Chapter 4.

In response to [WZ3, §2.3, end of ¶2], we show in Chapter 5 some examples of hypergeometric sums whose recurrence have leading coefficients that vanish at positive integers where the sums are valid. We devote most of the chapter to the proof—using results from Chapters 1 and 2—of our

Main Theorem. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term, and $P(n,k)$ be a polynomial with coefficients *in $\mathbb{Z}$. Let*

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\},$$

$$y := \max\{p, q\},$$

$$z := \max_{0\le i,j} \left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\},$$

and let n_0 be a given integer. If $\sum_k F(n,k) = 1$ for

$$n_0 \le n \le (3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3},$$

then $\sum_k F(n,k) = 1$ for <u>all</u> $n \ge n_0$.

In the last section of Chapter 5 we generalize the Main Theorem to the cases where the equations $\sum_k F(n,k) = f(n)$ (for some hypergeometric term $f(n)$) and $\sum_k F(n,k) = \sum_k G(n,k)$ (for some admissible proper-hypergeometric term) are the objects of interest.

We generalize Theorem 5.1 to multiple summation indices in Chapter 6.

Chapter 7 contains a sufficient condition on $F(n,k)$ for the sum, $\sum_k F(n,k)$, to be hypergeometric—or equivalently, to be summable in closed form. The sum $\sum_k F(n,k) =: f(n)$ is hypergeometric if $f(n)/f(n+1) = P(n)/Q(n)$ for some polynomials, P and Q, in n. Notice that in this case, $P(n)f(n+1) - Q(n)f(n) = 0$, so $f(n)$ is a solution to a first order recurrence relation (in n) with polynomial-in-n coefficients.

Petkovšek, in his Ph. D. dissertation [P], gives an algorithm that solves the following decision problem:

Given a linear recurrence relation of order h with polynomial coefficients, decide whether the recurrence has a solution that satisfies another recurrence of order 1; and if so, find that recurrence of order 1.

In other words, Petkovšek gives necessary conditions on the polynomial coefficients of the recurrence for the existence of a hypergeometric solution to the recurrence. Petkovšek's algorithm works only if the recurrence contains no free parameters. We still do not know any necessary condition on an admissible proper-hypergeometric term, $F(n,k)$, for the sum $\sum_k F(n,k)$ to be hypergeometric.