Math 126
Summer 2015
Dr. Lily Yen

Test 2

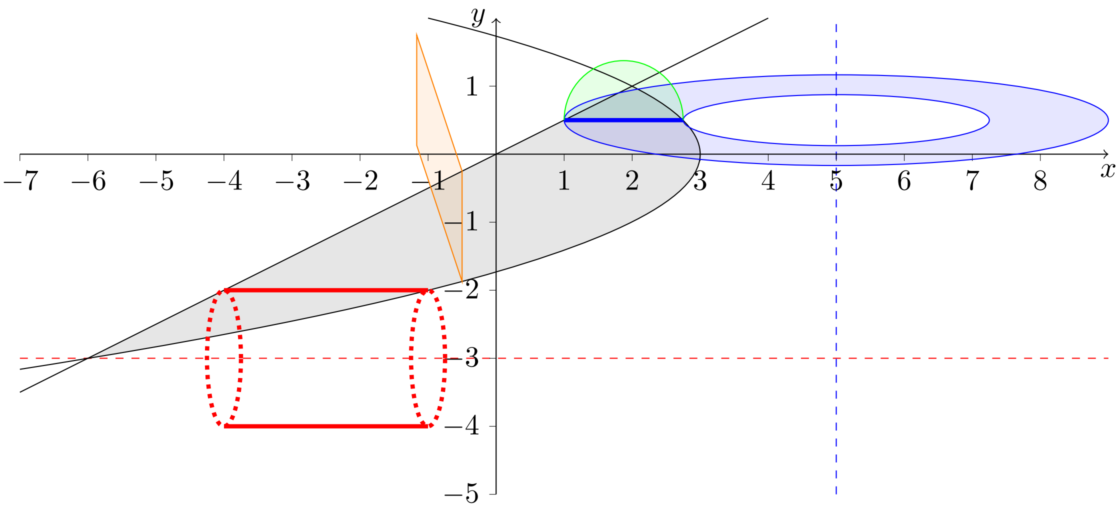
Show all your work

Name: ____________________

Score: ___/41

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: First draw $x + y^2 = 3$ and $x - 2y = 0$. Shade the region enclosed by these two graphs.



Score: /2

Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid of rotation.

a. The area of the shaded region.

If $x + y^2 = 3$ and $x - 2y = 0$, then $y^2 + 2y = 3$, so $y^2 + 2y - 3 = 0$, so $(y-1)(y+3) = 0$, so $y = 1$ or $y = -3$. Thus the intersection points are $(2, 1)$ and $(-6, -3)$. The area is then $\int_{-3}^{1} (3 - y^2) - (2y)\, dy$ which is easy to evaluate to $\frac{32}{3}$.

Score: /2

b. The volume obtained if the region is rotated about $y = -3$, using the method of cylindrical shells.

$\int_{-3}^{1} 2\pi(y+3)(3 - y^2 - 2y)\, dy$ which is easy to evaluate to $\frac{128}{3}\pi$.

Score: /2

c. The volume obtained if the region is rotated about $x = 5$, using the method of slicing (washers/disks).

$\pi \int_{-3}^{1} (5 - 2y)^2 - (5 - (3 - y^2))^2\, dy$ which is easy to evaluate to $\frac{576}{5}\pi$.

Score: /2

d. The volume of a solid that has the shaded region as its base, and cross-sections perpendicular to the y-axis are semi-circles.

$\frac{\pi}{2} \int_{-3}^{1} (\frac{(3-y^2)-2y}{2})^2\, dy$ which is easy to evaluate to $\frac{3814}{15}\pi$.

Score: /2

e. The volume of a solid with the shaded region as its base, and cross-sections perpendicular to the x-axis are rectangles whose length above the x–y plane is twice its width on the x–y plane.

$\int_{-6}^{2} 2(\frac{1}{2}x + \sqrt{3-x})^2\, dx + \int_{2}^{3} 2(2\sqrt{3-x})^2\, dx$ which is easy to evaluate to $32 + 4 = 36$.

Score: /2

Problem 2: Evaluate the following integrals exactly.

a. $\displaystyle\int xe^{2x}\,dx$

Use integration by parts: $\int xe^{2x}\,dx = x \cdot \frac{1}{2}e^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

Score: /3

b. $\displaystyle\int \frac{4x^2+2x-3}{(2x-5)(x+2)}\,dx$

Use the method of partial fractions: If $\frac{4x^2+2x-3}{(2x-5)(x+2)} = 2 + \frac{A}{2x-5} + \frac{B}{x+2}$, then $4x^2+2x-3 = 2(2x-5)(x+2) + A(x+2) + B(2x-5)$. Substituting -2 for x yields that $B = -1$. Substituting $\frac{5}{2}$ for x yields that $27 = \frac{9}{2}A$, so $A = 6$.
Thus $\int \frac{2x^2+2x-3}{(2x-5)(x+2)}\,dx = \int 2 + \frac{A}{2x-5} + \frac{B}{x+2}\,dx = 2x + \frac{A}{2}\ln|2x-5| + B\ln|x-2| + C = 2x + 6\ln|2x-5| - \ln|x+2| + C$.

Score: /4

c. $\displaystyle\int_2^3 \frac{1}{(x-2)^{3/4}}\,dx$

Needs an improper integral because 2 is not in the domain.
$\lim_{t\to 2^+} \int_t^3 \frac{1}{(x-2)^{3/4}}\,dx = \lim_{t\to 2^+} 4(x-2)^{1/4}\Big|_t^3 = 4\sqrt[4]{1} - 4\sqrt[4]{0} = 4.$

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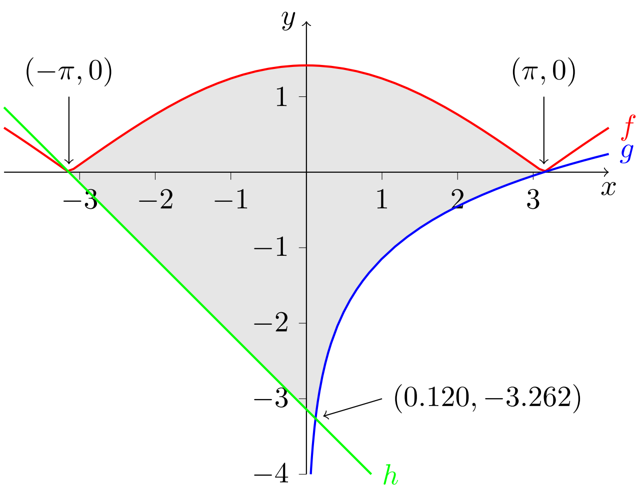
Calculators permitted from here on.

Problem 3: First, draw the following graphs on the coordinate system, then shade the region bounded by them:

$$f(x) = \sqrt{1+\cos(x)}, \quad g(x) = \ln(\frac{x}{\pi}), \quad \text{and} \quad h(x) = -x - \pi.$$

Set up the integral for the area of the shaded region, then approximate the integral to 3 decimal places.



The intersection points are as labelled. The desired area is therefore

$$\int_{-\pi}^{0.120} f(x) - h(x)\,dx + \int_{0.120}^{\pi} f(x) - g(x)\,dx \approx 8.319 + 5.287 = 13.606$$

Score: /5

Problem 4: Find the point(s) at which the value of $f(x) = e^x/r$, $r > 0$ equals the average value of the function on the interval $[0, r]$. Hint: The coordinates are expressed in terms of r.

The average value of $f(x)$ on $[0, r]$ is $\frac{1}{r}\int_0^r f(x)\,dx = \frac{1}{r}\int_0^r \frac{1}{r}e^x\,dx = \frac{1}{r^2}e^x\Big|_0^r = \frac{e^r-1}{r^2}$. Therefore $f(x)$ equals its average when $f(x) = \frac{e^r-1}{r^2}$, so $\frac{e^x}{r} = \frac{e^r-1}{r^2}$, so $e^x = \frac{e^r-1}{r}$, so $x = \ln(\frac{e^r-1}{r})$.

Score: /4

Problem 5: Find the work done by steadily lifting a leaking bucket which is full at the bottom weighing 100 pounds but only weighing 40 pounds at the top 30 feet above. Assume that the rope weighs 0.07 lb/ft and the bucket leaks at a steady rate.

At x feet from the top, the mass of the bucket is $40 + 2x$ and the mass of the rope is $0.07x$. The work required to lift bucket and rope the short distance dx is therefore $(40 + 2.07x)\,dx$. The total work is then $\int_0^{30} 40 + 2.07x\,dx = 40x + 1.035x^2\Big|_0^{30} = 2131.5$lb-ft

Score: /5

Problem 6: Built around 2600 BC, the Great Pyramid of Giza in Egypt is 146 m high and has a square base of side 230 m. Find the work (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m^3.

Each horizontal slice of the pyramid is a square. At height h from the ground, the side length of the square is $230(1 - \frac{h}{146})$, so if the slice has thickness dh, it has volume $\big(230(1 - \frac{h}{146})\big)^2\,dh$ and mass $2000\big(230(1 - \frac{h}{146})\big)^2\,dh$.
The slice has to be lifted the distance h, which requires the work $2000\big(230(1 - \frac{h}{146})\big)^2 gh\,dh$, where g is the acceleration due to gravity.
The total amount of work is then
$\int_0^{146} 2000\big(230(1 - \frac{h}{146})\big)^2 gh\,dh = 2000g\int_0^{146} 52\,900h - 724.7h^2 + 2.482h^3\,dh =$
$2000g \times 9.397 \times 10^7 = 1.879 \times 10^{11} g = 1.842 \times 10^{12}\,\text{J} = 1.842\,\text{TJ}$ (using $9.8\,\text{m/s}^2$ for g).

Score: /4