Math 126
Summer 2014
Dr. Lily Yen

Test 1

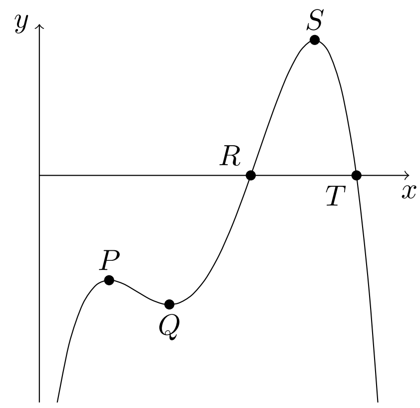
Show all your work

Name: ____________________

Score: ___/46

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Let $A(x) = \int_0^x f(t)\,dt$, where f is the function graphed below.



a. $A(x)$ has a local maximum at the x-coordinates of (circle all that apply):

P Q R S T

b. $A(x)$ has a local minimum at the x-coordinates of (circle all that apply):

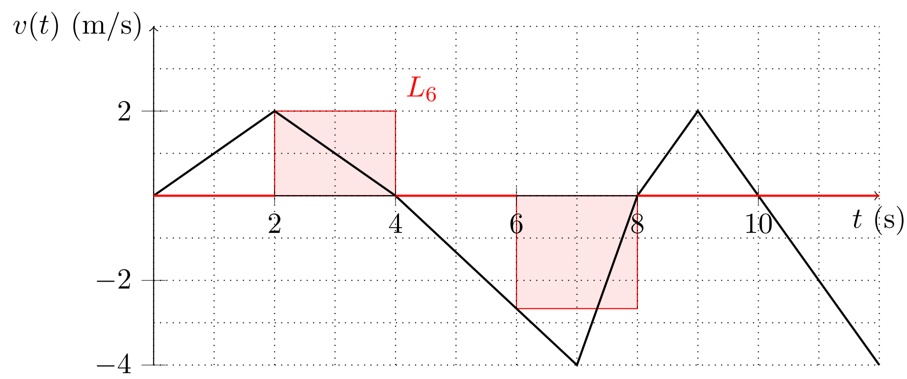
P Q R S T

c. Which inequality is true for all x in the interval over which the graph of f is shown? (Circle all that apply.)

$A(x) \leq 0$ $A(x) \geq 0$ neither both

Score: /3

Problem 2: A particle begins at the origin at time $t = 0$ and moves with velocity $v(t)$ as shown below.



a. Find the distance travelled by the particle after 7 seconds? 10 metres

b. How many times does the particle return to the origin in the first 12 seconds?

Once, $t = 6.45$

c. Find the largest time interval when the particle is moving towards the origin.

$t \in [4, 6.45]$

d. What is the particle's maximum distance from the origin? 6 m, $s(12) = -6$

e. Directly on the graph, draw in and shade the approximating rectangles representing L_6.

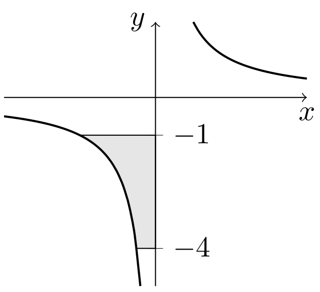
Score: /7

Problem 3: Evaluate exactly $\int_{-1}^{2} |(x-1)(x+2)|\,dx$

$\int_{-1}^{2}|(x-1)(x+2)|\,dx = \int_{-1}^{1}|(x-1)(x+2)|\,dx + \int_{1}^{2}|(x-1)(x+2)|\,dx =$
$\int_{-1}^{1} -(x-1)(x+2)\,dx + \int_{1}^{2}(x-1)(x+2)\,dx = \int_{-1}^{1} -x^2 - x + 2\,dx + \int_{1}^{2} x^2 + x - 2\,dx =$
$-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\Big|_{-1}^{1} + \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\Big|_{1}^{2} =$
$(-\frac{1}{3} - \frac{1}{2} + 2) - (\frac{1}{3} - \frac{1}{2} - 2) + (\frac{8}{3} + 2 - 4) - (\frac{1}{3} + \frac{1}{2} - 2) = \frac{5}{3} - \frac{1}{2} + 4 = \frac{10}{6} - \frac{3}{6} + \frac{24}{6} = \frac{31}{6} = 5\frac{1}{6}.$

Score: /4

Problem 4: Determine the exact area of the shaded region bounded by the y-axis, the curve given by $x = \frac{1}{y}$, and the lines $y = -1$ and $y = -4$.



Integrating on y, the area is
$\int_{-4}^{-1} 0 - \frac{1}{y}\,dy = -\int_{-4}^{-1}\frac{1}{y}\,dy = -\ln|y|\Big|_{-4}^{-1} = (-\ln(1)) - (-\ln(4)) = \ln(4) = 2\ln(2) \approx 1.386$

Score: /2

Problem 5: Integrate the following analytically.

a. $\int 3u\sqrt{4+u}\,du$

Substitute $x = 4 + u$. Then $\frac{dx}{du} = 1$, so $du = dx$ and
$\int 3u\sqrt{4+u}\,du = \int 3(x-4)\sqrt{x}\,dx = \int 3x^{3/2} - 12x^{1/2}\,dx = \frac{6}{5}x^{5/2} - 8x^{3/2} + C =$
$\frac{6}{5}(4+u)^{5/2} - 8(4+u)^{3/2} + C$

Score: /3

b. $\int_{\pi/2}^{\pi} \sin\left(\frac{5x-\pi}{6}\right)\,dx$

$\int_{\pi/2}^{\pi} \sin\left(\frac{5x-\pi}{6}\right)\,dx = -\frac{6}{5}\cos(\frac{5x-\pi}{6})\Big|_{\pi/2}^{\pi} = (-\frac{6}{5}\cos(\frac{2}{3}\pi)) - (-\frac{6}{5}\cos(\frac{1}{4}\pi)) =$
$-\frac{6}{5} \times \frac{-1}{2} + \frac{6}{5}\frac{\sqrt{2}}{2} = \frac{3+3\sqrt{2}}{5}$

Score: /3

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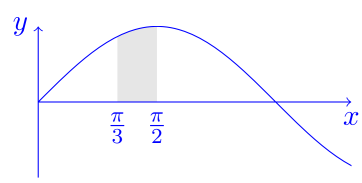
Calculators permitted from here on.

Problem 6: Express the following limit as an integral and evaluate.

$$\lim_{N\to\infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right)$$

Sketch and shade the corresponding region whose area the limit represents. Compute the integral exactly.

Since $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$, this is the Riemann sum (evaluated at the right-hand endpoint) for $\int_{\pi/3}^{\pi/2} \sin(x)\,dx = -\cos(x)\Big|_{\pi/3}^{\pi 2} = -\cos(\frac{\pi}{2}) + \cos(\frac{\pi}{3}) = 0 + \frac{1}{2} = \frac{1}{2}$.



Score: /3

Problem 7: Evaluate the following integrals exactly using the Fundamental Theorem of Calculus.

a. $\displaystyle \frac{d}{dx} \int_2^x \sqrt{t^5 - 1}\,dt$ Score: /1

By the Fundamental Theorem of Calculus,

$\sqrt{x^5 - 1}$

b. $\displaystyle \int \frac{d}{dt} \sin(e^t)\,dt$

$\sin(e^t) + C$

Score: /1

c. $\displaystyle \frac{d}{dt} \int_{e^t}^{t^4} u^3 + \cos(u^2)\,du$

Let $g(x) = \int_0^x u^3 + \cos(u^2)\,du$. Then $g'(x) = x^3 + \cos(x^2)$. Therefore $\frac{d}{dt}\int_{e^t}^{t^4} u^3 + \cos(u^2)\,du = \frac{d}{dt}\big(g(t^4) - g(e^t)\big) = g'(t^4)\cdot 4t^3 - g'(e^t)\cdot e^t = \big(t^{12} + \cos(t^8)\big)\cdot 4t^3 - \big(e^{3t} + \cos(e^{2t})\big)\cdot e^t = 4t^{15} + 4t^3\cos(t^8) - e^{4t} - e^t\cos(e^{2t})$

Score: /3

Problem 8: Evaluate the following integral exactly.

$$\int_1^{\sqrt{3}} \frac{dx}{\tan^{-1}(x)(1+x^2)}$$

Substitute $u = \tan^{-1}(x)$. Then $\frac{du}{dx} = \frac{1}{1+x^2}$, so $dx = (1+x^2)\,du$. Therefore $\int_1^{\sqrt{3}} \frac{dx}{\tan^{-1}(x)(1+x^2)} = \int_{\pi/4}^{\pi/3} \frac{(1+x^2)\,du}{u(1+x^2)} = \int_{\pi/4}^{\pi/3} \frac{1}{u}\,du = \ln|u|\Big|_{\pi/4}^{\pi/3} = \ln(\frac{\pi}{3}) - \ln(\frac{\pi}{4}) = \ln(\pi) - \ln(3) - \ln(\pi) + \ln(4) = \ln(4) - \ln(3) = 2\ln(2) - \ln(3) \approx 0.2877$

Score: /3

Problem 9: Evaluate the following integral analytically.

$$\int_0^{3/4} \frac{dx}{\sqrt{9-16x^2}}$$

If $u = \frac{4}{3}x$, then $\frac{du}{dx} = \frac{4}{3}$, so $dx = \frac{3}{4}\,du$ and
$\int_0^{3/4} \frac{dx}{\sqrt{9-16x^2}} = \int_0^1 \frac{\frac{3}{4}\,du}{\sqrt{9-9u^2}} = \int_0^1 \frac{1}{3\sqrt{1-u^2}} \cdot \frac{3}{4}\,du = \frac{1}{4}\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{4}\sin^{-1}(u)\Big|_0^1 = \frac{1}{4}\cdot\frac{\pi}{2} - \frac{1}{4}\cdot 0 = \frac{\pi}{8}.$

Score: /3

Problem 10: On a hot summer day, Burnaby consumes water at the rate of

$$r(t) = 100 + 72t - 3t^2$$

(in millions of litres per hour), where t is the number of hours past midnight. Answer the following by writing logical and coherent steps which lead to your final answer. No mark will be given to a single number answer.

a. When is water consumed at the highest rate?

Since $0 = r'(t) = 72 - 6t$ when $t = 12$, and $r''(t) = -6 < 0$, the rate is maximal when $t = 12$, so 12 hours after midnight, so at noon.

Score: /1

b. Find the daily water consumption.

In a (very) short interval of length dt hours, Burnaby consumes $r(t)\,dt$ millions of litres of water. Therefore the daily consumption is
$\int_0^{24} r(t)\,dt = \int_0^{24} 100 + 72t - 3t^2\,dt = 100t + 36t^2 - t^3\Big|_0^{24} = 9312$ million litres.

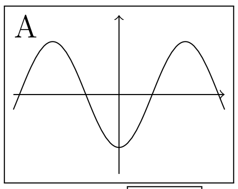
Score: /2

c. How much water is consumed between 6 pm and midnight?

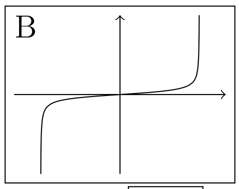
As in the previous part,
$\int_{18}^{24} r(t)\,dt = \int_{18}^{24} 100 + 72t - 3t^2\,dt = 100t + 36t^2 - t^3\Big|_{18}^{24} = 1680$ million litres.

Score: /2

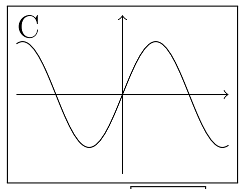
Problem 11: Match each graph of $y = f'(x)$ (A–E) with the graph of a possible anti-derivative, $y = f(x)$ (1–10).



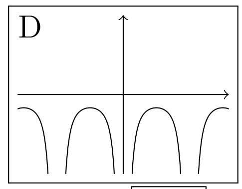
Answer: 10



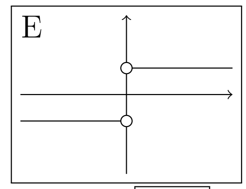
Answer: 1



Answer: 4

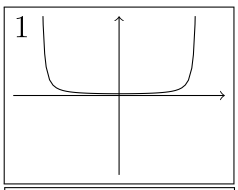


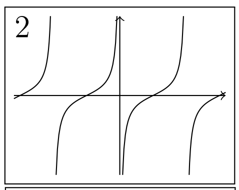
Answer: 9

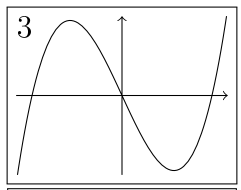


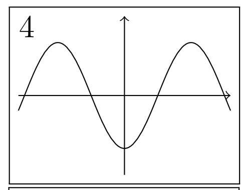
Answer: 5

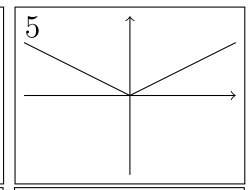
Anti-derivative graphs:

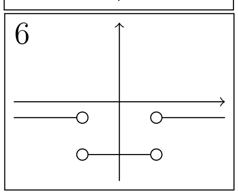


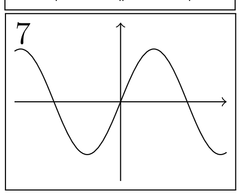


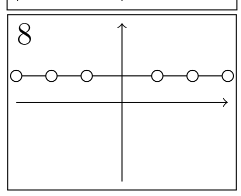


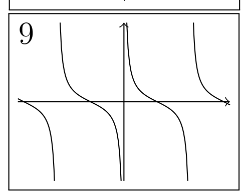


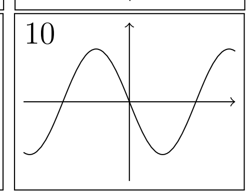












Score: /5