Math 126
Summer 2007
Dr. Lily Yen

Test 2

Show all your work

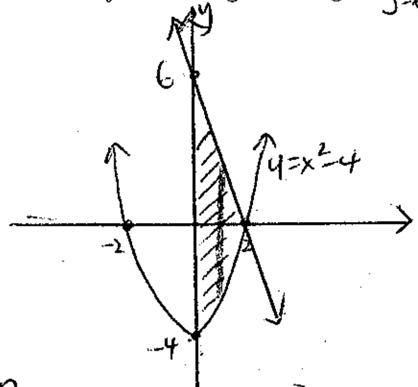
Name: ______________

Score: ___/34

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

9:50
10:15

Problem 1: Write a Riemann sum and then a definite integral representing the area of the region enclosed by the y-axis, the line $3x + y = 6$, and the parabola $y = x^2 - 4$. Find the area by evaluating the integral. $y=6-3x$ Score: /3



$$\int_0^2 6-3x-(x^2-4)\, dx$$

$$=\int_0^2 -x^2-3x+10\, dx$$

$$=-\frac{x^3}{3}-\frac{3x^2}{2}+10x\Big|_0^2$$

sign error careless -1

-1/2

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[(6-3x_i)-(x_i^2-4)\right]\Delta x$$

$$\Delta x = \frac{2}{n}$$

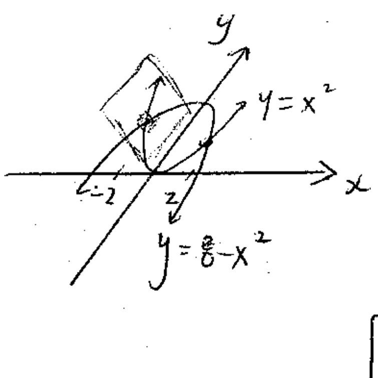
$$=-\frac{8}{3}-6+20-0$$

$$=14-2\tfrac{2}{3}$$

$$=11\tfrac{1}{3}$$ -1/2 for eval.

Problem 2: Find the volume of the solid whose base is the region in the x–y-plane bounded by the curves $y = x^2$ and $y = 8 - x^2$ and whose cross-sections perpendicular to the x-axis are squares with one side in the x–y-plane. Score: /3

Intersections of $\begin{cases} y=x^2 \\ y=8-x^2 \end{cases}$



$x^2=8-x^2$

$2x^2=8$

$x^2=4$

$x=\pm 2$

$$\int_{-2}^{2}\left[(8-x^2)-x^2\right]^2 dx$$ even function

$$=2\int_0^2 (8-2x^2)^2\, dx$$ $\frac{6}{5}-\frac{2}{3}$

$$=2\int_0^2 2\cdot(16-8x^2+x^4)\,dx$$ -1/2 $=\frac{18-10}{15}=\frac{8}{15}$

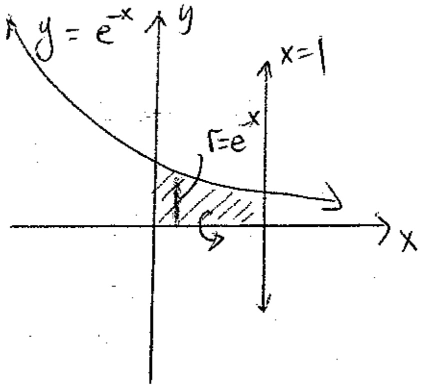
$$=8\left(16x-\frac{8}{3}x^3+\frac{x^5}{5}\Big|_0^2\right)$$

wrong factor -1/2

$$=8\left(32-\frac{64}{3}+\frac{32}{5}\right)=8(32)\left(1-\frac{2}{3}+\frac{1}{5}\right)=2^8\cdot\frac{8}{15}=\frac{2^{11}}{15}$$

Problem 3: The region bounded by the curve $y = e^{-x}$ and the x-axis between $x = 0$ and $x = 1$ is revolved around the x-axis. Find the volume of this solid of revolution.

Score: /3



$$\int_0^1 \pi (e^{-x})^2 \, dx$$

$$= \pi \int_0^1 e^{-2x} \, dx$$

$$= \pi \cdot \left. \frac{e^{-2x}}{-2} \right|_0^1$$

$$= \pi \cdot \left(-\frac{e^{-2}}{2} - \left(-\frac{1}{2} \right) \right)$$

$$= \pi \left(\frac{1}{2} - \frac{1}{2e^2} \right) = \frac{1}{2} \pi \left(1 - \frac{1}{e^2} \right)$$

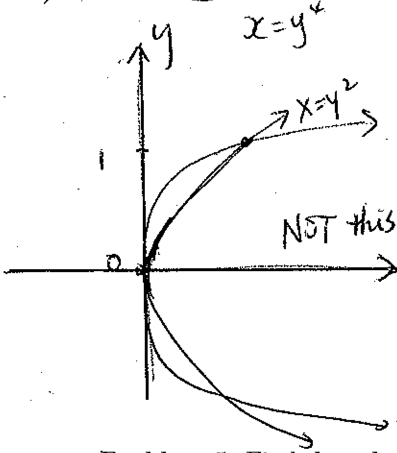
or $\frac{1}{2} \pi \left(\frac{e^2 - 1}{e^2} \right)$

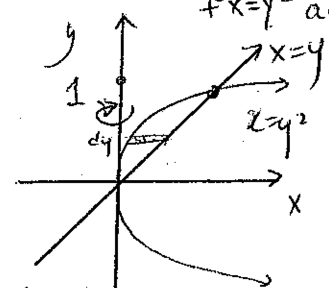
Problem 4: Describe the solid whose volume is represented by the integral below.

$$\pi \int_0^1 (y^4 - y^2) \, dy$$

Solid obtained by rotating (revolving) region enclosed by $x = y$ + $x = y^2$ about y-axis

Score: /3





$$\int_0^1 \pi y^2 - \pi y^4 \, dy$$

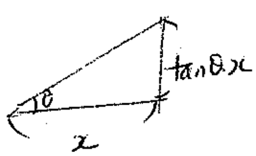
Problem 5: Find the volume of a wedge formed by a plane slicing a right-circular cylinder of radius r if the plane meets the base at an angel θ and the line formed by its intersection with the base forms a diameter of the cylinder.

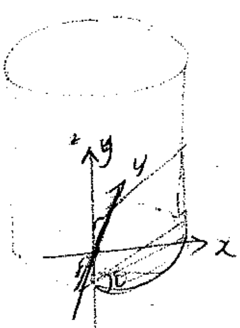
Score: /3 4

$x^2 + y^2 = r^2$

$x = \pm \sqrt{r^2 - y^2}$

We only take $x = \sqrt{r^2 - y^2}$ to the right of y-axis.





$$\frac{1}{2} \int_{-r}^{r} \sqrt{r^2 - y^2} \cdot \tan\theta \left(\sqrt{r^2 - y^2} \right) dy$$

$$= 2 \cdot \frac{1}{2} \int_0^r \tan\theta \cdot (r^2 - y^2) \, dy$$

$$= \tan\theta \left. \left(r^2 y - y^3/3 \right) \right|_0^r$$

$$= \tan\theta \left(r^3 - r^3/3 \right) = \tan\theta \frac{2r^3}{3}$$

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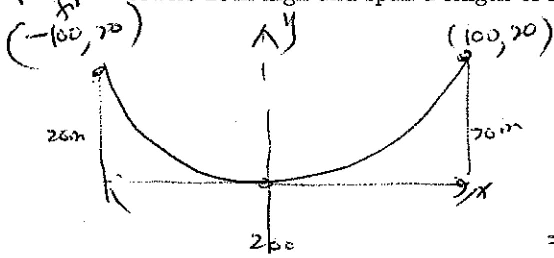
10:17
10:47
~~Calculators~~ permitted from here on.

not able to find.

Problem 6: Find the length of the parabolic support cable for a suspension bridge with ~~towers~~ 20 m high and span a length of 200 m. Score: /3



$y=\frac{1}{500}x^2$, $\frac{dy}{dx}=\frac{1}{250}x\,dx$

$$\int_{-100}^{100}\sqrt{1+\left(\frac{1}{250}x\right)^2}\,dx$$

$$=2\int_0^{100}\sqrt{1+\left(\frac{x}{250}\right)^2}\,dx$$

$$\approx 205.21$$

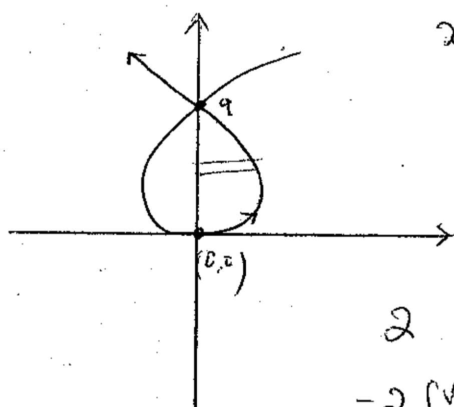
$$y=ax^2$$

$$20=a\cdot 100^2$$

$$a=\frac{\cancel{20}}{\cancel{100}\ 100\ 100}=\frac{1}{500}$$

around 205.21 m of one support cable.

Problem 7: Find the length of the loop of the curve $x(t)=3t-t^3$ and $y(t)=3t^2$. Score: /4



$x=0$ if $3t-t^3=0$

$t(3-t^2)=0$

$t=0,\ t=\pm\sqrt{3}$

$y(\sqrt{3})=9 \qquad y(-\sqrt{3})=9$

$x'(t)=3-3t^2\,dt \qquad y'(t)=6t\,dt$

$$2\int_0^{\sqrt{3}}\sqrt{(3-3t^2)^2+(6t)^2}\,dt$$

$$=2\int_0^{\sqrt{3}}\sqrt{9-18t^2+9t^4+36t^2}\,dt$$

$$=2\int_0^{\sqrt{3}}3t^2+3\,dt=6\left(\frac{t^3}{3}+t\right)\Big|_0^{\sqrt{3}}=6(\sqrt{3}\times 2)=12\sqrt{3}$$

Problem 8: If Lily's cup of tea has temperature 95°C in Cedar 136 where the temperature is 22°C, then, according to Newton's Law of Cooling, the temperature of the tea after t minutes is $T(t)=22+73e^{-t/50}$. What is the average temperature of the tea during the first half hour? Score: /4

half an hour = 30 minutes

$$\text{average}=\frac{1}{30}\int_0^{30}22+73e^{-t/50}\,dt$$

$$=\frac{1}{30}\cdot 22t-\frac{73e^{-t/50}}{\cancel{50}}\cdot 50\Big|_0^{30}$$

$$=\frac{1}{30}\left(660-73\cdot 50\cdot e^{-30/50}-0+73\cdot 50\cdot 1\right)$$

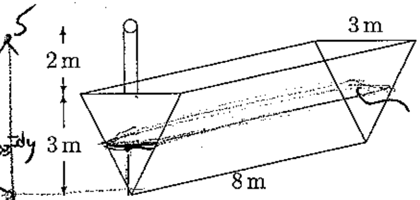
$$\cong\frac{1}{30}(4310-2003)\cong 76.89$$

around $\approx$ 76.89 °C is the average temperature

$$W = 1000\cdot(-g)\int_3^0 8y(5-y)\,dy \qquad \frac{5y^2}{2} - \frac{y^3}{3}\Big|_3^0$$

$$= -1000g\cdot 8\int_3^0 5y - y^2\,dy = -8000g\cdot$$

Problem 9: The tank in the shape of a triangular prism is full of water. $1000\ kg/m^3$



$y \quad 5-y \quad dy \quad 0$

$A = y\cdot 8 \qquad = -8000g\left(9 - \frac{45}{2}\right)$

$V = 8y\,dy \qquad \approx 1.06\,J$

a. Find the work required to pump the water out of the spout. Score: /3

$$W = \int_3^0 8\cdot\tfrac{1}{2}y^2\cdot 1000(-g)\,(5-y)\,dy$$

area $= \frac{1}{2}y^2$

volume $= 8\times\frac{1}{2}y^2\,dy = 1000g\,4\int_3^0 5y^2 - y^3\,dy$

$g = -9.8\ m/s^2 \qquad = -4000\times 9.8\left(\frac{5y^3}{3} - \frac{y^4}{4}\Big|_3^0\right) = -4000\times 9.8\left(-45 + \frac{81}{4}\right)$

$= 9.702\times 10^5\ J$

b. Suppose that the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank? Score: /3

$$\cancel{-4000\times 9.8\times\left(\frac{y^4}{4} - \frac{5y^3}{3}\Big|_d^3\right) = 4.7\times 10^5}$$

d = depth $\quad \cancel{\frac{81}{4} - 45 - \frac{d^4}{4} + \frac{5d^3}{3} = 11.99}$

$\frac{5b^2}{2} - \frac{b^3}{3} - \frac{45}{2} + 9 = -6 \qquad -3 + 22\tfrac{1}{2} \qquad -15$

$$-8000g\left(\frac{5y^2}{2} - \frac{y^3}{3}\right)\Big|_3^b = 4.7\times 10^5 \qquad \frac{5b^2}{2} - \frac{b^3}{3} = 19\tfrac{1}{2}$$

$x \approx 2.028,\ 4.22 \qquad x \approx 7.05,\ -1.58$

take $x \approx 2.03$ m

Problem 10: The mouse population in Lily's neighbourhood is modelled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

a. For what values of P is the population increasing?

More up down

$1 - \frac{P}{4200} > 0 \qquad$ so $1 > \frac{P}{4200}$

$0 < P < 4200$, $P(t)$ increases.

b. For what values of P is the population decreasing?

$1 - \frac{P}{4200} < 0 \qquad$ or $1 < \frac{P}{4200}$

$P > 4200$

c. What are the equilibrium solutions?

$P = 0$ or $P = 4200$

Score: ~~4~~ 3

Problem 11: Find the orthogonal trajectories of the family of curves given by $xy = k$.
Score: /4

$$y + xy' = 0$$

$$y' = -\frac{y}{x}$$

So $\perp$ family $y' = \frac{x}{y}$

$$y \cdot y' = x$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

or $y^2 = x^2 + C_1$

$$y = \pm\sqrt{x^2 + C_1}$$

-1 for $\sqrt{}$ incorrectly.

Problem 12: Solve the differential equation $\cos(y)y' - \tan(x) = 0$ with initial condition $y(0) = \pi/6$.
Score: /4

$$\cos(y)y' = \tan(x)$$

$$\int \cos(y)\,dy = \int \tan(x)\,dx = -\int \frac{-\sin(x)}{\cos(x)}\,dx$$

$$\sin(y) = -\ln|\cos(x)| + C$$

$-\frac{1}{2}$

$x=0$, $y = \pi/6$

$$\sin(\pi/6) = -\underbrace{\ln 1}_{0} + C$$

$$C = 1/2$$

$$y = \arcsin\left(-\ln|\cos(x)| + \tfrac{1}{2}\right)$$

-1

Problem 13: The air in our classroom with volume $400\,\text{m}^3$ initially contains 0.15% carbon dioxide. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of $2\,\text{m}^3/\text{min}$ and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run? Score: /5

$V = 400\ m^3$

CO_2 % falls to 0.05% ①

$t \to \infty \quad 0.1\, e^{-t/200} \to 0$

initial concentration $0.15\%\ CO_2$.

So initial $CO_2 = 400 \times \frac{0.15}{100} = 0.6\ m^3$

concentration in $= 0.05\%$

rate in $= 2\ m^3/min$

rate out $= 2\ m^3/min$

Not concentrate but amount.

$-\frac{1}{2}$.

Let the amount of CO_2 at time t be $A(t)$

$$\frac{dA(t)}{dt} = \frac{0.05}{100} \times 2 - \frac{A(t)}{400} \cdot 2$$

$$\frac{dA}{dt} = \frac{0.1}{100} - \frac{A(t)}{200}$$

$$= \frac{0.2 - A(t)}{200}$$

$$-\int \frac{-dA}{0.2-A} = \int \frac{dt}{200}$$

$$-\ln|0.2-A| = \frac{1}{200}t + C$$

Since A is dropping, $0.2-A \lessgtr 0$, so OK to remove $| \ |$.

$$\ln \frac{1}{\cancel{0.2-}A-0.2} = \frac{1}{200}t + C$$

$$\frac{1}{\cancel{0.2-}A-0.2} = e^{\frac{1}{200}t + C} = k\, e^{\frac{1}{200}t}$$

at $t=0$, $A_0 = 0.6$, Substitute to get k.

$$\frac{1}{\cancel{0.2}\ 0.4} = k \cdot e^0 = k \qquad \text{So } k = \frac{1}{0.4} = 5/2$$

$$\frac{1}{A-0.2} = 5/2\ e^{t/200} \quad \text{or} \quad A - 0.2 = 2/5\ e^{-t/200}$$

or $A(t)$ $\frac{0.21}{}$ + $\frac{21}{}$ $e^{-t/200}$ ~~A(t)~~ $0.2 + 2/5\ e^{-t/200}$