

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Determine a_3 , a_4 , and a_5 of the sequence starting with $n = 1$.

$$a_1 = 1, \quad a_2 = 2, \quad a_n = 3a_{n-1} - 2a_{n-2}.$$

$$\begin{aligned} a_3 &= 3a_2 - 2a_1 = 3 \times 2 - 2 \times 1 = 4 \\ a_4 &= 3a_3 - 2a_2 = 3 \times 4 - 2 \times 2 = 8 \\ a_5 &= 3a_4 - 2a_3 = 3 \times 8 - 2 \times 4 = 16 \end{aligned}$$

Actually, $a_n = 2^{n-1}$, but you were not asked or expected to find or prove that.

Score: /3

Problem 2: Solve the following initial-value problem.

$$t \frac{dy}{dt} = \sqrt{1 - y^2}, \quad (t, y) = (e, 0).$$

If $t \frac{dy}{dt} = \sqrt{1 - y^2}$, then $\int \frac{1}{\sqrt{1 - y^2}} dy = \int \frac{1}{t} dt$, so $\sin^{-1}(y) = \ln|x| + C$. Since $y(e) = 0$, $\sin^{-1}(0) = \ln|e| + C$, so $0 = 1 + C$, so $C = -1$. Thus $\sin^{-1}(y) = \ln|x| - 1$, so

$$y = \sin(\ln|x| - 1),$$

for non-zero x .

Score: /3

Problem 3: Determine the limit of the sequence or show that it diverges.

a. $G_n = \frac{3}{5} - \frac{9}{25} + \frac{27}{125} - \frac{81}{625} + \cdots + (-1)^{n-1} \frac{3^n}{5^n}$, for integers $n \geq 1$.

For partial marks, write G_1 and G_2 without computing.

$G_1 = \frac{3}{5}$, $G_2 = \frac{6}{25}$.
Since $\{G_n\}$ is a geometric sequence with common ratio $-3/5$, the (infinite sum is

$$\frac{3/5}{1 - (-3/5)} = \frac{3/5}{8/5} = \frac{3}{8}$$

Score: /4

b. $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$

This is divergent because p -test says that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent for $0 < p \leq 1$ (and in the problem, $p = \frac{1}{2}$).

Score: /3

Problem 4: Determine whether the series converges conditionally, converges absolutely, or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}.$ Write out a few terms before starting your analysis.
Since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}} = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{8} + \frac{1}{5\sqrt{5}} - \cdots$$

is alternating and the terms decrease in size, the series converges.

Moreover, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$ which is convergent by p -Test. Thus the original sum absolutely convergent.

Score: /4

b. $\sum_{n=1}^{\infty} \frac{n!}{5^n}.$ Hint: Consecutive term ratio.

$$\sum_{n=1}^{\infty} \frac{n!}{5^n} = \frac{1}{5} + \frac{2}{25} + \frac{6}{125} + \frac{24}{625} + \frac{120}{3125} + \cdots + a_n + \cdots$$

where $a_n = \frac{n!}{5^n}.$ Since, $\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!/5^{n+1}}{n!/5^n} = \frac{(n+1)}{5} \rightarrow \infty > 1$ as $n \rightarrow \infty,$ it follows from Ratio Test that the series diverges to $\infty.$

Score: /3

Calculators permitted in this part.

Problem 5: Find the orthogonal trajectories of the families of curves where k is a parameter. Sketch at least three members of each family (the original and the orthogonal) on the same coordinate system. Label each curve with its corresponding parameter (k) value.

$$y = \frac{1}{x + k}$$

If $y = \frac{1}{x+k}$, then $\frac{1}{y} = x + k$,
so $\frac{1}{y} - x = k$. Hence
 $-\frac{1}{y^2} \frac{dy}{dx} - 1 = 0$, so
 $\frac{dy}{dx} = -y^2$.
Therefore the orthogonal trajectories satisfy the equation

$$\frac{dy}{dx} = \frac{1}{y^2},$$

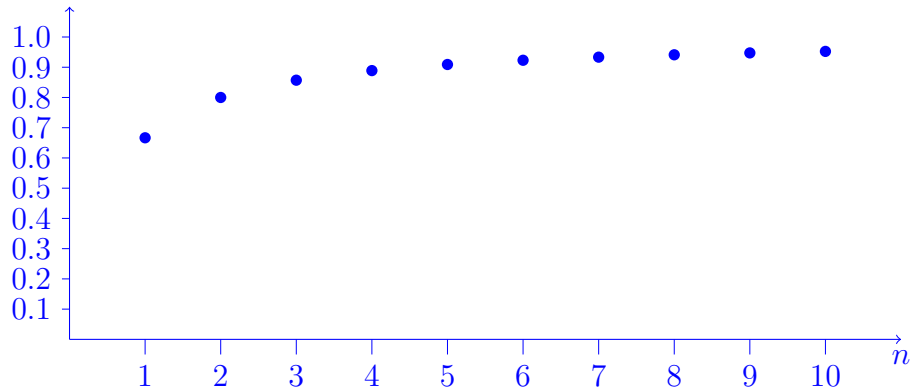
so $\int y^2 dy = \int dx$, so
 $\frac{1}{3}y^3 = x + C$, so
 $y^3 = 3x + 3C$, so
 $y = \sqrt[3]{3x + 3C}$.

Score: /5

Problem 6: Find the sum of the series below by first graphing the n th partial sum for the first 10 n 's. State your function entry and window dimensions in your graphing calculator.

$$\frac{2}{1 \times 3} + \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \frac{2}{7 \times 9} + \cdots$$

The denominator of each term in the summand is the product of consecutive odd integers.



Note that $\frac{1}{2i-1} - \frac{1}{2i+1} = \frac{2i+1}{(2i-1)(2i+1)} - \frac{2i-1}{(2i-1)(2i+1)} = \frac{2}{(2i-1)(2i+1)}$. Therefore
 $\frac{2}{1 \times 3} + \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \frac{2}{7 \times 9} + \cdots = \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{2n+1} = 1 - \frac{1}{2n+1} \rightarrow 1$ as $n \rightarrow \infty$.
TI-84 entries:

$$nMin = 1, \quad u(n) = u(n-1) + \frac{2}{(2n-1)(2n+1)}, \quad u(nMin) = \left\{\frac{2}{3}\right\}$$

with window dimensions: $(0, 11) \times (0, 1.2)$, $nMin = 1$, $nMax = 10$.

Score: /5

Problem 7: Jasmine bought a cup of Tim Horton's coffee which started with a temperature of 75°C . She let it cool for 10 minutes and measured the temperature to be 70°C . Suppose the surrounding temperature was 20°C , find the temperature of her coffee an hour after her purchase.

If $T(t)$ is the temperature in $^\circ\text{C}$ of the coffee at time t minutes, then $\frac{dT}{dt} = k(20 - T)$, so

$$\int \frac{1}{20 - T} dT = \int k dt$$

so $-\ln|20 - T| = kt + C$, so $|20 - T| = e^{-kt-C}$, so $20 - T = \pm e^{-C} e^{-kt} = Ae^{-kt}$, so

$$T = 20 - Ae^{-kt}$$

Now, $75 = T(0) = 20 - Ae^0 = 20 - A$, so $A = -55$. Moreover,

$70 = T(10) = 20 - Ae^{-10k} = 20 + 55e^{-10k}$, so $\frac{50}{55} = e^{-10k}$, so $\ln(10/11) = -10k$, so

$k = -\frac{1}{10} \ln(10/11) \approx 0.00953$.

Hence $T(60) = 51.0^\circ\text{C}$.

Score: /4

Problem 8: A pool holds 5000 litres of pure water. A tap is opened and a solution containing 0.2 g of chlorine per litre is fed into the pool for disinfection at a rate of 2 litres per minute. At the same time, a drain plug in the bottom of the pool is opened and the thoroughly mixed solution leaves at the same rate. How much chlorine is in the pool after two hours?

If $y(t)$ is the amount of chlorine in the pool after t minutes, then the concentration is $\frac{y}{5000}$. Thus 0.4 g of chlorine enters the pool per minute and $2 \times \frac{y}{5000} = \frac{y}{2500}$ grams leave. Thus

$$\frac{dy}{dt} = 0.4 - \frac{y}{2500} = \frac{1000 - y}{2500}$$

Therefore $\int \frac{1}{1000 - y} dy = \int \frac{1}{2500} dt$, so $-\ln(1000 - y) = \frac{t}{2500} + C$, so $\ln(1000 - y) = -\frac{t}{2500} - C$, so $1000 - y = e^{-t/2500 - C} = Ae^{-t/2500}$, where $A = e^{-C}$. Hence $y = 1000 - Ae^{-t/2500}$, and since $0 = y(0) = 1000 - Ae^0 = 1000 - A$, it follows that $A = 1000$.

The amount of chlorine after two hours is then $y(120) = 1000 - 1000e^{-120/2500} \approx 46.9$ g.

Score: /6