Math 126
Fall 2025
Dr. Lily Yen

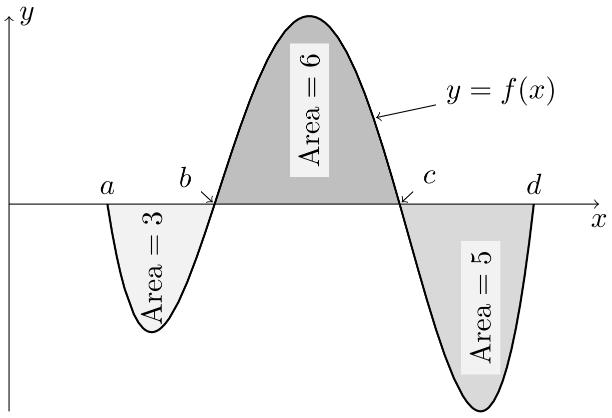
# Test 1

Show all your work

Full Name: __________
Student Number: __________
Signature: __________
Score: ___/38

**No Calculator permitted** in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

**Problem 1**: Use the figure to find the values of the following.




a. $\int_b^d f(x)\,dx = \boxed{6\text{-}5=1}$

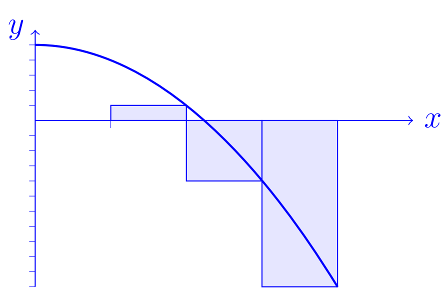
b. $\left|\int_a^d f(x)\,dx\right| = \boxed{|-3+6-5|=2}$

c. $\int_a^d |f(x)|\,dx = \boxed{3+6+5=14}$

Score: /3

**Problem 2**: Use the definition of the definite integral to write $\int_1^4 5-x^2\,dx$ as a limit of a Riemann sum for $R_n$. Draw $R_3$ on a coordinate system. Do not evaluate

$$\int_1^4 5-x^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{3}{n}\sum_{j=1}^{n}\left(5-\left(1+\frac{3j}{n}\right)^2\right)$$




Score: /4

**Problem 3**: Use the Fundamental Theorem of Calculus to find the following.

a. If $F(x) = \int_\pi^{\sqrt[3]{x}} t\,dt$, find $F'(8)$ exactly.

Let $h(x) = \int_\pi^x t\,dt$. Then $h'(x) = x$ by the FTC. Moreover, $F(x) = h(\sqrt[3]{x})$, so $F'(x) = h'(\sqrt[3]{x})\frac{1}{3}x^{-2/3} = \sqrt[3]{x}\frac{1}{3}x^{-2/3}$, so $F'(8) = \sqrt[3]{8}\frac{1}{3}8^{-2/3} = 2\times\frac{1}{3}\times\frac{1}{4} = \frac{1}{6}$.

Score: /2

b. If $G(x) = \int_{\ln(x)}^{\sin(\pi/6)} e^t - \sin^{-1}(t)\,dt$, find $dG(x)/dx$. Simplify.

Let $h(x) = \int_0^x e^t - \sin^{-1}(t)\,dt$. Then $h'(x) = e^x - \sin^{-1}(x)$ by the FTC. Moreover, $G(x) = \int_{\ln(x)}^0 e^t - \sin^{-1}(t)\,dt + \int_0^{\sin(\pi/6)} e^t - \sin^{-1}(t)\,dt =$ $-\int_0^{\ln(x)} e^t - \sin^{-1}(t)\,dt + \int_0^{\sin(\pi/6)} e^t - \sin^{-1}(t)\,dt = -h(\ln(x)) + C$, where $C$ is a constant because of the definite integral; so

$$G'(x) = -h'(\ln(x))\frac{1}{x} + 0 = -\left(e^{\ln(x)} - \sin^{-1}(\ln(x))\right)\frac{1}{x}$$
$$= -\left(x - \sin^{-1}(\ln(x))\right)\frac{1}{x} = \frac{\sin^{-1}(\ln(x))}{x} - 1$$

Score: /2

/11

**Problem 4**: Integrate each of the following indefinite integrals. Use correct notation and show all steps.

a. $\displaystyle\int \frac{\sqrt{u}-u^2}{u^3}\,du$

$$\int \frac{u^{1/2}-u^2}{u^3}\,du = \int u^{-5/2}-u^{-1}\,du = -\frac{2}{3}u^{-3/2}-\ln|u|+C$$

Score: /2

b. $\displaystyle\int \frac{\sin(x)}{4+\cos^2(x)}\,dx$

If $u = \frac{1}{2}\cos(x)$, then $\frac{du}{dx} = -\frac{1}{2}\sin(x)$, so $dx = \frac{-2}{\sin(x)}\,du$. Moreover, $2u = \cos(x)$. Therefore

$$\begin{aligned}\int \frac{\sin(x)}{4+\cos^2(x)}\,dx &= \int \frac{\sin(x)}{4+(2u)^2}\,\frac{-2}{\sin(x)}\,du \\ &= \int \frac{-1/2}{1+u^2}\,du = -\frac{1}{2}\tan^{-1}(u)+C = -\frac{1}{2}\tan^{-1}\left(\frac{\cos(x)}{2}\right)+C\end{aligned}$$

Score: /2

c. $\displaystyle\int \frac{e^{-t}}{(5-e^{-t})^4}\,dt$

If $u = 5-e^{-t}$, then $\frac{du}{dt} = e^{-t}$, so $dt = e^{t}\,du$. Therefore

$$\begin{aligned}\int \frac{e^{-t}}{(5-e^{-t})^4}\,dt = \frac{e^{-t}}{u^4}\,e^{t}\,du &= \int u^{-4}\,du = -\tfrac{1}{3}u^{-3}+C \\ &= -\tfrac{1}{3}\left(5-e^{-t}\right)^{-3}+C = \frac{-1}{3\left(5-e^{-t}\right)^3}+C\end{aligned}$$

Score: /2

d. $\displaystyle\int \cot(\theta)\,d\theta$

If $u = \sin(\theta)$, then $\frac{du}{d\theta} = \cos(\theta)$, so $d\theta = \frac{1}{\cos(\theta)}\,du$. Therefore

$$\begin{aligned}\int \cot(\theta)\,d\theta = \int \frac{\cos(\theta)}{\sin(\theta)}\,d\theta &= \int \frac{\cos(\theta)}{u}\,\frac{1}{\cos(\theta)}\,du \\ &= \int \frac{1}{u}\,du = \ln|u|+C = \ln|\sin(\theta)|+C\end{aligned}$$

Score: /2

$\boxed{\quad /8}$

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# Test 1

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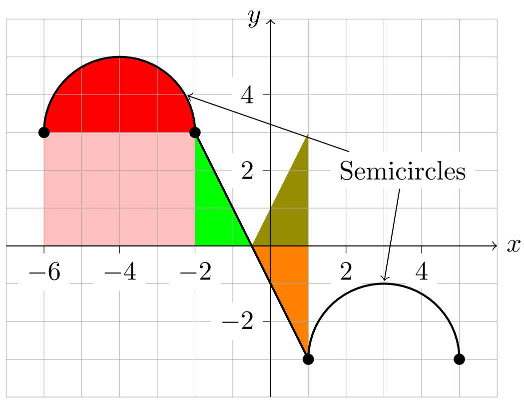
**A TI83 or TI84 Calculator permitted** in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

**Problem 5**: Given the graph of $f$, evaluate the following definite integrals exactly (with $\pi$ and fractions). Show all steps.




a. $\displaystyle\int_{-6}^{-2} f(x)\,dx$

The red and pink areas under the curve add up to $4 \times 3 + \frac{1}{2}\pi 2^2 = 12 + 2\pi$.

b. $\displaystyle\int_{-2}^{1} |f(x)|\,dx$

The (congruent) green and olive triangles amount to $2 \times \frac{1}{2}(\frac{3}{2} \times 3) = 9/2$.

c. $\displaystyle\left|\int_{-2}^{5} f(x)\,dx\right|$

The green and orange triangles cancel each other, so $\int_{-2}^{5} f(x)\,dx = \int_{1}^{5} f(x)\,dx = -(4 \times 3 - \frac{1}{2}\pi 2^2) = 2\pi - 12$.

Hence $\left|\int_{-2}^{5} f(x)\,dx\right| = 12 - 2\pi$.

Score: /6

**Problem 6**: A car comes to a stop six seconds after the driver applies the brakes. While the brakes are on, the following velocities are recorded. Give an upper estimate for the distance the car travelled after the brakes were applied. Specify which approximation you use for the estimate.

Score: /3

| Time since brakes applied (s) | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| Velocity (m/s) | 30 | 17 | 6 | 0 |

Let $v(t)$ denote the velocity at time $t$ since the brakes were applied. Then the distance travelled is $\int_0^6 v(t)\,dt$ which can be approximated by a Riemann sum with three intervals. Since $v(t)$ is decreasing, the left sum overestimates while the right sum underestimates because the function is decreasing. Now

$$L_3 = (v(0) + v(2) + v(4))2 = (30 + 17 + 6)2 = 106\,\text{m}$$
$$R_3 = (v(2) + v(4) + v(6))2 = (17 + 6 + 0)2 = 46\,\text{m}$$

Trapezoidal rules gives

$$T_3 = \left(\frac{v(0)+v(2)}{2} + \frac{v(2)+v(4)}{2} + \frac{v(4)+v(6)}{2}\right)2 = 76\,\text{m}$$

/9

**Problem 7**: Find a function $f$ such that $f(1) = 5$ and the derivative of $f$ is

$$f'(t) = 3\sqrt{t} + 4t^3 = 3t^{1/2} + 4t^3$$

The antiderivative is

$$f(x) = 2t^{3/2} + t^4 + C$$

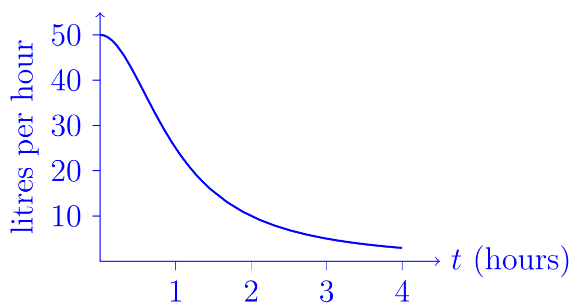
Now $5 = f(1) = 2 \times 1^{3/2} + 1^4 + C = 2 + 1 + C = 3 + C$, so $C = 2$.

$$f(x) = 2t^{3/2} + t^4 + 2$$

Score: /2

**Problem 8**: Water is leaking out of a tank at a rate of $R(t) = \frac{50}{1+t^2}$ litres per hour, $t$ hours after the leak starts. Include units.

a. Use your calculator to sketch the graph of $y = R(t)$ over $[0, 4]$.




b. At what time was the water leaking out at the fastest rate?

$R'(t) = -\frac{100t}{(1+t^2)^2}$, so $R'(t) = 0$ when $t = 0$. The graph confirms that $t = 0$ is a maximum.
Even if you interpret the question as implying a closed interval, $t \in [0, 4]$, the maximum occurs at $t = 0$ since $R(4) = 50/17$ while $R(0) = 50$.

c. Use $R_{50}$ to estimate the volume of water that leaked out of the tank in the first hour. Provide 4-decimal place accuracy.

We need to take the definite integral

$$\int_0^1 \frac{50}{1+t^2}\, dt \approx R_{50} \approx 39.02$$

Thus, about 39.02 litres of water leaked in the first hour.
Actually, $\int_0^1 \frac{50}{1+t^2}\, dt = \frac{25}{2}\pi \approx 39.27$.

d. Find the average rate of leakage in the first four hours. Provide 4-decimal place accuracy.

$$\frac{1}{4}\int_0^4 R(t)\, dt = \frac{1}{4}\int_0^4 \frac{50}{1+t^2}\, dt \approx 16.57$$

litres per hour on average.

Score: /8

/10