Math 126 Fall 2022 Dr. Lily Yen	# Test 2 Show all your work	Name: ______ Number: ______ Signature: ______ Score: ___/35

No Calculator permitted in this part. Read the questions carefully. Show all your work and clearly indicate your final answer. Use proper notation.

Problem 1: Evaluate the following integrals analytically.

a. $\displaystyle\int \tan^{-1} x\, dx$

Using integration by parts—and remembering that the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$,

$$\int 1 \cdot \tan^{-1}(x)\, dx = x\tan^{-1}(x) - \int x\frac{1}{1+x^2}\, dx.$$

Substituting $u = 1 + x^2$, so $\frac{du}{dx} = 2x$, yields

$$x\tan^{-1}(x) - \tfrac{1}{2}\int \frac{1}{u}\, du = x\tan^{-1}(x) - \tfrac{1}{2}\ln|u| + C = x\tan^{-1}(x) - \tfrac{1}{2}\ln(1+x^2) + C$$

Score: /2

b. $\displaystyle\int e^x \sin(3x)\, dx$

Using integration by parts twice,

$$\begin{aligned}\int e^x \sin(3x)\, dx &= e^x\sin(3x) - 3\int e^x\cos(3x)\, dx\\ &= e^x\sin(3x) - 3\left(e^x\cos(3x) + 3\int e^x\sin(3x)\, dx\right)\\ &= e^x\sin(3x) - 3e^x\cos(3x) - 9\int e^x\sin(3x)\, dx,\end{aligned}$$

so $10\int e^x\sin(3x)\, dx = e^x\sin(3x) - 3e^x\cos(3x)$, so

$$\int e^x\sin(3x)\, dx = \tfrac{1}{10}\big(\sin(3x) - 3\cos(3x)\big)e^x + C$$

Score: /4

Problem 2: Evaluate if possible; otherwise, state why the integral does not exist.

$$\int_2^4 \frac{1}{(x-3)^2}\, dx$$

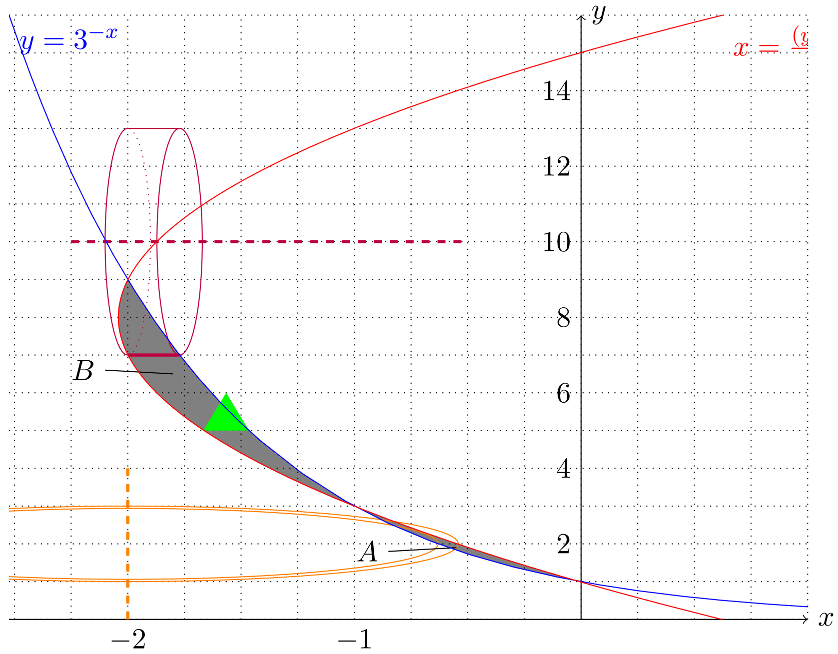
The integrand is not defined at $x = 3$, so you must evaluate on both sides separately. However,

$$\int_2^3 \frac{1}{(x-3)^2}\, dx = \lim_{z\to 3^-}\int_2^z \frac{1}{(x-3)^2}\, dx = \lim_{z\to 3^-} \frac{-1}{x-3}\bigg|_2^z = \lim_{z\to 3^-}\frac{-1}{z-3} - 1 = \infty,$$

so $\int_2^3 \frac{1}{(x-3)^2}\, dx$ does not exist, and therefore $\int_2^4 \frac{1}{(x-3)^2}\, dx$ does not exist either.

Score: /3

Problem 3: Shown are the graphs of $y = 3^{-x}$ and $x = \dfrac{(y-1)(y-15)}{24}$ on the grid including all three intersections for these two curves: $(0, 1)$, $(-1, 3)$, and $(-2, 9)$. The two regions enclosed by the curves are A and B.



Use integrals to express the following. Do not evaluate your integrals. Draw a cross-sectional strip for each solid of rotation.

a. The total area of the shaded regions A and B.

If $y = 3^{-x}$, then $-x = \log_3(y)$, so $x = -\log_3(y)$. Therefore the area is

$$\int_1^3 \frac{(y-1)(y-15)}{24} + \log_3(y)\,dy + \int_3^9 -\log_3(y) - \frac{(y-1)(y-15)}{24}\,dy$$

$$= \cdots = \left(\frac{35}{18} - \frac{2}{\ln(3)}\right) + \left(-\frac{9}{2} + \frac{6}{\ln(3)}\right) = -\frac{23}{9} + \frac{4}{\ln(3)} \approx 0.1240 + 0.9614 \approx 1.0854$$

Score: /2

b. The volume of a solid that has the shaded region B as its base, and cross-sections perpendicular to the y-axis are equilateral triangles.

$$V_B = \frac{\sqrt{3}}{4}\int_3^9 \left(-\log_3(y) - \frac{(y-1)(y-15)}{24}\right)^2 dy$$

$$= \cdots = \left(\frac{169}{80} - \frac{5}{\ln(3)} + \frac{3}{(\ln(3))^2}\right)\sqrt{3} \approx 0.0813$$

Score: /2

c. The volume of the solid obtained by rotating the region A around $x = -2$. State the method used.

Washer method yields

$$V_{Ar} = \pi\int_1^3 \left(\frac{(y-1)(y-15)}{24} + 2\right)^2 - (-\log_3(y) + 2)^2\,dy$$

$$= \cdots = \pi\left(\frac{2971}{540} - \frac{2}{\ln(3)} - \frac{4}{(\ln(3))^2}\right) \approx 1.1537$$

Score: /2

d. The volume of the solid obtained by rotating the region B around $y = 10$. State the method used.

Cylindrical shells yield

$$V_{Br} = 2\pi\int_3^9 \left(-\log_3(y) - \frac{(y-1)(y-15)}{24}\right)(10 - y)\,dy$$

$$= \cdots = 3\pi\left(\frac{28}{\ln(3)} - 23\right) \approx 23.4366$$

Score: /2

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Calculators permitted from here on.

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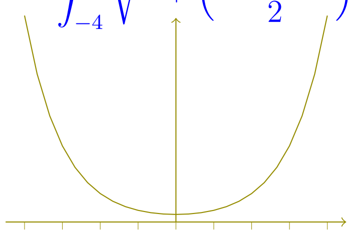
Problem 4: A hanging cable has the shape of $y = \dfrac{e^x + e^{-x}}{2}$ for x in the interval $[-4, 4]$. Draw the curve and set up the integral that expresses the length of the cable. DO NOT EVALUATE YOUR INTEGRAL.

Since $y' = \frac{1}{2}(e^x - e^{-x})$, the length is

$$\int_{-4}^{4} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx = \int_{-4}^{4} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx = \int_{-4}^{4} \frac{e^x + e^{-x}}{2}\, dx$$

$$= \left.\frac{e^x - e^{-x}}{2}\right|_{-4}^{4} = e^4 - e^{-4} \approx 54.58$$



Score: /3

Problem 5: A variable force of $f(x) = \sqrt{3 + x}$ lb is applied in moving an object from $x = 6$ to $x = 13$ feet. How much work is performed? Include units.

$$W = \int_6^{13} \sqrt{3 + x}\, dx = \tfrac{2}{3}(3 + x)^{3/2}\Big|_6^{13} = \tfrac{2}{3}(16^{3/2} - 9^{3/2}) = \tfrac{2}{3}(64 - 27) = \frac{74}{3} \text{ lb-ft} \approx 24.7 \text{ lb-ft}$$

Score: /3

Problem 6: Find the integral analytically: $\displaystyle\int \frac{x^2 + 4x - 1}{(x - 3)(x^2 + 2)}\, dx$

By the method of partial fractions, $\frac{x^2+4x-1}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$, so $x^2 + 4x - 1 = A(x^2 + 2) + (Bx + C)(x - 3)$. If $x = 3$, this yields $20 = 11A$, so $A = 20/11$. If $x = 0$ (and $A = 20/11$), the equation yields that $-1 = \frac{40}{11} - 3C$, so $C = 17/11$. Using this and, say, $x = 1$ you get that $4 = \frac{26}{11} - 2B$, so $B = -9/11$.
Thus

$$\int \frac{x^2 + 4x - 1}{(x - 3)(x^2 + 2)}\, dx = \int \frac{20}{11(x - 3)} + \frac{-9x}{11(x^2 + 2)} + \frac{17}{11(x^2 + 2)}\, dx$$

$$= \tfrac{20}{11} \ln|x - 3| - \tfrac{9}{22} \ln(x^2 + 2) + \tfrac{17}{11\sqrt{2}} \tan^{-1}(x/\sqrt{2}) + C$$

Score: /5

Problem 7: A 20 m uniform chain with a mass density of 1.5 kg/m is dangling from the roof of a building. How much work is required to pull a quarter (1/4) of the chain up onto the top of the building? Use $g = 9.81\,\text{m/s}^2$. Provide your answer accurate to 2 decimal places.

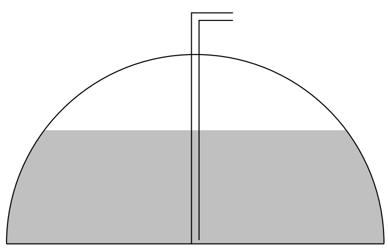
The bottom 3/4 of the chain simply need to be lifted 5.00 m. The mass of the bottom 3/4 of the chain is $1.50\,\text{kg/m} \times 15.00\,\text{m} = 22.50\,\text{kg}$, so lifting it requires
$22.50\,\text{kg}g \times 5.00\,\text{m} = 1103.63\,\text{J}$.
For the top 1/4 of the chain, the short piece of length dy at distance y from the roof has mass $1.5\,dy$ and (obviously) needs to be lifted the distance y which required $1.5gy\,dy$ amount of work. In total (for the top quarter of the chain)

$$\int_0^5 1.5gy\,dy = \frac{1.5g}{2}y^2\Big|_0^5 = 18.75g \approx 183.94\,\text{J}$$

Grand total 1287.56 J

Score: /3

Problem 8: A hemispherical dome with radius 5 m is filled with water to a depth of 3 m. A spout rises 2 m from the top at the centre. Express as a definite integral the work required to pump all the water out of the dome through the spout. Find work required up to 2-decimal-place accuracy. (The density of water is $\rho = 1000\,\text{kg/m}^3$ and the acceleration due to gravity is $g = 9.81\,\text{m/s}^2$.)



Set the origin at the centre of the bottom of the dome with positive y-axis pointing up so that the spout is at $y = 7$.
Recall that $W = Fd$, so we need to get a cross-sectional area of water to find a small volume element.

$$A = \pi(\sqrt{25 - y^2})^2, \qquad \Delta V = A\Delta y$$

Thus, force on a small volume element is $1000\,\text{kg/m}^3 \cdot 9.81\,\text{m/s}^2 \cdot \pi(\sqrt{25-y^2})^2\,dy$.
Since the distance travelled for a volume element at position y is $7 - y$ metres, work needed is

$$W = 9810\pi \int_0^3 (25 - y^2)(7 - y)\,dy \approx 1.14 \times 10^7\,\text{J} = 11.4\,\text{MJ}$$

Score: /4