Math 124
Spring 2008
Dr. Lily Yen

Final Exam

Show all your work

Name: ____________________

Number: ____________________

Signature: ____________________

Score: ___/100

- No book.
- No notes.
- Show all relevant work.
- Anything irrelevant that is wrong will be counted against you.

Problem 1: The following questions are one mark each. Provide short answers for each.

a. State the converse of $p \to q$.

$q \to p$.

b. Negate the proposition $\forall x\big(p(x) \to q(x)\big)$.

$\exists x \neg\big(p(x) \to q(x)\big)$ so $\exists x\big(p(x) \wedge \neg q(x)\big)$.

c. Evaluate $P(8,2)$.

$\boxed{56}$

d. How many edges are in the complete bipartite graph $K_{3,5}$?

$\boxed{15}$

e. In how many ways can you arrange the letters in the word TRINOMIAL?

$\frac{9!}{2!} = 181\,440$

f. How many different ways can 10 questions on a true–false test be answered?

$2^{10} = 1024$

g. In how many ways can you arrange 10 keys on a key ring?

$\frac{9!}{2} = 181\,440$

h. If 12 points lie on the circumference of a circle, how many inscribed pentagons can you draw having these points as vertices?

$\binom{12}{5} = 792$

i. In how many ways can 40 things be divided into 5 equal groups?

$\binom{40}{8,8,8,8,8} = 7.66 \cdot 10^{24}$

j. A deck of cards has thirteen cards of each suit. How many 11-card hands exist that have exactly 3 aces and 4 queens?

$\binom{4}{3}\binom{44}{4} = 135\,751$

k. Simplify the sum $\sum_{k=0}^{27} \binom{27}{k} = \boxed{2^{27} = 134\,217\,728}$

l. Convert the binary number 11001 to decimal.

$\boxed{25}$

m. State an inverse for 4 modulo 7.

$\boxed{2}$

n. How many integer triples (i, j, k) exist that satisfy the inequality $1 \le i \le j \le k \le 50$?

$$\binom{52}{3} = \boxed{22\,100}$$

o. Find the coefficient of x^2y^3 in $(2x - 3y)^5$.

$$2^2(-3)^3\binom{5}{2} = \boxed{1\,080}$$

p. How many spanning subgraphs does $G = (V, E)$ have if $|V| = n$ and $|E| = 5$.

$$\boxed{2^5 = 32}$$

Score: /16

Problem 2: Suppose $p(x, y)$ and $q(x, y)$ are propositional functions over the real numbers:

$$p(x, y) : x^2 < y + 1$$
$$q(x, y) : x^2 + y^2 = 9$$

For each of the following propositions, state whether they are true (T), false (F), or cannot be determined (?). If true, explain. If false, give a counterexample. If it cannot be determined, say why. One mark is for the right answer, and one mark for your explanation.

a. $p(-4, 10) \wedge q(0, -3)$ True or False or ?

Here $p(-4, 10) \equiv (-4)^2 < 10 + 1 \equiv 16 < 11$ is false, so $p(-4, 10) \wedge$ *anything* is false.

b. $\neg p(2, 7) \vee q(-1, -2)$ True or False or ?

Here $p(2, 7) \equiv 2^2 < 7 + 1 \equiv 4 < 8$ is true, so $\neg p(2, 7)$ is false. Moreover, $q(-1, -2) \equiv (-1)^2 + (-2)^2 = 9 \equiv 5 = 9$ is false. Thus $\neg p(2, 7) \vee q(-1, -2)$ is false.

c. $\forall x \exists y [p(x, y)]$ True or False or ?

For any x, choose $y = x^2$. Then $p(x, y) = p(x, x^2) \equiv x^2 < x^2 + 1$ which is true.

d. $\exists x \forall y [q(x, y)]$ True or False or ?

Let, say, $y = 4$. Then $q(x, y) \equiv x^2 + 16 = 9 \equiv x^2 = -7$ is false for all real numbers, x.

e. $\exists x \exists y [p(x, y) \to q(x, y)]$ True or False or ?

Choose $x = 0$ and $y = -10$. Then $p(x, y) = p(0, -10) \equiv 0^2 < -10 + 1 \equiv 0 < -9$ which is false. Then $p(x, y) \to q(x, y)$ is true, since false implies anything.

Score: /10

Problem 3: Consider the statement: "The square of an odd integer is an odd integer."

- Restate the statement in logical symbols using "if–then." Score: /2

  "If x is an odd integer, then x^2 is an odd integer." Or, using symbols, let $p(x)$ be the proposition "x is an odd integer." Then the statement is $p(x) \to p(x^2)$.

- Write the negation of the statement symbolically. Score: /2

  $\neg\big(p(x) \to p(x^2)\big) \equiv p(x) \wedge \neg p(x^2)$.

- Prove the correct statement. (Declare which one you are going to prove.)
  Score: /4

  The original statement is true: If x is an odd integer, then some integer, n, exists such that $x = 2n+1$. Then $x^2 = (2n+1)^2 = 4n^2+4n+1 = 2(2n^2+2n)+1$, so x^2 is odd.

Problem 4: Consider the following Pascal program segment, where i, and j are integer variables.

```
For i := 1 to n do
    For j :=1 to i do
         Writeln (i*j);
```

a. If n is 10, how many times is the statement Writeln executed? Score: /2

   The inner loop runs i times. Thus the Writeln is executed $1+2+3+\cdots+10 = 55$ times.

b. Find a Θ notation in terms of n for the number of times the statement Writeln is executed. Score: /2

   $1+2+3+\cdots+n = \frac{(1+n)n}{2} = \Theta(n^2)$.

Problem 5: Let $A = \{a, b, c\}$ and $B = \{0, 1\}$.

a. Find the Cartesian product of A and B, i.e. $A \times B$. Score: /2

$$\{(a, 0), (b, 0), (c, 0), (a, 1), (b, 1), (c, 1)\}$$

b. Find the cardinality of the power set of $A \cup B$, i.e. $|\mathcal{P}(A \cup B)|$. Score: /1

Here $A \cup B = \{a, b, c, 0, 1\}$ which has five elements, thus the powerset has $2^5 = 32$ elements.

Problem 6: Let the universe U consist of all finite length binary strings. For $i \geq 1$, define A_i to be the set of all binary strings with exactly i zeroes and fewer than i ones.

a. List the elements of the set A_2. Score: /1

$A_2 = \{00, 001, 010, 100\}$.

b. Give the set : $A_4 \cap A_9$. Score: /1

All elements of A_4 have exactly 4 zeroes, and all elements of A_9 have exactly 9 zeroes. No string can have both 4 and 9 zeroes, so the intersection is empty, $A_4 \cap A_9 = \emptyset$.

c. Describe the set $\overline{A_3}$. Score: /1

All finite length binary strings that do not have exactly 3 zeroes or that do have exactly 3 zeroes and at least 3 ones.

d. Describe the set: $\bigcup_{i \in \mathcal{N}} A_i$ Score: /2

All finite length binary strings with fewer ones than zeroes.

Problem 7: Let $g = \{(1,a),(2,c),(3,c)\}$ be a function from $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$.

a. Is g one-to-one? $\boxed{\text{No}}$

b. Is g onto Y? $\boxed{\text{No}}$

c. Let $T = \{1,3\}$. Find $g(T)$ $\boxed{\{a,c\}}$

d. Let $U = \{c\}$. Find $g^{-1}(U)$. $\boxed{\{2,3\}}$

Score: /4

Problem 8: Use the Binomial Theorem to expand $(a-3b)^5$ completely. Simplify all coefficients and write them as integers.

Score: /4

$$a^5 - 15a^4b + 90a^3b^2 - 270a^2b^3 + 405ab^4 - 243b^5$$

Problem 9:

a. Use the Euclidean algorithm to find the $\gcd(1365, 550)$. Score: /2

$$\gcd(1365, 550) = 5$$

b. Use your work above to find integers a and b such that $1365a + 550b = \gcd(1365, 550)$.
Score: /3

$$a = 27 + 110n, b = -67 - 273n, \quad n \in \mathbb{Z}$$

Problem 10: Perform the following addition in hexadecimal number system.

$$A2C + D5F7$$

Express your answer in both hexadecimal and decimal numbers. Score: /4

$$A2C_{16} + D5F7_{16} = E023_{16} = 57379_{10}$$

Problem 11: Use Mathematical Induction to prove the following proposition for all integers $n \geq 1$:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Score: /5

The claim is trivially true if $n = 1$.
Assume the claim holds for n. Then
$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$, so the claim holds for $n + 1$, which proves the claim.

Problem 12: Use the Pigeonhole Principle to prove that given any set of 52 distinct integers, there must be two integers in the set whose sum or difference is divisible by 100. A complete answer includes a description of the objects, the boxes, the rule for placing objects in the boxes and an explanation of your answer. Score: /5

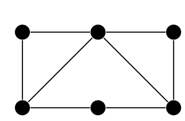
Create 51 boxes labelled 0 through 50. Box j is to contain those numbers that are equivalent to j modulo 100 or equivalent to $-j$ mod 100. Since you have 52 numbers but only 51 boxes, some box contains at least two numbers; say x and y are both in box j. If x and y are both equivalent to j modulo 100, then their difference is equivalent to 0 modulo 100 (so the difference is divisible by 100). Likewise, if x and y are both equivalent to j modulo 100, then their difference is equivalent to 0 modulo 100. Otherwise x, say, is equivalent to j and y is equivalent to $-j$. In that case, their sum is equivalent to 0 modulo 100.

Problem 13: For each of the following, draw a picture of a simple graph with the desired property or explain why no such graph exists.

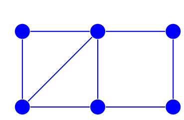
a. A connected graph with one cycle and more vertices than edges. Score: /2

A connected graph on n vertices and the fewest possible number of edges is a tree and thus has $n-1$ edges. To create a cycle from the tree, you must add at least on edge to a total of at least n edges. Thus the required graph does not exist.

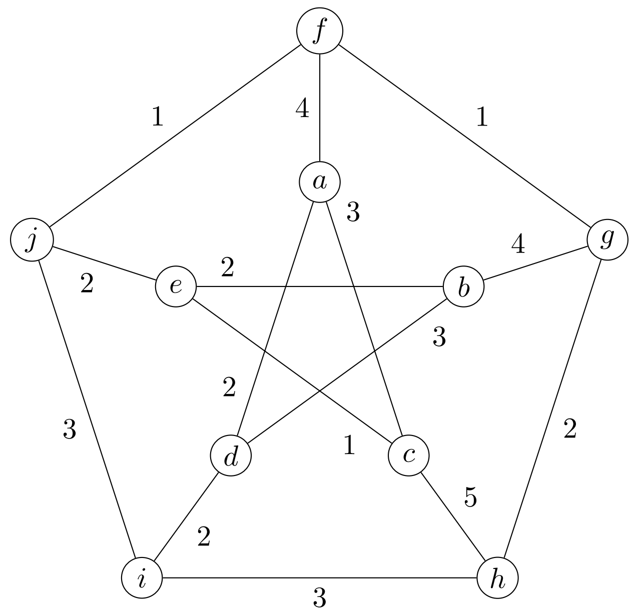
b. A graph with vertex degrees 4, 3, 3, 2, 2, and 2 that is not isomorphic to the graph shown.



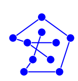
Score: /2



Problem 14: Below is a weighted Petersen Graph.



a. Find a minimal spanning tree. Score: /3

For example  The sum of the weights of the edges of the spanning tree should be 16.

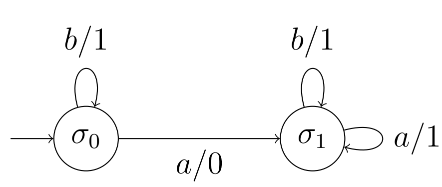
b. Does the graph have an Euler cycle? Explain. Score: /1

No: each vertex degree is odd.

c. Is the Petersen Graph bipartite? Score: /1

No: it contains a cycle of odd length.

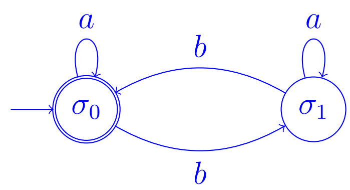
Problem 15: Describe the language accepted by the finite-state machine given below.



Score: /3

All finite strings over $\{a, b\}$ except those that contain a single a and that a in the final position.

Problem 16: Draw the transition diagram of a finite-state automaton that accepts an even number of b's over all strings over $\{a, b\}$. Score: /4



Problem 17: Give four different yet equivalent ways of characterizing a tree. Choose two of the four to prove that they are equivalent. Score: /5

Suppose G is a graph on n vertices. Then G is a tree if

a. for any two vertices there exists a *unique* simple path connecting them.

b. G is connected and acyclic.

c. G is connected and has $n - 1$ edges.

d. G is acyclic and has $n - 1$ edges.

If (a) holds, then G is clearly connected. If G had a cycle, then two paths around the cycle would connect any two vertices on the cycle. Thus G is acyclic, and (b) holds.
If (b) holds, then clearly a path connects any two vertices. If two path connect two vertices, the two paths together would form a cycle. This contradiction shows the path is unique, and (a) holds.

Problem 18: Prove that if G is a connected graph and every vertex has even degree, then G has an Euler cycle. For partial credits, define an Euler cycle. Score: /6

An Euler cycle is a cycle that uses every edge in the graph. A cycle is a path that begins and ends at the same vertex.
Choose a vertex. Since the graph is connected, you may choose an edge incident to that vertex. Follow the edge to its other endpoint. Since this vertex has even degree, you may choose an unused edge to follow from this vertex. Continue until you arrive at a vertex where you have used all the edges. This must be the vertex where you began, since all vertices have even degree.
If you have used all edges in the graph, you have constructed an Euler cycle. Otherwise, some vertex on the cycle you have constructed must be the endpoint of an unused edge. Beginning with that vertex and edge construct a cycle as above. Then tie the old and new cycles to form a longer cycle. Repeat.
This process ends as the graph is finite.