Math 108-01
Summer 2025
Dr. Lily Yen

Midterm II

Show all your work

Name: ____________
Number: ____________
Signature: ____________
Score: ___/30

One calculator (TI-83 or 84) is allowed for this test.

Problem 1: Find $g'(x)$ for $g(x) = \cos^2(5-x)\big((\tan^{-1}(\ln(x)) + e^{x^2-3x}\big)$.

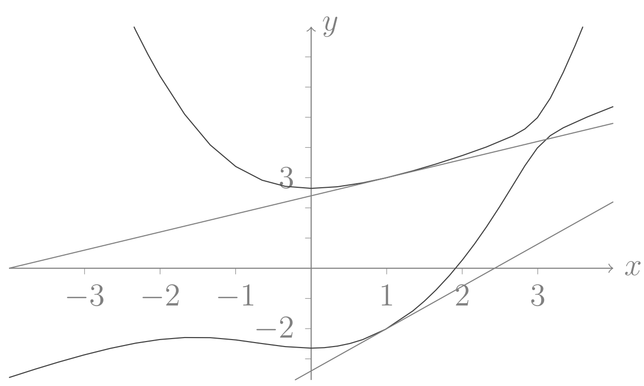
$$g'(x) = 2\cos(5-x)\sin(5-x)\big((\tan^{-1}(\ln(x)) + e^{x^2-3x}\big) + \cos^2(5-x)\left(\frac{1}{(1+(\ln(x))^2)x} + (2x-3)e^{x^2-3x}\right)$$

Score: /3

Problem 2: Determine dy/dx for $x^3 + y^2 - x^2y = 7$.

If $x^3 + y^2 - x^2y = 7$, then $3x^2 + 2yy' - 2xy - x^2y' = 0$, so
$3x^2 - 2xy = x^2y' - 2yy' = (x^2 - 2y)y'$, so

$$y' = \frac{3x^2 - 2xy}{x^2 - 2y}$$



a. For the implicitly defined function, find all points on the curve where $x = 1$. Find an equation of the tangent line for one of the points you found.

If $x = 1$ in $x^3 + y^2 - x^2y = 7$, then $y^2 - y + 1 = 7$, so $y^2 - y - 6 = 0$, so
$(y-3)(y+2) = 0$, so $y = 3$ or $y = -2$.
If $(x, y) = (1, 3)$, then $y' = \frac{3}{5}$, so the tangent line is

$$y - 3 = \frac{3}{5}(x-1) \quad \text{or} \quad y = \frac{3}{5}x + \frac{12}{5} \quad \text{or} \quad 3x - 5y + 12 = 0$$

If $(x, y) = (1, -2)$, then $y' = \frac{7}{5}$, so the tangent line is

$$y + 2 = \frac{7}{5}(x-1) \quad \text{or} \quad y = \frac{7}{5}x - \frac{17}{5} \quad \text{or} \quad 7x - 5y - 17 = 0$$

Score: /4

/7

Problem 3: The concentration (in µg/L) of a certain drug in the bloodstream x hours after being administered is approximately $C(x) = \dfrac{2x}{5+x^2}$. Use the differential to approximate the change in concentration from 2 hours after to 2.1 hours after the drug was administered. Provide 6 decimal places.

$C'(x) = \frac{2(5+x^2)-2x(2x)}{(5+x^2)^2} = \frac{10-2x^2}{(5+x^2)^2}$, so $C'(2) = \frac{2}{81}$. Therefore the change in concentration is approximately $C'(2)\cdot(2.1-2) = \frac{2}{81}\cdot 0.1 \approx 0.002\,469$ µg/L.
Actually, $C(2.1) - C(2) \approx 0.001\,889$.

Score: /3

Problem 4: Use logarithmic differentiation to find y' for y.

$$y = \frac{(x+5)^2(3^x + \sin^{-1}(x))\log(x-2)}{x^2 - \sqrt{x+1}}$$

$$\ln(y) = 2\ln(x+5) + \ln(3^x + \sin^{-1}(x)) + \ln(\log(x-2)) - \ln(x^2 - \sqrt{x+1})$$

so

$$\frac{y'}{y} = \frac{2}{x+5} + \frac{3^x\ln(3) + \frac{1}{\sqrt{1-x^2}}}{3^x + \sin^{-1}(x)} + \frac{1}{(x-2)\ln(10)\log(x-2)} - \frac{2x - \frac{1}{2\sqrt{x+1}}}{x^2 - \sqrt{x+1}}$$

so

$$y' = \frac{(x+5)^2(3^x + \sin^{-1}(x))\log(x-2)}{x^2 - \sqrt{x+1}} \cdot \left(\frac{2}{x+5} + \frac{3^x\ln(3) + \frac{1}{\sqrt{1-x^2}}}{3^x + \sin^{-1}(x)} + \frac{1}{(x-2)\ln(10)\log(x-2)} - \frac{2x - \frac{1}{2\sqrt{x+1}}}{x^2 - \sqrt{x+1}}\right)$$

Score: /3

Problem 5: The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousand automobile tires is approximated by

$$P(x) = -x^3 + 9x^2 + 120x - 400, \quad x \geq 5$$

Find the number of hundred thousands of tires that must be sold to maximize profit. Find the maximum profit.

The derivative is $P'(x) = -3x^2 + 18x + 120 = -3(x^2 - 6x - 40) = -3(x+4)(x-10)$, so $p'(x) = 0$ if $x = -4$ or $x = 10$. Since $x \geq 5$, the only critical value is $x = 10$.

x	5		10	
$P'(x)$		$+$	0	$-$
$P(x)$		$\nearrow$	700	$\searrow$

Thus the maximal profit is \$700 000 which occurs when selling 1 000 000 tires.

Score: /3

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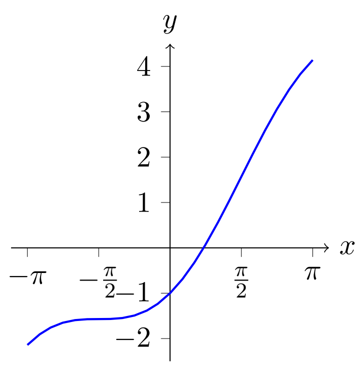
Problem 6: Graph $f(x) = x - \cos(x)$ in $[-\pi, \pi]$, and determine the exact x coordinates where f is increasing most rapidly and also where f is increasing least rapidly. Your work needs to include calculus techniques.



f is changing the least rapidly when f' has a minimum. Since $f'(x) = 1 + \sin(x) \geq 0$ (so increasing in the given domain. Taking the derivative of f' to get critical numbers of f', we get $f''(x) = \cos(x)$; thus the critical values of f' are when $f''(x) = 0$, so when $x = \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$. Looking at the sign of $f''(x) = \cos(x)$, f' has a maximum when n is even and a minimum when n is odd. Therefore, the graph of f is changing the most rapidly at $x = \frac{\pi}{2} + 2n\pi$ and least rapidly when $x = -\frac{\pi}{2} + 2n\pi$, where $n \in \mathbb{Z}$.
Within the domain, f increases most rapidly at $x = \pi/2$ and least rapidly at $x = -\pi/2$.

Alternatively, it is clear that $f'(x) = 1 + \sin(x) \geq 0$, so the minimal value of f' is 0, which happens when $\sin(x) = -1$, so when $x = -\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. However, this method does not find the maximum value of f'.

Score: /4

Problem 7: A tiny spherical balloon is inserted into a clogged artery and is inflated at a rate of 0.002π cubic millimetres per minute. How fast is the radius of the balloon increasing when the diameter is 0.010 mm?

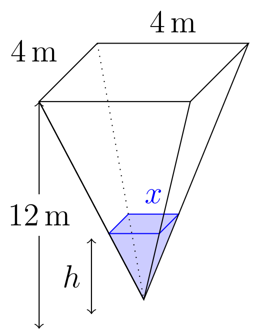
The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Therefore $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$.
When the diameter is 0.010 mm, $r = 0.005$ mm and $\frac{dV}{dt} = 0.002\pi\ \mathrm{mm}^3/\mathrm{min}$. Hence

$$\frac{dr}{dt} = \frac{0.002\pi\ \mathrm{mm}^3/\mathrm{min}}{4\pi \times 0.000\,025\ \mathrm{mm}^2} = 20\ \mathrm{mm/min}.$$

Score: /3

/7

Problem 8: A tank shaped like an upside-down square ($4\,\mathrm{m} \times 4\,\mathrm{m}$) pyramid with height $12\,\mathrm{m}$ is leaking water at the rate of $0.2\,\mathrm{m}^3/\mathrm{s}$. How fast does the height decrease when the water is $2\,\mathrm{m}$ deep? Give 2 decimal places.



Let x be the side length of the square at the top of the water, and let h be the depth of the water. By similar triangles, $\frac{x}{4} = \frac{h}{12}$, so $x = h/6$. The volume of the water is $V = \frac{1}{3}x^2h = \frac{1}{3}(h/3)^2h = h^3/27$. Therefore

$$\frac{dV}{dt} = \frac{h^2}{9}\frac{dh}{dt}.$$

When $h = 2\,\mathrm{m}$,

$$\frac{dh}{dt} = \frac{dV/dt}{h^2/9} = \frac{-0.2\,\mathrm{m}^3/\mathrm{s}}{(2\,\mathrm{m})^2/9} = -0.45\,\mathrm{m/s}.$$

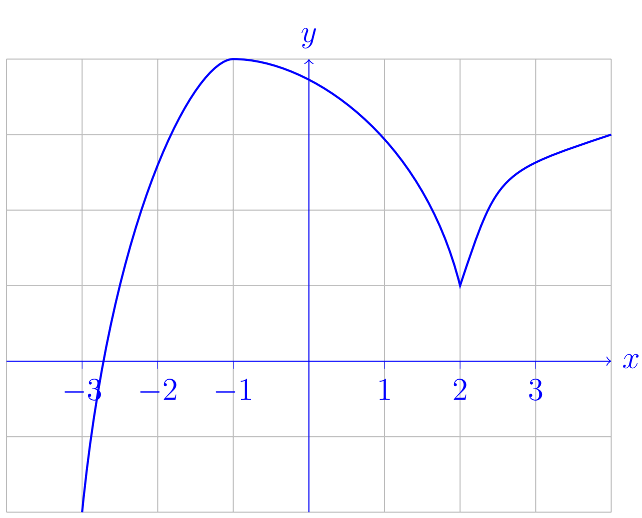
Score: /4

Problem 9: Graph a continuous function f satisfying all the following conditions over the domain $[-3, 3]$.

a. The function f is increasing for $(-3,-1) \cup (2,\infty)$.

b. The first derivative of f is negative, namely, $f'(x) < 0$ for $(-1, 2)$.

c. The function is concave down for $x \in (-3, 2) \cup (2, 3)$.



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