Math 108-01
Summer 2024
Dr. Lily Yen

Midterm One

Show all your work

Name: ____________

Number: ____________

Signature: ____________

Score: ___/45

No Calculator allowed in this part.

Problem 1: Determine the following limits analytically showing all steps. Use the symbols DNE, ∞, and $-\infty$ where appropriate.

a. $\displaystyle\lim_{x\to 6^+} \frac{x^2-5x-6}{|6-x|} =$ **7**

If $x > 6$, then $|6-x| = -(6-x) = x-6$, so

$$\lim_{x\to 6^+} \frac{x^2-5x-6}{|6-x|} = \lim_{x\to 6^+} \frac{(x-6)(x+1)}{x-6} = \lim_{x\to 6^+} x+1 = 7$$

Score: /2

b. $\displaystyle\lim_{x\to 3} \frac{x-7}{(x-3)^2} =$ $-\infty$

$\lim_{x\to 3} x-7 = -4$ and $\lim_{x\to 3}(x-3)^2 = 0$ (but positive), so

$$\lim_{x\to 3} \frac{x-7}{(x-3)^2} = -\infty$$

Score: /2

Problem 2: Answer the following using derivative rules. Do NOT simplify.

a. Find $h'(x)$ where $h(x) = \sin\left(\frac{1}{x} - \log_3(x^2)\right)$

$$h'(x) = \cos\left(\frac{1}{x} - \log_3(x^2)\right)\left(\frac{-1}{x^2} - \frac{1}{x^2\ln(3)}2x\right)$$

$$= -\cos\left(\frac{1}{x} - \log_3(x^2)\right)\left(\frac{1}{x^2} + \frac{2}{x\ln(3)}\right)$$

Score: /3

b. Find $d(g(x))/dx$ where

$$g(x) = \frac{\tan^{-1}(e^{6x})}{(\pi + 2x^3 - 5\sqrt[4]{x})}$$

$$g'(x) = \frac{\frac{1}{1+e^{12x}}e^{6x}6\left(\pi + 2x^3 - 5\sqrt[4]{x}\right) - \tan^{-1}(e^{6x})\left(6x^2 - \frac{5}{4}x^{-3/4}\right)}{\left(\pi + 2x^3 - 5\sqrt[4]{x}\right)^2}$$

Score: /3

/10

Midterm One

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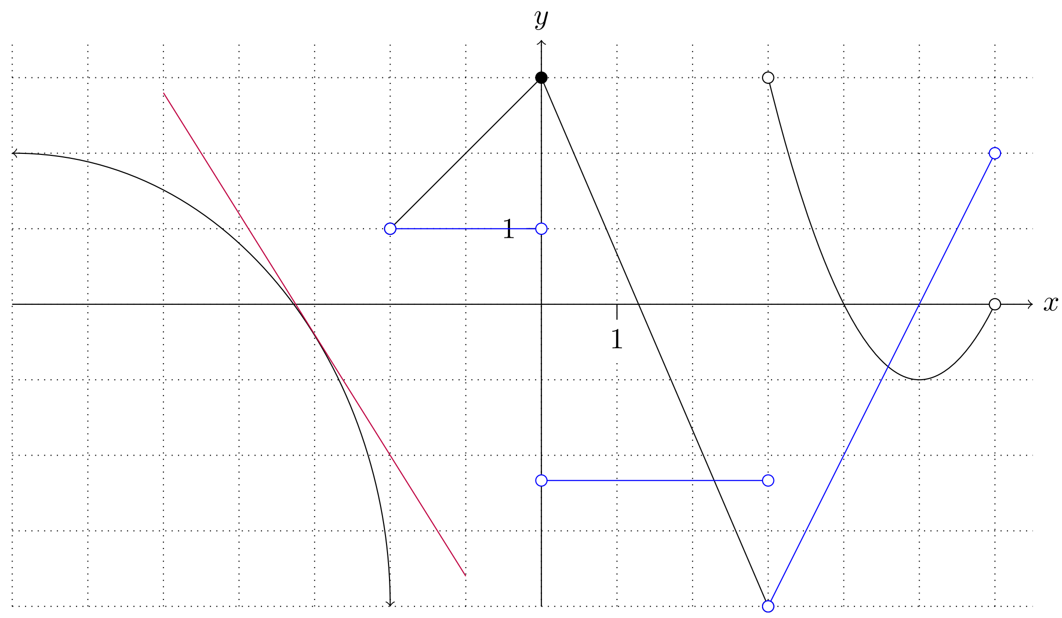
Calculators allowed from here on.

Problem 3: The graph of $y = f(x)$ is shown. Use the graph to answer the questions. Use the symbols DNE, ∞, and $-\infty$ where appropriate.



a. Express in as few intervals as possible where f is continuous in $(-\infty, 6)$.

$(-\infty, -2) \cup (-2, 3) \cup (3, 6)$

b. List the x values where f is continuous but not differentiable.

$x = 0$

c. $\lim_{x\to 2^-} f(x) =$ $-5/3$

d. $\lim_{x\to 3^+} f(x) =$ 3

e. $\lim_{x\to -\infty} f(x) =$ 2

f. $\lim_{x\to 0} \frac{f(x)-f(0)}{x} =$ DNE

g. $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} =$ $-7/3$

h. Estimate $f'(-3)$ by drawing a tangent line at the point in question and approximating its slope. -1.6

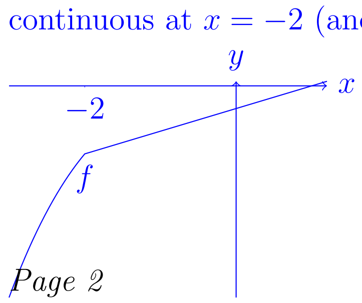
i. In the same grid above, graph $y = f'(x)$ for the interval $(-2, 6)$ where you see a piece-wise linear function and a parabola.

Score: /10

Problem 4: Use the limit definition of continuity to find a value c that makes the piece-wise defined function continuous everywhere. Draw your resulting function to check. From the graph, is the function differentiable at $x = -2$?

$$f(x) = \begin{cases} x^3 - 1, & x \le -2 \\ 3x + c, & x > -2 \end{cases}$$

Polynomials and roots are continuous. Compositions of continuous functions are continuous. Therefore each piece of f is continuous.
Note that $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x^3 - 1 = -9$, that $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} 3x + c = -6 + c$, and that $f(-2) = -9$. Therefore f is continuous at $x = -2$ (and hence everywhere) if $-6 + c = -9$, so $c = -3$.



The graph looks like it has a cusp at $(-2, 0)$, so f is likely NOT DIFFERENTIABLE and further analysis bears this out.

Score: /4

/14

Problem 5: Use correct notation, show all steps and leave your answer in simplified form.

a. Use the limit definition of the derivative to find the derivative of $f(x) = \dfrac{1}{x+3}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{x+3}{(x+h+3)(x+3)} - \frac{x+h+3}{(x+h+3)(x+3)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-h}{(x+h+3)(x+3)}}{h} = \lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)}$$
$$= \frac{-1}{(x+3)^2}$$

b. Find an equation of the tangent line to f at $x = 2$.

Since $f(2) = \frac{1}{5}$ and $f'(2) = -\frac{1}{25}$, the tangent line is

$$y - \frac{1}{5} = -\frac{1}{25}(x-2) \quad \text{or} \quad y = -\frac{1}{25}x + \frac{7}{25} \quad \text{or} \quad x + 25y - 7 = 0$$

Score: /5

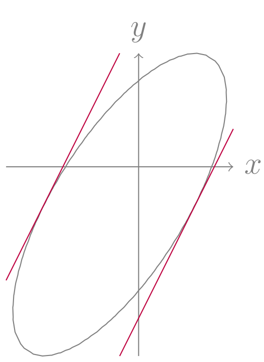
Problem 6: Given the following implicitly defined function:

$$y^2 + 2x^2 + 2y - 2xy = 30$$

a. Solve for $\dfrac{dy}{dx}$.

Here $2yy' + 4x + 2y' - 2(y + xy') = 0$, so $2yy' + 4x + 2y' - 2y - 2xy' = 0$, so $(2y + 2 - 2x)y' = 2y - 4x$, so

$$\frac{dy}{dx} = \frac{2y-4x}{2y+2-2x} = \frac{2x-y}{x-y-1}$$



b. Find all point(s) on the curve with a tangent slope of 2.

If $\frac{2x-y}{x-y-1} = 2$, then $2x - y = 2(x-y-1) = 2x - 2y - 2$, so $y = -2$ is the only solution.
If $y = -2$ and $y^2 + 2x^2 + 2y - 2xy = 30$, then $2x^2 + 4x = 30$, so $x^2 + 2x - 15 = 0$, so $(x-3)(x+5) = 0$, so $x = 3$ or $x = -5$.
Therefore the desired points of tangency are $(3, -2)$ and $(-5, -2)$.

Score: /5

/10

Problem 7: The spread of an avian flu virus is modelled by $V(t)$ where $V(t)$ is the number of people (in hundreds) with the virus, and t is the number of weeks since the first case was observed at Capilano University's main campus. Carefully interpret the following mathematical statements regarding the virus.

a. $V'(3) = 0.4$

Three weeks after the initial case, the number of infected people increases at 40 persons per week.

b. $\dfrac{\Delta V}{\Delta t} = 0.3$ for $t = 0$ and $t = 5$.

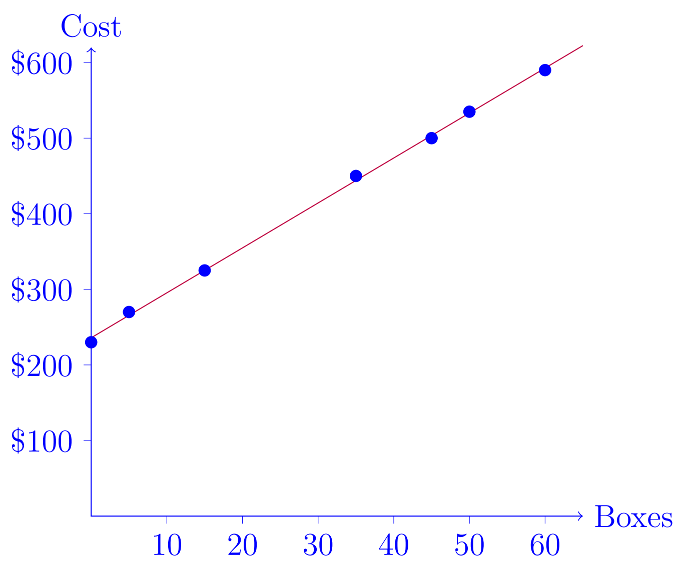
During the first five weeks after the initial case, the number of infected people grew at an average rate of 30 people per week.

Score: /2

Problem 8: Capilano University East Indian Truck food company has found the following cost/production information:

Lunch boxes produced:	0	5	15	35	45	50	60
Total cost ($) of production:	230	270	325	450	500	535	590

a. Sketch the scatterplot and find the linear model.



$y = 5.947x + 235.9$

Score: /4

b. Find the marginal cost function from the model.

$y' = 5.947$ dollars per lunch box

Score: /2

c. With your model, approximate the cost of producing the 16th lunch box.

5.95 dollars for the 16th lunch box.

Score: /1

d. Find the average cost of producing x lunch boxes.

$x \cdot \Delta y/\Delta x = 5.95x$ dollars for x boxes

Score: /2

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