Math 108
Spring 2024
Dr. Lily Yen

Midterm Two

Show all your work

Name: ____________________
Number: ____________________
Signature: ____________________
Score: ___/33

One calculator (TI-83 or 84) is allowed for this test.

Problem 1: The base of a triangle is shrinking at a rate of 2 cm/min while the height of the triangle is increasing at a rate of 0.5 cm/min. Find the rate at which the area of the triangle changes when the base is 10 cm and the height is 8 cm.

The area is $A = \frac{1}{2}bh$, so

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + b\frac{dh}{dt}\right) = \frac{1}{2}\left(-2\,\text{cm/min} \times 8\,\text{cm} + 10\,\text{cm} \times 0.5\,\text{cm/min}\right)$$
$$= -5.5\,\text{cm}^2/\text{min}$$

Score: /4

Problem 2: Estimate $\sqrt{3.99}$ using linear approximation. Your steps need to include a proposed f and a corresponding anchor point a, and its linear approximation $L(x)$. Find the error between your calculator's evaluation and your estimate.

Let $f(x) = \sqrt{x}$ and $a = 4$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, and $f(x) \approx f(a) + f'(a)(x-a)$, so

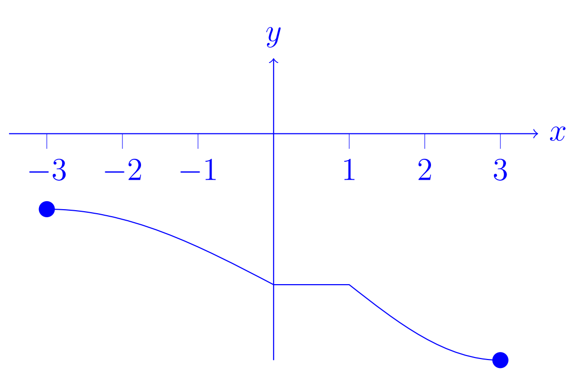
$$\sqrt{3.99} = f(3.99) \approx f(4) + f'(4)(3.99-4)$$
$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(-0.01) = 2 - \frac{1}{4}(0.01) \approx 1.9975$$

Calculator computes $\sqrt{3.99} \approx 1.997\,498$, so the difference is 1.564×10^{-6}.

Score: /4

Problem 3: Draw a graph that satisfies the given specifications for the domain $[-3, 3]$. The function does not have to be continuous or differentiable.

- $f(x) < 0$ and $f'(x) < 0$ for $x > 1$ and $-3 < x < 0$.
- $f'(x) = 0$ for $0 < x < 1$.
- $f''(x) < 0$ for $-3 < x < 0$.



Score: /4

/12

Problem 4: In the Salmon Creek of Ketchikan, Alaska, the number of salmon swimming upstream to spawn is approximated by

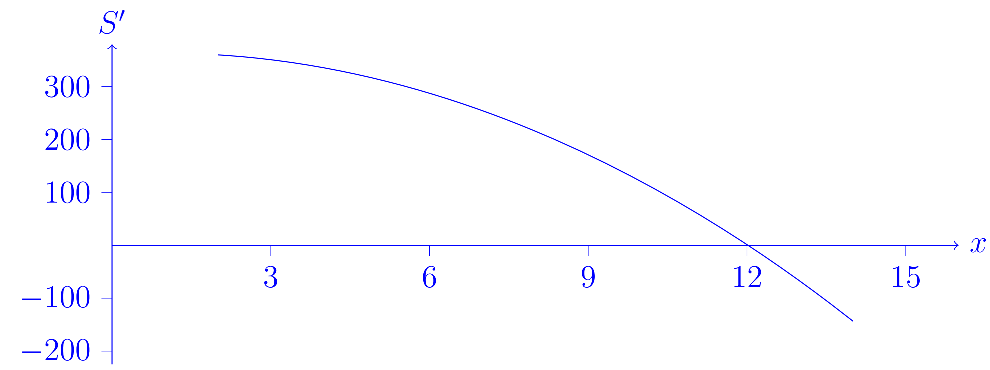
$$S(x) = -x^3 + 3x^2 + 360x + 7000, \quad 2 \le x \le 14,$$

where x represents the temperature of the water in degrees Celsius.

a. Find and graph S' in the given domain.

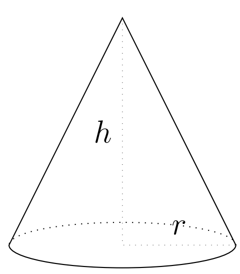
b. Find the water temperature that produces the maximum number of salmon swimming upstream.

$S'(x) = -3x^2 + 6x + 360$, so $S'(x) = 0$ when $x = \frac{-6 \pm \sqrt{4356}}{-6} \frac{-6 \pm 66}{-6} = -10, 12$.
Reject $x = -10$ because it is out of the domain $[2, 14]$. Since $S(2) = 7724$, $S(12) = 10024$, and $S(14) = 9884$, the optimal temperature is $12\,^\circ$C.



Score: /5

Problem 5: A sand storage tank used by BC's highway department for snow storms is leaking. As the sand leaks out, it forms a conical pile. The height of the pile increases at the rate of 0.735 cm/minute. The radius of the base is always twice the height of the pile. Find the rate at which the volume of the pile is increasing at the instant the height of the pile is 10 cm. Hint: The volume of a cone is $V = \pi r^2 h/3$.



The volume is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3$, so $\frac{dV}{dt} = 4\pi h^2 \cdot \frac{dh}{dt}$.
With the given data,

$$\frac{dV}{dt} = 4\pi(10\,\text{cm})^2 \cdot 0.735\,\text{cm/min} = 924\,\text{cm}^3/\text{min}$$

The volume increases at $924\,\text{cm}^3$/min when the height of the coneis at 10 cm.

Score: /5

/10

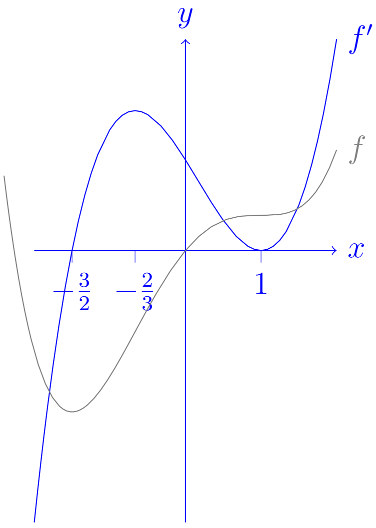
Problem 6: Let the derivative of f be defined by $f'(x) = (2x+3)(x-1)^2$.

a. Draw f'.

b. List all intervals where the original function f is INCREASING.

c. List all inflection points of f.

d. List all intervals where the graph of f is concave DOWN.



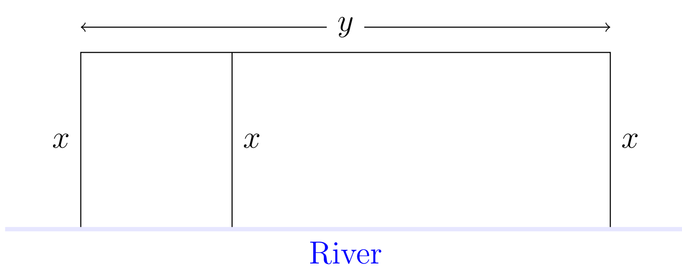
f is increasing where $f'(x) > 0$ so on $(-\frac{3}{2}, \infty)$. And, yes, f is also increasing at $x = 1$.

f has an inflection point where $f''(x) = 0$. Since $f''(x) = 2(x-1)^2 + (2x+3)2(x-1) = 2\big((x-1)+(2x+3)\big)(x-1) = 2(3x+2)(x-1)$, the inflection points are at $x = -\frac{2}{3}$ and at $x = 1$.

f is concave down where $f''(x) < 0$ so on $(-\frac{2}{3}, 1)$.

Score: /5

Problem 7: A Rocha directors have 1200 m of fencing and would like to enclose a rectangular field bordering Little Campbell River, with no fencing needed along the river. Suppose the fencing material also needs to subdivide the field into two both of which with water access, find the dimensions of the entire rectangular field that maximizes its area.



Let the the fence parallel to the river have length y and the other parts each have length x as in the figure. Then $3x + y = 1200$, so $y = 1200 - 3x$, so the area is $A = xy = x(1200 - 3x) = 1200x - 3x^2$. Since $A' = 1200 - 6x = 0$ when $x = 200$ and $A'' = -6 < 0$, the maximal area is attained when $x = 200\,\text{m}$ and $y = 1200 - 3x = 600\,\text{m}$. In that case, the area is $120\,000\,\text{m}^2$.

Score: /6

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