Math 108
Spring 2024
Dr. Lily Yen

Midterm One

Show all your work

Name: ____________
Number: ____________
Signature: ____________
Score: ___/43

No Calculator allowed in this part.

Problem 1: Determine the following limits analytically showing all steps. Use the symbols DNE, ∞, and $-\infty$ where appropriate.

a. $\lim_{h\to 0} \frac{(a+h)^2+4-(a^2+4)}{h} =$ $\boxed{2a}$

$$\lim_{h\to 0} \frac{(a+h)^2+4-(a^2+4)}{h} = \lim_{h\to 0} \frac{a^2+2ah+h^2+4-a^2-4}{h}$$
$$= \lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0} 2a+h = 2a$$

Alternatively: Let $f(x) = x^2+4$. Then the required limit is the definition of $f'(a)$. But $f'(x) = 2x$, so $f'(a) = 2a$.

Score: /2

b. $\lim_{x\to 3^-} \frac{2|x-3|}{x-3} =$ $\boxed{-2}$

If $x < 3$, then $x-3 < 0$, so $|x-3| = -(x-3)$. Therefore

$$\lim_{x\to 3^-} \frac{2|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-2(x-3)}{x-3} = \lim_{x\to 3^-} -2 = -2$$

Score: /2

Problem 2: Answer the following using derivative rules. Do NOT simplify.

a. Find $g'(x)$ where $g(x) = \cos^{-1}(3x^4 - 5\sqrt{x} + e)$

$$g'(x) = -\frac{12x^3 - \frac{5}{2\sqrt{x}}}{\sqrt{1-(3x^4-5\sqrt{x}+e)^2}}$$

Score: /3

b. Find $d(f(x))/dx$ where

$$f(x) = \frac{\tan(\ln(x^6))}{\left(\frac{1}{x}-7\right)}$$

$$f'(x) = \frac{\sec^2(\ln(x^6)) \cdot \frac{6x^5}{x^6}\left(\frac{1}{x}-7\right) - \tan(\ln(x^6))\frac{-1}{x^2}}{\left(\frac{1}{x}-7\right)^2}$$
$$= \frac{\sec^2(\ln(x^6)) \cdot \frac{6}{x}\left(\frac{1}{x}-7\right) + \tan(\ln(x^6))\frac{1}{x^2}}{\left(\frac{1}{x}-7\right)^2}$$

Score: /3

/10

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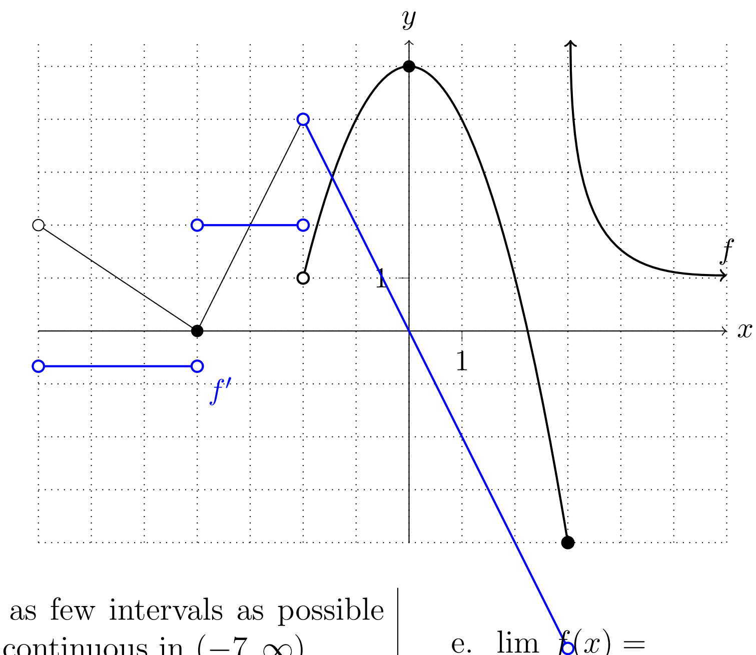
Calculators allowed from here on.

Problem 3: The graph of $y = f(x)$ is shown. Use the graph to answer the questions. Use the symbols DNE, ∞, and $-\infty$ where appropriate.



a. Express in as few intervals as possible where f is continuous in $(-7, \infty)$.

$(-7, -2) \cup (-2, 3) \cup (3, \infty)$

b. List the x values where f is continuous but not differentiable.

$x = -4$

c. $\lim_{x \to -2^+} f(x) =$ **1**

d. $\lim_{x \to 3^+} f(x) =$ **∞**

e. $\lim_{x \to \infty} f(x) =$ **1**

f. $\lim_{x \to 0} \frac{f(x) - 5}{x} =$ **0**

g. $\lim_{h \to 0} \frac{f(-\pi + h) - f(-\pi)}{h} =$ **2**

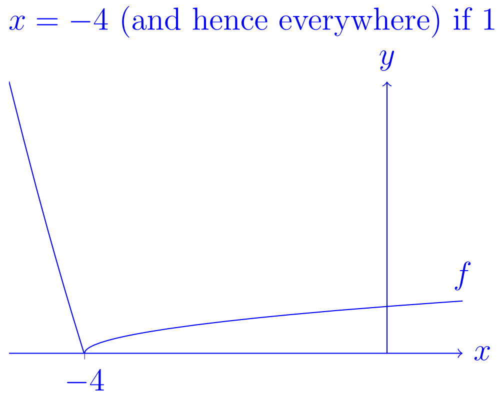
h. In the same grid above, graph $y = f'(x)$ for the interval $(-7, 3)$ where you see a parabola and a piece-wise linear function.

Score: /9

Problem 4: Use the definition of continuity to find a value c that makes the piece-wise defined function continuous everywhere. Draw your resulting function to check. From the graph, is the function differentiable at $x = -4$?

$$f(x) = \begin{cases} \sqrt{x+4}, & x \geq -4 \\ x^2 + c, & x < -4 \end{cases}$$

Polynomials and roots are continuous. Compositions of continuous functions are continuous. Therefore each piece of f is continuous.
Note that $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} x^2 + c = 16 + c$, that $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \sqrt{x+4} = 0$, and that $f(-4) = 0$. Therefore f is continuous at $x = -4$ (and hence everywhere) if $16 + c = 0$, so $c = -16$.



The graph looks like it has a cusp at $(-4, 0)$, so f is likely NOT DIFFERENTIABLE and further analysis bears this out.

Score: /5

/14

Problem 5: Use correct notation, show all steps and leave your answer in simplified form.

a. Use the limit definition of the derivative to find the derivative of $f(x) = \sqrt{x+3}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h+3}-\sqrt{x+3})(\sqrt{x+h+3}+\sqrt{x+3})}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\ &= \lim_{h\to 0} \frac{(x+h+3)-(x+3)}{h(\sqrt{x+h+3}+\sqrt{x+3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}} \end{aligned}$$

b. Find an equation of the tangent line to f at $x = 6$.

Since $f(6) = \sqrt{6+3} = 3$ and $f'(6) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$, the tangent line is

$$y - 3 = \frac{1}{6}(x-6) \quad \text{or} \quad y = \frac{1}{6}x + 2 \quad \text{or} \quad x - 6y + 12 = 0$$

Score: /5

Problem 6: The concentration of antibiotic in the bloodstream t hours after being injected is given by

$$C(t) = \frac{2t^2+t}{t^3+40},$$

where C is measured in milligrams per litre of blood. Answer the following.

a. Find the earliest time when the concentration reaches a local maximum.

Your graphing calculator finds that the maximum occurs when $t = 4.09$, so after 4 hours and 5 minutes.

b. Find the average rate of change in concentration over the first 3 hours after injection.

$$\frac{C(3)-C(0)}{3-0} = 0.104\,\text{mL/h}$$

c. Estimate the instantaneous rate of change in concentration 3 hours after injection. Use the chart below to document your estimate from average rate of change (ARC) to its limiting value. Include all calculator functions and appropriate units for the answer.

interval	ARC	interval	ARC

The limit should be about $0.0677\,\text{mL/h}$.

Score: /5

/10

Problem 7: The number of cell phones produced when x dollars is spent on capital invested and y dollars is spent on labour by a manufacturer can be modelled by the equation

$$60x^{1/4}y^{3/4} = 3240.$$

a. Use implicit differentiation to find $\dfrac{dy}{dx}$ and evaluate at the point $(16, 81)$.

b. Interpret the result from the previous part.

If $60x^{1/4}y^{3/4} = 3240$, then $x^{1/4}y^{3/4} = 54$, so $xy^3 = 54^4$, so $y^3 = 54^4/x$, so $y = \left(54^4/x\right)^{1/3} = 54^{4/3}/x^{1/3}$. Therefore

$$\frac{dy}{dx} = -\frac{1}{3} \cdot \frac{54^{4/3}}{x^{4/3}} = -\frac{18 \cdot 54^{1/3}}{x^{4/3}}$$

Alternatively, if $xy^3 = 54^4$, then $y^3 + x \cdot 3y^2y' = 0$, so $3xy^2y' = -y^3$, so

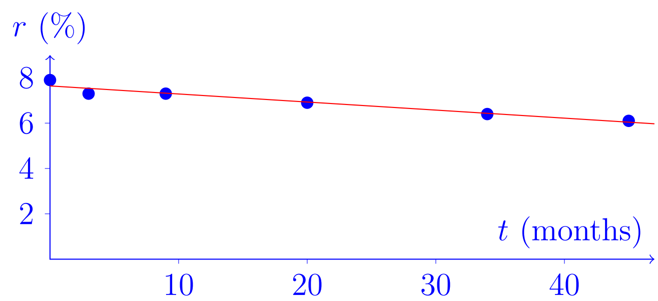
$$y' = -\frac{y^3}{3xy^2} = -\frac{y}{3x}.$$

If $(x, y) = (16, 81)$, then $\frac{dy}{dx} = -\frac{27}{16}$. For every \$27 extra you spend on labour, you can save \$16 in capital invested (while maintaining production).

Score: /5

Problem 8: Consider the following Canadian unemployment data.

	Months	Rate (%)
Sep. 2003	0	7.9
Dec. 2003	3	7.3
June 2004	9	7.3
May 2005	20	6.9
July 2006	34	6.4
June 2007	45	6.1



a. State the best linear model for the given data. Next to the given table, draw a scatterplot with axes correctly labelled including units and dimensions of the window from your graphing calculator.

$$r = 7.640 - 0.035\,52t$$

Score: /2

b. Use your model to predict the unemployment rate as of August 2006. Comment on the accuracy of this prediction.

August 2006 is 35 months after September 2003, so $t = 35$ in the formula above.

$$r = 6.4\,\%$$

Interpolating should be ok.

Score: /1

c. Use your model to predict the unemployment rate as of January 2008. Comment on the accuracy of this prediction.

January 2008 is $t = 52$, so

$$r = 5.8\,\%$$

Extrapolation is uncertain.

Score: /1

/9