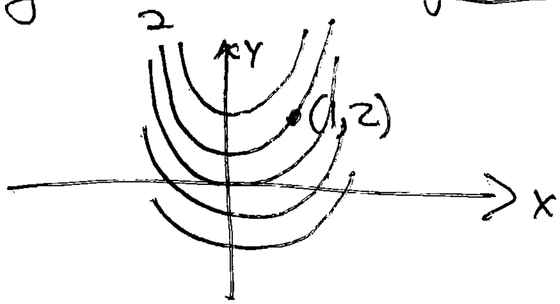
9.1 Differential Equations

A *differential equation* is any equation involving a derivative.

Example: Solve $\frac{dy}{dx} = x$ with *initial condition* $y(1) = 2$

$$\int dy = \int x\,dx$$

$$y = \frac{x^2}{2} + C \quad \text{general solution}$$



$$2 = \frac{1}{2} + C, \quad C = \frac{3}{2}$$

$$y = \frac{x^2}{2} + \frac{3}{2} \quad \text{particular solution}$$

If f, g are solutions to the DE

$$10y'' + 17y' + 3y = 0$$

is $f+g$ also a solution?

$$10(f''+g'') + 17(f'+g') + 3(f+g)$$
$$= 10f'' + 17f' + 3f \quad + \quad 10g'' + 17g' + 3g$$
$$= \quad 0 \quad + \quad 0 \quad = 0 \quad \text{YES}$$

Is cf also a solution for any real c? YES

Is $cf + dg$ " " " " c, d? YES

9.1 Solving differential equations

Omit the material after example 2 to the end of the section.

Some first order differential equations:

1. Find the general solution to $\frac{dy}{dx} = 5e^{-x}$.

$$\frac{dy}{dx} = 5e^{-x}$$
$$\int dy = \int 5e^{-x}\,dx \rightarrow y = -5e^{-x} + C$$

An initial value problem:

2. Find the particular solution to $(y^2 + e^{\pi y})\frac{dy}{dx} = 3x - \frac{5}{x}$ with initial condition $y(1) = 2$.

$$\int (y^2 + e^{\pi y})\,dy = \int \left(3x - \frac{5}{x}\right)dx$$
$$\frac{y^3}{3} + \frac{e^{\pi y}}{\pi} = \frac{3x^2}{2} - 5\ln|x| + C \quad \text{general solution}$$
$$\frac{8}{3} + \frac{e^{2\pi}}{\pi} = \frac{3}{2} + C, \quad C = \frac{8}{3} + \frac{e^{2\pi}}{\pi} - \frac{3}{2} = \frac{7}{6} + \frac{e^{2\pi}}{\pi}$$
$$\frac{y^3}{3} + \frac{e^{\pi y}}{\pi} = \frac{3x^2}{2} - 5\ln|x| + \frac{7}{6} + \frac{e^{2\pi}}{\pi} \quad \text{particular solution}$$

Some second order differential equations.

3. Verify that $y = e^{-5x} + 2x$ is a solution to the differential equation $50y + 5y' - y'' = 100x + 10$.

$$y' = -5e^{-5x} + 2$$
$$y'' = 25e^{-5x}$$

$$50y + 5y' - y''$$
$$= 50(e^{-5x} + 2x) + 5(-5e^{-5x} + 2) - 25e^{-5x}$$
$$= 50e^{-5x} - 25e^{-5x} - 25e^{-5x}$$
$$+ 100x + 10 = 100x + 10$$

4. If $y = e^{wt}$ is a solution to the differential equation $ay'' + by' + cy = 0$ then the constant w must satisfy some quadratic equation. Find the quadratic equation.

$$\left.\begin{aligned} y' &= we^{wt} \\ y'' &= w^2e^{wt} \end{aligned}\right\} \text{ so } ay'' + by' + cy$$
$$= a(w^2e^{wt}) + bwe^{wt} + ce^{wt}$$
$$= e^{wt}(aw^2 + bw + c) = 0$$
$$e^{wt} > 0 \text{ so } aw^2 + bw + c = 0$$

Thus w is a solution to $ax^2 + bx + c = 0$

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

5. Use the result of the last question to find all solutions involving the natural exponential function to the DE $10y'' + 17y' + 3y = 0$.

$$10x^2 + 17x + 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-17 \pm \sqrt{17^2 - 4(10)3}}{2(10)}$$

$$= \frac{-17 \pm 13}{20}$$

$$= \frac{-4}{20}, \frac{-30}{20}$$

$$= -\frac{1}{5}, -\frac{3}{2}$$

Solutions are $e^{-\frac{1}{5}t}$, $e^{-\frac{3}{2}t}$

Separable differential equations $\frac{dy}{dx} = f(x)g(y)$

1. Solve the equations $(1+x^2)y' = \dfrac{1}{\sqrt{2y+1}}$.

$$(1+x^2)\frac{dy}{dx} = \frac{1}{\sqrt{2y+1}}$$

$$\int \sqrt{2y+1}\, dy = \int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$

$$\frac{(2y+1)^{3/2}}{\frac{3}{2} \cdot 2} = \frac{1}{3}(2y+1)^{3/2},$$

or sub $u = 2y+1$

$$\frac{1}{3}(2y+1)^{3/2} = \tan^{-1}(x) + C$$

$$(2y+1)^{3/2} = 3\tan^{-1}(x) + 3C$$

$$2y + 1 = (3\tan^{-1}(x) + C)^{2/3}$$

$$y = \frac{1}{2}(3\tan^{-1}(x) + C)^{2/3} - \frac{1}{2}$$

2. $\frac{dy}{dx} = 2^{3y}\tan(x) + x2^{3y}$

$$\frac{dy}{dx} = 2^{3y}(\tan(x) + x)$$

$$\int 2^{-3y}\, dy = \int (\tan(x) + x)\, dx$$

$$\boxed{\frac{2^{-3y}}{-3\ln(2)} = \ln|\sec(x)| + \frac{x^2}{2} + C}$$

$$\frac{d}{dx}2^{f(x)} = 2^{f(x)}\ln(2)f'(x)$$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

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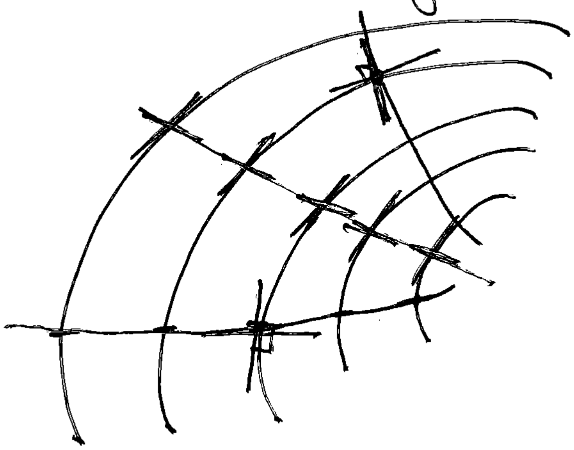
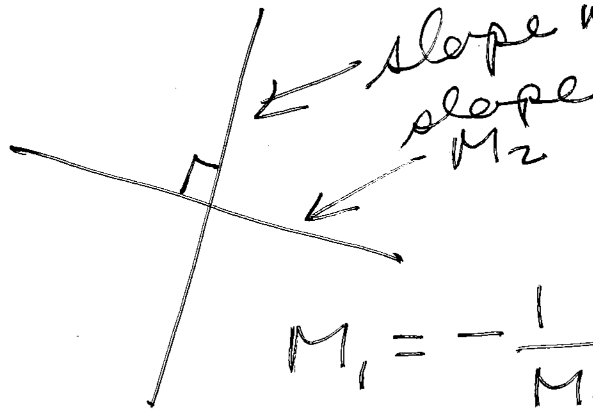
Orthogonal Trajectories





$$M_1 = -\frac{1}{M_2}$$

P.77 #2 $\quad y = ke^{-x}, \; k \neq 0 \quad \longrightarrow \quad k = \underline{ye^{x}}$

$$\frac{dy}{dx} = \underline{k}e^{-x}(-1)$$

$$\frac{dy}{dx} = ye^{x}e^{-x}(-1) = -y$$

$$\perp \quad \frac{dy}{dx} = \frac{-1}{-y} = \frac{1}{y}$$

$$\int y\,dy = \int dx$$

O.T.: $\quad \dfrac{y^2}{2} = x + C$

To graph

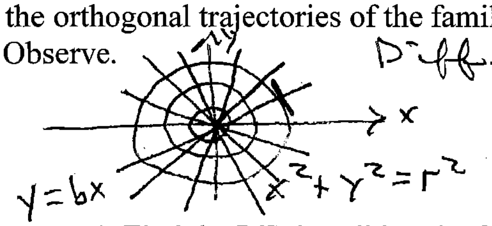
$$y^2 = 2x + C_1, \quad C_1 = 2C$$

$$y = \pm\sqrt{2x + C_1}$$

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles.

1. Find the orthogonal trajectories of the family of circles $x^2 + y^2 = r^2$.
Graph. Observe.



Diff. implicitly $2x + 2yy' = 0$

$y' = -\frac{x}{y}$

$\perp \quad \frac{dy}{dx} = -\frac{1}{-x/y} = y/x$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln|y| = \ln|x| + C$

$e \qquad e$

$|y| = e^C |x|,$

$a = e^C$

$|y| = a|x|$

$y = \pm ax$

$\boxed{y = bx \text{ for all } b \neq 0}$

$e^{\ln|x| + C}$

$= e^{\ln|x|} e^C$

$= |x| e^C$

Method:
1. Find the DE describing the family
2. Solve the DE for dy/dx.
3. Write dy/dx for the orthogonal family.
4. Solve the DE

2. Find the orthogonal trajectories of the family $y - ke^{-x}, \quad k \neq 0$.

3. Find the orthogonal trajectories of the family $y^2 = kx^2$.

$\cancel{2}yy' = \cancel{2}kx \qquad k = y^2x^{-2}$

$yy' = y^2x^{-2}x = y^2/x$

$y' = y/x$

$\perp \quad \frac{dy}{dx} = \frac{-x}{y}, \quad \int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + C, \quad y^2 + x^2 = C$

4. Find the orthogonal trajectories to the family of curves $xy = c, c \neq 0$.

$y + xy' = 0$

$y' = -y/x$

$\perp \quad \frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 - x^2 = C$

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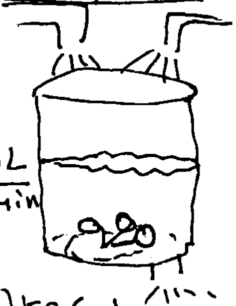
Mixture problem

A tank ~~which holds~~ with capacity 500 L ~~of orange juice mixture~~ is half full, containing water and 2 kg of pure juice. At 9 am two taps open and out of one tap pours 3 L/min of mixture containing 0.3 kg/L of pure juice, while the other tap produces 7 L/min of mixture containing 0.7 kg/L of pure juice. The mixture is well stirred, thoroughly mixed and leaves the tank at the rate of 10L/min.

a) Set up a differential equation to model this activity, letting y be the amount kg of pure juice in the tank t minutes after 9 am.



$$\frac{dy}{dt} = \text{amount in} - \text{amount out}$$

$$= \frac{3L}{min}\cdot\frac{0.3kg}{L} + \frac{7L}{min}\cdot\frac{0.7kg}{L} - \frac{y\,kg}{250L}\cdot\frac{10L}{min}$$

$$= (.9 + 4.9 - y/25)\ kg/min$$

$$= (5.8 - 0.04y)\ kg/min$$

b) Solve the differential equation.

$$\frac{dy}{dt} = 5.8 - 0.04y$$

$$\int \frac{dy}{5.8-0.04y} = \int dt = t + C$$

sub $u = 5.8 - 0.04y$

$du = -0.04\,dy$

$-25\,du = dy$

$$\int \frac{-25\,du}{u} = -25\ln|u| + C_1$$

$$-25\ln|5.8-0.04y| = t + C$$

$$e^{\ln|5.8-0.04y|} = e^{-0.04t + C}$$

$$|5.8-0.04y| = C_1 e^{-0.04t},\ C>0$$

$$5.8 - 0.04y = C_2 e^{-0.04t},\ C \neq 0$$

$$y = \frac{5.8 - C_2 e^{-0.04t}}{+0.04}$$

$$\rightarrow y = 145 - C_3 e^{-0.04t}\ kg$$

general solution

When $t=0, y=2$

$$2 = 145 - Ce^0 = 145 - C$$

$$C = 143$$

$$y = 145 - 143e^{-0.04t}\ kg$$

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P.78 c) Find the quantity of pure juice in the tank after 2 hr. Answer in kg to 2 decimal accuracy.

$$2 \text{ hr} = 120 \text{ min}$$

$$y = 145 - 143\,e^{-0.04(120)} \approx \underline{143.82 \text{ kg}}$$

d) After a very long time, how many kg of pure juice are in the tank?

$$\lim_{t \to \infty} y = 145 \text{ kg}$$