8.6 Alternating Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

$$\sum(-1)^k a_k \text{ or } \sum(-1)^{k+1} a_k,\ a_k>0$$

Alternating Series Test (AST)

$\sum(-1)^k a_k$ converges if

(i) $a_k \geq a_{k+1} > 0$ for all k greater than some N.

(ii) $\lim_{k\to\infty} a_k = 0$.

Example Alternating harmonic series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \sum_{1}^{\infty}(-1)^{k+1}\frac{1}{k},\ a_k=\frac{1}{k}$$

AST (i) $a_k=\frac{1}{k} \geq a_{k+1}=\frac{1}{k+1}>0$ for $k\geq 1$

(ii) $\lim_{k\to\infty}\frac{1}{k}=0$

so $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges.

Error in estimating the sum of an alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k,\ a_k>0$

$S=\sum(-1)^{k+1}a_k,\ S_n=a_1-a_2+a_3-a_4+\cdots+(-1)^n a_n$

Then $|S-S_n|\leq a_{n+1}$, $|S-S_n|$ is the error of the estimate

8.5

Series: AST and Ratio Test

1. Estimate the error (to three decimal places) in approximating the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$ by the sum of the first six terms. (S_6) Is the error less than 0.05? Yes What is the smallest number of terms you can use to estimate the sum to within 0.001 of the exact value?

$a_{n+1} < .001$

$\frac{1}{n+1} < .001$

$n+1 = 1000$

$n = 999$

$$|S - S_6| \le a_7 = \frac{2^7}{7!} \approx 0.025 < 0.05$$

2. Determine if the following series converge: (a) $\sum_{n=1}^{\infty} \frac{(-1)^n (n^3-1)}{(n^2+3)(n^2+8)}$, alternating

AST (i) $a_n = \frac{n^3-1}{(n^2+3)(n^2+8)} \overset{?}{\geq} a_{n+1} = \frac{((n+1)^3-1)}{((n+1)^2+3)((n+1)^2+8)}$

(ii) $\lim_{n\to\infty} a_n = 0$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n)}{5\ln(n+1)}$

(c) $\sum_{1}^{\infty} \frac{12^n}{(n+1)5^n}$ NOT Alt Series

Div test $\lim_{n\to\infty} \left(\frac{12^n}{(n+1)5^n}\right) = \lim_{n\to\infty} \left(\frac{1}{n+1}\right)\left(\frac{12}{5}\right)^n$

$\frac{1}{n+1} \to 0$, $\left(\frac{12}{5}\right)^n \to \infty$

2. Approximate $\sum_{0}^{\infty} \frac{(-1)^n}{3^n n!}$ to 4 decimal accuracy.

alt series

$$|S - S_n| \le a_{n+1} < .0001$$

$a_n = \frac{1}{3^n n!}$ on calculator

 Understand the methods so you can solve similar problems.
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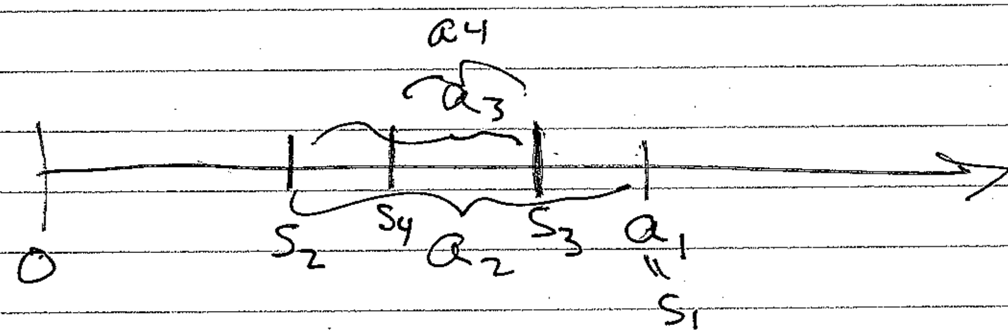
$a_1 - a_2 + a_3 - a_4 + \cdots$



$a_{n+1} < .001$

$\frac{2^{n+1}}{(n+1)!} < .001$

for $n+1 = 10$

$\frac{2^{10}}{\cancel{10!}\ 11!} \approx 5.\cancel{8}E-5 = .000051$

$n = 9 \quad \frac{2^{n+1}}{(n+1)!} \approx .00028 < .001$

S_9 will be in error by less than 0.001.

ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{((n+1)^3-1)/((n+1)^2+3)((n+1)^2+8)}{(n^3-1)/(n^2+3)(n^2+8)}$$

$$= 1 \quad \text{inconclusive}$$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{5\ln(n+1)}$ alt series

AST (i) $a_n \geq a_{n+1} > 0$ $\qquad \ln(n+1)^2 = 2\ln(n+1)$

$$\frac{\ln(n)}{5\ln(n+1)} \overset{?}{\geq} \frac{\ln(n+1)}{5\ln(n+2)}$$

$$\ln(n)\ln(n+2) \overset{?}{\geq} (\ln(n+1))^2$$

$$e \qquad \qquad e$$

$$n^{\ln(n+2)} \geq (n+1)^{\ln(n+1)}$$

$$\ln(n)\ln(n+2) \overset{?}{\geq} (\ln(n+1))^2$$

(ii) $\displaystyle\lim_{n\to\infty}\frac{\ln(n)}{5\ln(n+1)} = \frac{1}{5} \neq 0$ AST fails

Ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\frac{\ln(n+1)/5\ln(n+2)}{\ln(n)/5\ln(n+1)}$$

$$= \lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)}\,\frac{\ln(n+1)}{\ln(n+2)} = 1$$

inconclusive

Div test $\displaystyle\lim_{n\to\infty}a_n = 0$ inconclusive

$$\frac{n^3-1}{n^4+11n^2+24} \overset{?}{\geq} \frac{n^3+3n^2+3n}{\underbrace{(n^2+2n+4)(n^2+2n+9)}}$$

$$
\begin{array}{r}
n^4+2n^3+9n^2 \\
+2n^3+4n^2+18n \\
+4n^2+8n+36 \\
\hline
n^4+4n^3+17n^2+26n+36
\end{array}
$$

$$(n^3-1)(n^4+4n^3+17n^2+26n+36) \geq (n^3+3n^2+3n)(n^4+11n^2+24)$$

$$
\begin{array}{r}
n^7+4n^6+17n^5+26n^4+36n^3 \\
-n^4-4n^3-17n^2-26n-36 \\
\hline
n^7+4n^6+17n^5+25n^4+32n^3-17n^2-26n-36
\end{array}
$$

$$
\begin{array}{r}
\geq n^7+11n^5+24n^3 \\
\cancel{+3n^5} \\
+3n^6+33n^4+72n^2 \\
+3n^5+33n^3+72n \\
\hline
n^7+3n^6+14n^5+33n^4+57n^3+72n^2+72n
\end{array}
$$

$$n^6+3n^5-8n^4-25n^3-89n^2-98n-36 \overset{?}{\geq} 0$$

true for $n \geq N$ for some positive N

3. Determine if the series converges. If the alternating series converges, determine the smallest value of n necessary to estimate the sum to within 0.01.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-3}(6n^2-2n+3)}{(5+n)(2n+7)}$

Div test $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{6n^2-2n+3}{2n^2+17+35} \frac{1/n^2}{1/n^2}$

$= \lim_{n\to\infty} \frac{6 - \overbrace{2/n + 3/n^2}^{\to 0}}{2 + \underbrace{17/n + 35/n^2}_{\to 0}} = 3$

Series diverges

b) $\sum_{1}^{\infty}(-1)^{n+1}5e^{-2n}$

$5e^{-2n} = \frac{5}{e^{2n}} \to 0$

Alt series (i) $a_k \geq a_{k+1} > 0$

$\frac{5}{e^{2k}} \geq \frac{5}{e^{2(k+1)}}$) $e^{2k+2} \geq e^{2k}$ ✓

(ii) $\lim_{n\to\infty} 5e^{-2n} = 0$. Series converges.

c) $\sum n^4 e^{-n}$

Ratio test

$\lim_{n\to\infty} \left| \frac{(n+1)^4 e^{-(n+1)}}{n^4 e^{-n}} \right|$

$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^4 \frac{1}{e} \right| = 0$, converges. Not alternating.

d) $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n \ln(n)}{4^n (n+1)\ln(n+1)}$

Ratio test

$\lim_{n\to\infty} \left| \frac{5^{n+1}\ln(n+1)/(4^{n+1}(n+2)\ln(n+2))}{5^n \ln(n)/(4^n (n+1)\ln(n+1))} \right|$

$= \lim_{n\to\infty} \left| 5 \underbrace{\left(\frac{\ln(n+1)}{\ln(n)}\right)}_{\to 1} \frac{1}{4} \underbrace{\left(\frac{n+1}{n+2}\right)}_{\to 1} \underbrace{\frac{\ln(n+1)}{\ln(n+2)}}_{\to 1} \right| = \frac{5}{4} > 1$

Series diverges

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