8.4 Taylor Polynomials

Omit the Error Bound from the bottom of p. 492 to the top of p. 495. Do error estimates with the GC.

Linear approximation: $T_1(x) = f(a) + f'(a)(x-a)$ $\Rightarrow$ $T_1(a) = f(a)$

$T_1'(x) = f'(a)$, $T'(a) = f'(a)$

Quadratic approximation: $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$T_2(a) = f(a)$, $T_2'(x) = f'(a) + f''(a)(x-a)$, $T_2''(x) = f''(a)$

$T_2'(a) = f'(a)$, $T_2''(a) = f''(a)$

Taylor Polynomials

$T_3(x) = T_2(x) + \frac{f^{(3)}(a)}{3!}(x-a)^3$ satisfies $T_3^{(k)}(a) = f^{(k)}(a) for\ k = 0, 1, 2, 3.$

The $\boldsymbol{n^{th}}$ **order Taylor polynomial centered at $\boldsymbol{x = a}$** is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

1. Find the 5th-order Taylor polynomial for $f(x) = \cos(x)\ centered\ at\ x = \frac{\pi}{2}$.

k	$f^{(k)}(x)$	$f^{(k)}(\pi/2)$	$f^{(k)}(\pi/2)/k!$
0	$\cos(x)$	0	0
1	$-\sin(x)$	-1	-1
2	$-\cos(x)$	0	0
3	$\sin(x)$	1	$1/6$
4	$\cos(x)$	0	0
5	$-\sin(x)$	-1	$-1/120$

$$\boxed{T_5(x) = -1\left(x-\frac{\pi}{2}\right) + \frac{1}{6}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{120}\left(x-\frac{\pi}{2}\right)^5}$$

Use the above Taylor polynomial to estimate $cos(1.5)$. Compare with the calculator value. The absolute value of the difference is approximately the *remainder.*

$$\cos(1.5) \approx T_5(1.5) = -\left(1.5-\frac{\pi}{2}\right) + \frac{1}{6}\left(1.5-\frac{\pi}{2}\right)^3 - \frac{1}{120}\left(1.5-\frac{\pi}{2}\right)^5$$

$$\approx 0.0707372017$$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

2. Approximate $\frac{1}{2.1^3}$ with the 4th-order Taylor polynomial for $f(x) = \frac{1}{(x+2)^3}$ centered at $a = 0$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(x+2)^{-3}$	$1/8$
1	$-3(x+2)^{-4}$	$-3/16$
2	$12(x+2)^{-5}$	$12/32$
3	$-60(x+2)^{-6}$	$-60/64$
4	$360(x+2)^{-7}$	$360/128$

$$T_4(x) = \frac{1}{8} - \frac{3}{16}(x) + \frac{12}{32}\frac{x^2}{2} - \frac{60}{64}\frac{x^3}{6} + \frac{360}{128}\frac{x^4}{24}$$

$$\frac{1}{2.1^3} = f(0.1) \approx T(0.1)$$

$$\approx 0.10798$$

3. Find the nth -order Taylor polynomial $T_n(x)$ for $f(x) = e^{5x}$ centered at $x = 1$.

k	$f^{(k)}(x)$	$f^{(k)}(1)$	$f^{(k)}(1)/k!$
0	e^{5x}		
1	$e^{5x}5$		
2	$e^{5x}5^2$		
3	$e^{5x}5^3$		
$\vdots$	$\vdots$		
k	$e^{5x}5^k$	$e^5 5^k$	$e^5 5^k/k!$

$$T_n(x) = \sum_{k=0}^{n} \frac{e^5 5^k}{k!}(x-1)^k$$

5. Approximate $\sqrt[5]{33}$ using an appropriate $\underline{3^{rd}}$–order Taylor polynomial $T_3(x)$.

$$f(x) = \sqrt[5]{x} = x^{1/5}$$

$$a = 32 = 2^5$$

k	$f^{(k)}(x)$	$f^{(k)}(2^5)$
0	$x^{1/5}$	2
1	$\frac{1}{5}x^{-4/5}$	$\frac{1}{5\cdot 2^4} = 1/80$
2	$-\frac{4}{25}x^{-9/5}$	$\frac{-4}{25\cdot 2^9} = \frac{-1}{25\, 2^7} \approx -1/3200$
3	$\frac{36}{125}x^{-14/5}$	$\frac{36}{125\cdot 2^{14}} = \frac{9}{512000}$

$$T_3(x) = 2 + \frac{1}{80}(x-32) - \frac{1}{3200}\cdot\frac{1}{2}(x-32)^2 + \frac{9}{512000}\cdot\frac{1}{6}(x-32)^3$$

$$\sqrt[5]{33} \approx T_3(33)$$

$$= 2 + \frac{1}{80} - \frac{1}{6400} + \frac{9}{512000}\cdot\frac{1}{6}$$

$$\approx 2.01235$$

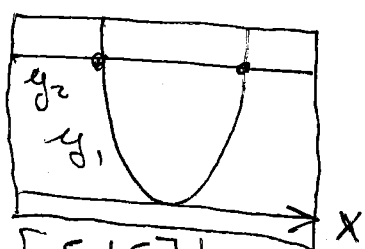
6. Find the 3^{rd} Taylor polynomial $T_3(x)$ for $f(x) = \ln(2x-1)$ at $a = 1$. Find the largest interval containing $a = 1$ on which $T_3(x)$ approximates $f(x)$ ~~within~~ with an error less than 0.01.

k	$f^{(k)}(x)$	$f^{(k)}(1)$	$f^{(k)}(1)/k!$
0	$\ln(2x-1)$	0	0
1	$\frac{2}{2x-1}$	2	2
2	$\frac{-4}{(2x-1)^2}$	-4	-2
3	$\frac{16}{(2x-1)^3}$	16	$\frac{16}{6} = 8/3$

$$T_3(x) = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3$$

$$y_1 = \mathrm{abs}(2(x-1) - 2(x-1)^2 + 8(x-1)^3/3 - \ln(2x-1))$$

$$y_2 = 0.01$$



$[.5, 1.5]$ by $[-.001, .01]$

$(0.798, 1.242)$

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8.4 Taylor Polynomials

$f^{(2)}(x) = -4(2x-1)^{-2}$

$f^{(3)}(x) = -4(-2)(2x-1)^{-3} \cdot 2 = \dfrac{16}{(2x-1)^3}$

Find the 5^{th} Taylor Polynomial $T_5(x)$ for
$f(x) = -7x^3+3x^2-2x+1$ centered at $x=1$.

k	$f^{(k)}(x)$	$f^{(k)}(1)$
0	$-7x^3+3x^2-2x+1$	-5
1	$-21x^2+6x-2$	-17
2	$-42x+6$	-36
3	-42	-42
4	0	0
5	0	0

$$T_5(x) = -5-17(x-1)-\frac{36}{2}(x-1)^2-\frac{42}{6}(x-1)^3$$

$$= -5-17(x-1)-18(x-1)^2-7(x-1)^3$$