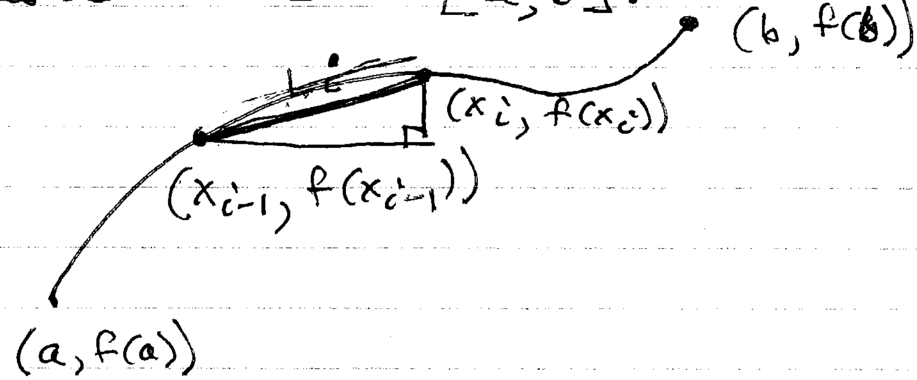
8.1 Arc Length

Assume $y=f(x)$ is continuous and has continuous derivative on $[a,b]$.



Partition $[a,b]$, $a = x_0 < x_1 < \cdots < x_n = b$

$$L_i = \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2}$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\text{slope} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

By the MVTh for some c_i in $[x_{i-1}, x_i]$, $f'(c_i) = \dfrac{f(x_i) - f(x_{i-1})}{\Delta x_i}$

so $\quad f(x_i) - f(x_{i-1}) = f'(c_i)\,\Delta x_i$

$$L_i = \sqrt{(f'(c_i)\Delta x_i)^2 + (\Delta x_i)^2}$$
$$= \sqrt{(f'(c_i))^2 + 1}\;\Delta x_i$$

As the norm of the partition tends to 0,

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\;dx$$

Chapter 8, Further applications of the integral, and Taylor Polynomials

8.1 Arc Length

If f is continuous and differentiable on [a,b] then $s = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

Give all approximations to 3 decimal accuracy.

1. Approximate the length of $x^2 - y^2 = 4,\ 3 \le x \le 5, y \ge 0$.

$$y^2 = x^2 - 4$$
$$y = \sqrt{x^2-4}$$
$$\frac{dy}{dx} = \frac{1}{2}(x^2-4)^{-1/2}\,2x = \frac{x}{\sqrt{x^2-4}}$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{x^2-4} = \frac{2x^2-4}{x^2-4}$$
$$S = \int_3^5 \sqrt{\frac{2x^2-4}{x^2-4}}\,dx \approx 3.084$$

2. Approximate the length of $x\,ln(y) = 1$ from (1,e) to $(2, \sqrt{e}\,)$. Set up two integrals for the exact length, one with respect to x and the other with respect to y.

$$\ln(y) = 1/x$$
$$y = e^{1/x}$$
$$\frac{dy}{dx} = e^{1/x}(-x^{-2})$$
$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2/x}}{x^4}$$
$$L = \int_1^2 \sqrt{1 + \frac{e^{2/x}}{x^4}}\,dx$$
$$L \approx 1.514$$

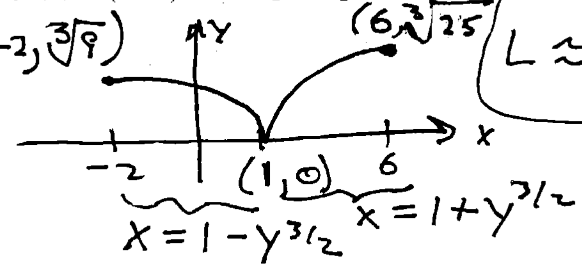
$$x = 1/\ln(y) = (\ln(y))^{-1}$$
$$\frac{dx}{dy} = -(\ln(y))^{-2}\cdot\frac{1}{y}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{y^2(\ln(y))^4}$$
$$L = \int_{\sqrt{e}}^{e} \sqrt{1 + \frac{1}{y^2(\ln(y))^4}}\,dy$$
$$L \approx 1.514$$

3. Approximate the length of $f(x) = (1-x)^{2/3}$ on [-2, 6]. Caution!

$$y = (1-x)^{2/3}$$

Solve for x:

$$\pm y^{3/2} = 1 - x$$
$$x = 1 \pm y^{3/2}$$



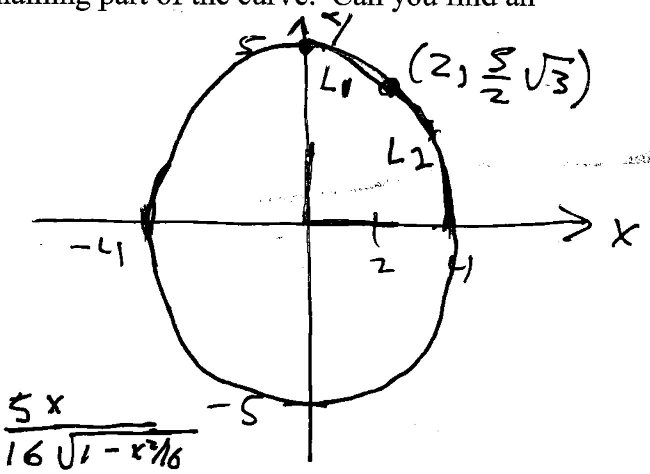
$$\frac{dx}{dy} = \pm\frac{3}{2}y^{1/2}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{9}{4}y$$
$$L = \int_0^{\sqrt[3]{9}} \sqrt{1 + 9/4\,y}\,dy + \int_0^{\sqrt[3]{25}} \sqrt{1 + 9/4\,y}\,dy$$
$$\approx 9.601$$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

4. Set up an integral expression for the exact length of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$.

Observe the tangent line is vertical when x = -4 and x = 4 so we cannot simply solve for y and integrate. Another approach: by symmetry we only need to find the length on ¼ of the curve, say from x = 0 to x = 4 with $y \geq 0$. We can find an integral with respect to x from x = 0 to x = 3, and another with respect to y for the remaining part of the curve. Can you find an even easier approach?



$$\frac{y^2}{25} = 1 - \frac{2^2}{16} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$y^2 = \frac{3 \cdot 25}{4}, \quad y = \frac{5}{2}\sqrt{3}$$

$$y^2 = 25\left(1 - x^2/16\right)$$

$$y = \pm 5\sqrt{1 - x^2/16}$$

$$\frac{dy}{dx} = \frac{\pm 5\,(-\cancel{2}x/16)}{\cancel{2}\sqrt{1-x^2/16}} = \mp \frac{5x}{16\sqrt{1-x^2/16}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{25x^2}{16^2(1 - x^2/16)} = 1 + \frac{25x^2}{256 - 16x^2}$$

$$L_1 = \int_0^2 \sqrt{1 + \frac{25x^2}{256 - 16x^2}}\, dx$$

$$x^2 = 16\left(1 - y^2/25\right)$$

$$x = \pm 4\sqrt{1 - y^2/25}$$

$$\frac{dx}{dy} = \frac{\pm 4\,(-\cancel{2}y/25)}{\cancel{2}\sqrt{1-y^2/25}} = \mp \frac{4y}{25\sqrt{1-y^2/25}}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{16y^2}{25^2\left(1 - \frac{y^2}{25}\right)} = 1 + \frac{16y^2}{625 - 25y^2}$$

$$L_2 = \int_0^{\frac{5}{2}\sqrt{3}} \sqrt{1 + \frac{16y^2}{625 - 25y^2}}\, dy, \quad L = 4(L_1 + L_2) \simeq 28.362$$

Omit surface area, from the bottom of p 469 to the top of p 471.