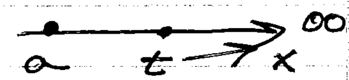
7.6 Improper Integrals

involve $-\infty$ or ∞ as a limit of integration or a discontinuity of the integrand.

Definition: If f is defined on the appropriate domain then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\left(\int_a^t f(x)\,dx\right)$$

converges if the limit exists, **diverges** otherwise.



$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty}\left(\int_t^a f(x)\,dx\right)$$

and $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Example

$$\int_1^{\infty} \frac{dx}{x^n} = \lim_{t\to\infty}\left(\int_1^t \frac{dx}{x^n}\right)$$

$$= \lim_{t\to\infty} \begin{cases} \left.\dfrac{x^{-n+1}}{-n+1}\right|_1^t & \text{if } n\neq 1 \\ \left.\ln|x|\right|_1^t & \text{if } n=1 \end{cases}$$

$$= \lim_{t\to\infty} \begin{cases} \dfrac{t^{-n+1}}{-n+1} - \dfrac{1}{-n+1} & \text{if } n\neq -1 \\ \ln|t| & \text{if } n=-1 \end{cases}$$

(2

$$= \begin{cases} 0 - \frac{1}{-n+1} & \text{if } n > 1 \\ \infty & \text{if } n < 1 \\ \infty & \text{if } n = 1 \end{cases}$$

if $n > 1$

$0 > -n+1$

so $t^0 > t^{-n+1}$

$1 > t^{-n+1}$

$n > 1$

$$-\frac{1}{-n+1} = \frac{1}{n-1}$$

if $n < 1$

$0 < -n+1$

so $t^0 < t^{-n+1}$

$1 < t^{-n+1}$

$$\text{so } \int_1^\infty \frac{dx}{x^n} = \begin{cases} \frac{1}{n-1} & \text{if } n > 1 \\ \infty, \text{ diverges} & \text{if } n \le 1 \end{cases}$$

Ex. $$\int_1^\infty \frac{dx}{x^2} = \lim_{t \to \infty} \int_1^t x^{-2}\,dx$$

$$= \lim_{t \to \infty} \left(\frac{x^{-1}}{-1} \right)_1^t$$

$$= -\lim_{t \to \infty} (\underbrace{t^{-1}}_{\to 0} - 1) = 1$$

7.6 Improper Integrals

Determine if each of the following integrals is convergent or divergent. If the integral converges, find its value if possible.

A. Infinite limits

1. $\int_1^{\infty} \frac{dx}{x}$ $\cancel{= \ln|x| \Big|_1^{\infty} = \infty \text{ diverges}}$

$$= \lim_{t \to \infty} \left(\int_1^t \frac{dx}{x} \right) = \lim_{t \to \infty} \left(\ln|x| \Big|_1^t \right) = \lim_{t \to \infty} \left(\ln|t| \right)$$

$$= \infty, \text{ diverges}$$

2. $\int_{-\infty}^{0} x\cos(x)dx$ $= I$

$\underline{IP}$ $\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$

$u = x \quad dv = \cos(x)\,dx$

$du = dx \quad v = \sin(x)$

$$I = \lim_{t \to -\infty} \left(\int_t^0 x\cos(x)\,dx \right)$$

$$= \lim_{t \to \infty} \left(x\sin(x) + \cos(x) \right)\Big|_t^0$$

$$= \lim_{t \to \infty} \left(0 + 1 - (t\sin(t) + \cos(t)) \right)$$

$$= \text{DNE, } \underline{\text{diverges}}$$

3. $\int_1^{\infty} \frac{dx}{x^2 + 2x + 5}$

P. 64 #3

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)+C$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

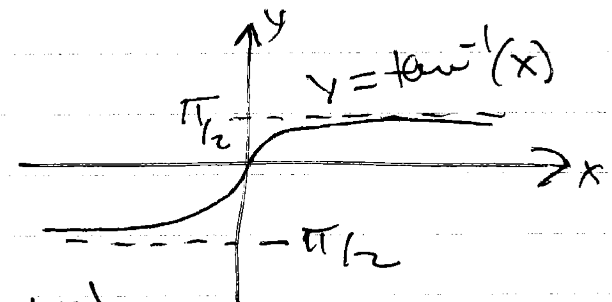
$x^2+2x+5 = (x^2+2x+1)+4$
$= (x+1)^2+4$

Sub $u = x+1$
$du = dx$

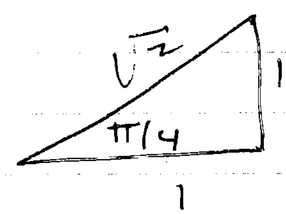
so $\displaystyle\int_1^{\infty} \frac{dx}{x^2+2x+5}$

$$= \lim_{t\to\infty}\left(\int_1^t \frac{dx}{x^2+2x+5}\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right)\Bigg|_1^t$$



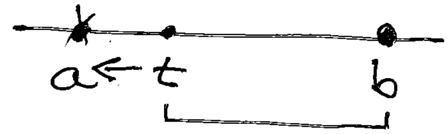
$$= \frac{1}{2}\lim_{t\to\infty}\left(\tan^{-1}\left(\frac{t+1}{2}\right) - \tan^{-1}(1)\right)$$



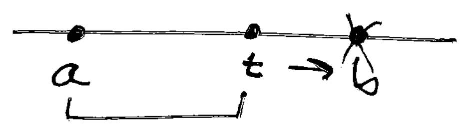
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Infinite Discontinuities

Definition: Suppose f is discontinuous at a and continuous on $(a, b]$.
Define
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \left(\int_t^b f(x)\,dx \right)$$

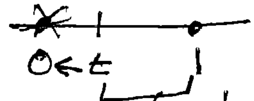


If f is discontinuous at b and continuous on $[a, b)$ then define
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \left(\int_a^t f(x)\,dx \right)$$



If f is discontinuous at c, $a < c < b$ and continuous on $[a, c)$ and on $(c, b]$ then define
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

P. 65 # 5.
$$\lim_{t \to 0^+} \left(t \ln(t) \right) \quad 0(-\infty)$$
$$= \lim_{t \to 0^+} \left(\frac{\ln(t)}{1/t} \right) \quad -\infty/\infty$$
$$\overset{L'H}{=} \lim_{t \to 0^+} \left(\frac{1/t}{-1/t^2} \right) = -\lim_{t \to 0^+} \left(\frac{1}{\cancel{t}} \cdot \frac{t^{\cancel{2}}}{1} \right) = 0$$



B. Discontinuities

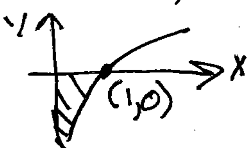
5. $\int_0^1 \ln(x)dx = \lim_{t\to 0^+}\left(\int_t^1 \ln(x)\,dx\right) = \lim_{t\to 0^+}\left(x\ln(x)-x\right)\Big|_t^1 = \lim_{t\to 0^+}\left(-1-(t\ln(t)-t)\right) = -1$

$\underbrace{t\ln(t)}_{\to 0}$, $\underbrace{t}_{\to 0}$

IP: $\int \ln(x)\,dx = x\ln(x) - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - x + C$

$u=\ln(x)$ $\quad dv=dx$

$du=\frac{1}{x}dx$ $\quad v=x$



C. More exercises. Determine if each of the following integrals converges or diverges. If possible, evaluate the convergent integrals.

6. $\int_0^\infty \cos(x)dx = \lim_{t\to\infty}\left(\int_0^t \cos(x)\,dx\right) = \lim_{t\to\infty}\left(\sin(x)\right)\Big|_0^t$

$= \lim_{t\to\infty}\left(\sin(t)-0\right)$ DIVERGES

$\sin(t) \to$ oscillates

7. $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+}\left(\int_t^1 x^{-1/2}\,dx\right) = \lim_{t\to 0^+}\left(2x^{1/2}\Big|_t^1\right)$

$= 2\lim_{t\to 0^+}\left(1-\underbrace{\sqrt{t}}_{\to 0}\right) = 2$

8. $\int_0^1 \frac{1-e^{-10x}}{1-e^{-5x}}dx = \int_0^1 \frac{\cancel{(1-e^{-5x})}(1+e^{-5x})}{\cancel{1-e^{-5x}}}\,dx$ [Hint: you can do some algebra and not need to take limits.]

$= \int_0^1 (1+e^{-5x})\,dx = \left[x + \frac{e^{-5x}}{-5}\right]_0^1$

$= 1 - \frac{e^{-5}}{5} - \left(0-\frac{1}{5}\right) = \frac{6}{5} - \frac{1}{5e^5}$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.