7.5 Partial Fractions

Rational function $= \dfrac{\text{polynomial}}{\text{polynomial}} = \dfrac{\text{numerator}}{\text{denominator}}$

The degree of a polynomial is the highest power of the variable.

eg

	degree
$3x - 8x^5 + 2$	5
$(x^2-4)(2x+3)$	3

Prime factor theorem: Any polynomial can be factored over the reals into a product of linear factors $(ax+b)$ and irreducible quadratic factors $(ax^2+bx+c,\ b^2-4ac<0)$.

A rational function is proper if degree of the numerator < degree of denominator.

P. 61 #1

PF

$$\int \frac{2x-3}{(2x-5)(x+2)}\,dx$$

Integrand is a proper rational function.
Substitution will not work.

Find a partial fraction decomposition of $\frac{2x-3}{(2x-5)(x+2)}$

$$\frac{2x-3}{(2x-5)(x+2)} = \frac{A}{2x-5} + \frac{B}{x+2}$$

Multiply by $(2x-5)(x+2)$

$$2x-3 = A(x+2) + B(2x-5)$$

$x=-2$: $\quad -7 = 0 + B(-9)$, $B = 7/9$

$x=\frac{5}{2}$: $\quad \underline{2} = A\left(\frac{5}{2}+2\right) + B(0) = \underline{\frac{9}{2}A}$, $A = 4/9$

The PF decomposition is

$$\frac{2x-3}{(2x-5)(x+2)} = \frac{4/9}{2x-5} + \frac{7/9}{x+2}$$

so

$$I = \int \frac{2x-3}{(2x-5)(x+2)}\,dx = \frac{4}{9}\int \frac{dx}{2x-5} + \frac{7}{9}\int \frac{dx}{x+2}$$

Sub $u = 2x-5$, $du = 2dx$, $\frac{1}{2}du = dx$ | $w = x+2$, $dw = dx$

$$I = \frac{4}{9}\int \frac{\frac{1}{2}du}{u} + \frac{7}{9}\int \frac{dw}{w}$$

$$= \frac{2}{9}\ln|u| + \frac{7}{9}\ln|w| + C$$

$$= \frac{2}{9}\ln|2x-5| + \frac{7}{9}\ln|x+2| + C$$

P. 61 #2

$$I = \int \frac{x^2-5}{(x-7)^2(x+4)}\,dx$$

Integrand is a proper rational function with repeated linear factors in the denominator

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PF decomposition

$$\frac{x^2-5}{(x-7)^2(x+4)} = \frac{A}{x-7} + \frac{B}{(x-7)^2} + \frac{C}{(x+4)}$$

$$x^2-5 = A(x-7)(x+4) + B(x+4) + C(x-7)^2$$

$x=7$: $\quad 44 = 0 + B\,11 + 0$

$$B = \frac{44}{11} = 4$$

$x=-4$: $\quad 11 = 0 + 0 + C(-11)^2$

$$C = \frac{1}{11}$$

$x=0$: $\quad -5 = A(-7)4 + 4(4) + \frac{1}{11}(7^2)$

$$-5 = -28A + 16 + \frac{49}{11}$$

$$28A = 21 + \frac{49}{11} \qquad A = \frac{10}{11}$$

$$I = \frac{10}{11}\int\frac{dx}{x-7} + 4\int\frac{dx}{(x-7)^2} + \frac{1}{11}\int\frac{dx}{x+4}$$

$$= \frac{10}{11}\ln|x-7| - \frac{4}{x-7} + \frac{1}{11}\ln|x+4| + C$$

7.5 Integration by Partial Fractions

Proper fractions: 1. $\int \frac{2x-3}{(2x-5)(x+2)}dx$

Repeated linear factors

2. $\int \frac{x^2-5}{(x-7)^2(x+4)}dx$

Improper fractions: Numerator degree $\geq$ denominator degree

Using long division: 3. $\int \frac{3x^3+9x^2-11x+6}{x^2+3x-4}dx = \int \left(3x + \frac{x+6}{x^2+3x-4}\right)dx = I$

$$\begin{array}{r} 3x \qquad\qquad\qquad \\ x^2+3x-4 \overline{\smash{)}\, 3x^3+9x^2-11x+6} \\ -(3x^3+9x^2-12x) \quad \\ \hline x+6 \end{array}$$

PF $\quad \frac{x+6}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

$x+6 = A(x-1) + B(x+4)$

$x=1$: $\quad 7 = 0 + B5, \quad B = 7/5$

$x=-4$: $\quad 2 = -5A + 0, \quad A = -2/5$

$I = \frac{3x^2}{2} + -\frac{2}{5}\int \frac{dx}{x+4} + \frac{7}{5}\int \frac{dx}{x-1} = \frac{3x^2}{2} - \frac{2}{5}\ln|x+4| + \frac{7}{5}\ln|x-1| + C$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

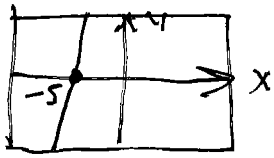
Use the arctangent for irreducible quadratic factors: $\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + C$ for $a > 0$.

4. $\int \frac{x-7}{x^3+5x^2+2x+10}\,dx$

$I =$

Proper rational function

$y_1 = x^3+5x^2+2x+10$



$[-10,10]$ by $[-10,10]$

Long division

$$
\begin{array}{r}
x^2 + 2 \\
x+5 \enclose{longdiv}{x^3+5x^2+2x+10} \\
-(x^3+5x^2) \\
\hline
2x \\
-(2x+10) \\
\hline
0
\end{array}
$$

$$I = \int \frac{x-7}{(x+5)(x^2+2)}\,dx$$

PF decomp

$$\frac{x-7}{(x+5)(x^2+2)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+2}$$

$$x-7 = A(x^2+2) + (Bx+C)(x+5)$$

$x=-5$: $\quad -12 = A(25+2) + 0, A = \frac{-12}{27} = -\frac{4}{9}$

$x=0$: $\quad -7 = -\frac{4}{9}(2) + C(5), \; -7+\frac{8}{9} = 5C$

$C = -11/9$

$x=1$: $\quad -6 = -\frac{4}{9}(3) + \left(B - \frac{11}{9}\right)(6)$

$-6 = -\frac{4}{3} + 6B - \frac{22}{3}, \; -6+\frac{4}{3}+\frac{22}{3} = 6B, \; B = 4/9$

$$I = -\frac{4}{9}\int \frac{dx}{x+5} + \int \frac{\frac{4}{9}x - \frac{11}{9}}{x^2+2}\,dx$$

$$= -\frac{4}{9}\ln|x+5| + \frac{4}{9}\int \frac{x\,dx}{x^2+2} - \frac{11}{9}\int\frac{dx}{x^2+2}$$

$u = x^2+2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= -\frac{4}{9}\ln|x+5| + \frac{4}{9}\int \frac{\frac{1}{2}du}{u} - \frac{11}{9}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$= -\frac{4}{9}\ln|x+5| + \frac{2}{9}\ln|x^2+2| - \frac{11}{9\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

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Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

Example: $I = \int \frac{23x^2+x+18}{(3x^2+4)(x-2)}\,dx$

Proper rational function

PF $\frac{23x^2+x+18}{(3x^2+4)(x-2)} = \frac{Ax+B}{3x^2+4} + \frac{C}{x-2}$

$$23x^2+x+18 = (Ax+B)(x-2) + C(3x^2+4)$$

x^2: $Ax^2 \quad +3Cx^2 = 23x^2$

x: $-2Ax + Bx \quad = x$

Const: $-2B \quad +4C \quad = 18$

$$\begin{pmatrix} 1 & 0 & 3 & 23 \\ -2 & 1 & 0 & 1 \\ 0 & -2 & 4 & 18 \end{pmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$A = 2,\ B = 5,\ C = 7$

$$I = \int \frac{2x+5}{3x^2+4}\,dx + 7\int\frac{dx}{x-2}$$

$$= 2\int \frac{x\,dx}{3x^2+4} + 5\int\frac{dx}{3x^2+4} + 7\ln|x-2|$$

Sub $u = 3x^2+4$
$du = 6x\,dx$
$\frac{1}{6}du = x\,dx$

$\frac{5}{3}\int\frac{dx}{x^2+4/3}$

$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)+C$

$$= 2\cdot\frac{1}{6}\int\frac{du}{u} + \frac{5}{3}\cdot\frac{1}{\sqrt{4/3}}\tan^{-1}\left(\frac{x}{\sqrt{4/3}}\right) + 7\ln|x-2|$$

$$= \frac{1}{3}\ln|3x^2+4| + \frac{5\sqrt{3}}{6}\tan^{-1}\left(\frac{x\sqrt{3}}{2}\right) + 7\ln|x-2| + C$$