6.3 Volumes of revolution: Disk method

If a region in the plane is rotated about a line, the resulting solid is called a *solid of revolution* and the line is called *the axis of revolution.*

Disks/Washers

Suppose the region A in the plane is bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis. If A is rotated about the x-axis a solid S is formed.

To find the volume of S, partition $[a, b]$, using $\Delta x = \dfrac{b-a}{n}$, where n is a positive integer.

Let $x_i = a + i\Delta x$ for $i = 0, 1, \ldots, n$. Then A is approximated with rectangles, each with base $x_i - x_{i-1}$ and height $f(x_i)$.

As we rotate each rectangle about the x-axis, we generate cylinders with volume

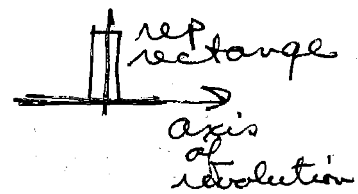
$$V = \pi R^2 h = \pi\left(f(x_i)\right)^2 \Delta x$$

Adding, we find the volume of S,

$$V(S) \approx \sum_{i=1}^{n} \pi\left(f(x_i)\right)^2 \Delta x$$

If f is continuous, the sum converges as n tends to infinity, and the sum converges to the integral, so

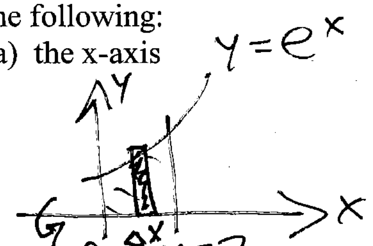
$$V(S) = \pi\int_a^b \left(f(x)\right)^2 dx$$



Examples

1. The region A is bounded by $y = e^x$, *the x- and y-axes and* $x = 2$. Sketch A.
Set up integrals (but do not evaluate) for the volume of the solid obtained by rotating A about the following:

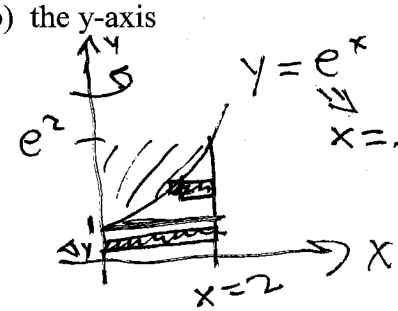
(a) the x-axis



$f(x) = e^x$

$$V = \pi\int_0^2 (e^x)^2\, dx$$

(b) the y-axis



$y = e^x \Rightarrow x = \ln(y)$

$$V = \pi\int_0^1 2^2\, dy + \pi\int_1^{e^2} 2^2\, dy - \pi\int_1^{e^2} (\ln(y))^2\, dy$$

$$= \pi\int_0^1 4\, dy + \pi\int_1^{e^2} \left(2^2 - (\ln(y))^2\right) dy$$

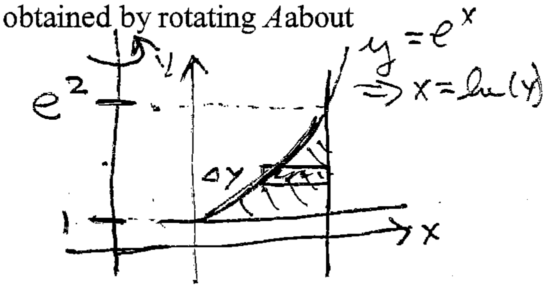
Washers: More generally, if the radius of the solid is $R(x)$ then the volume is $V(S) = \pi\int_a^b (R(x))^2\,dx$. If the region A has an outer radius $R(x)$ and an inner radius $r(x)$ then the volume is

$$V(S) = \pi\int_a^b \left[(R(x))^2 - (r(x))^2\right]dx$$

2. The region A is bounded by $y = e^x$, $y = 1$, $x = 2$. Sketch A.
Set up integrals (but do not evaluate) for the volume of the solid obtained by rotating A about the following:

(a) x = -1;

$$V = \pi\int_1^{e^2} \left(3^2 - (\ln(y)+1)^2\right) dy$$



$R(y) = (2-(-1)) = 3$
$r(y) = \ln(y) - (-1)$
$= \ln(y) + 1$

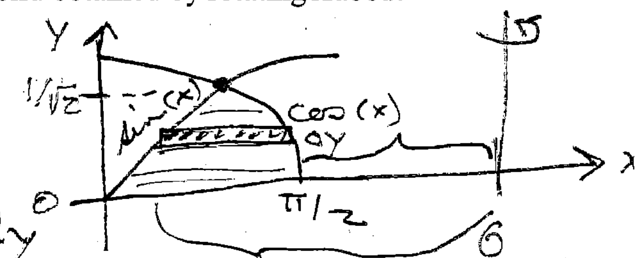
(b) y = 12.

3. The region A bounded by the x-axis, y = cos(x), y = sin(x), $0 \le x \le \pi/2$. Sketch A.
Set up integrals (but do not evaluate) for the volume of the solid obtained by rotating A about the following:

(a) x = 6;

$y = \sin(x)$
$x = \sin^{-1}(y)$

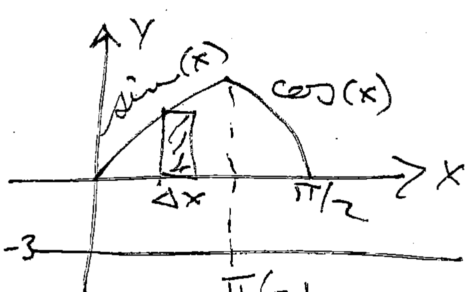
$$V = \pi\int_0^{1/\sqrt{2}} \left[(6 - \sin^{-1}(y))^2 - (6 - \cos^{-1}(y))^2\right] dy$$



$R(y) = 6 - \sin^{-1}(y)$
$r(y) = 6 - \cos^{-1}(y)$

(b) $y = -3$.

From $x = 0$ to $\pi/4$
$R(x) = \sin(x) + 3$
From $x = \pi/4$ to $\pi/2$
$R(x) = \cos(x) + 3$
$r(x) = 3$

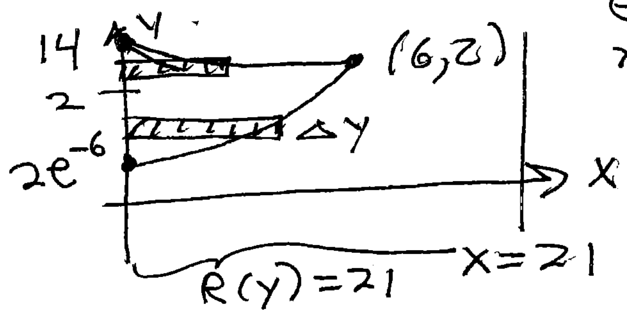


$$V = \pi\int_0^{\pi/4} \left[(\sin(x)+3)^2 - 9\right] dx + \pi\int_{\pi/4}^{\pi/2} \left[(\cos(x)+3)^2 - 9\right] dx$$

4. The region A bounded by the y-axis, $y = 2e^{x-6}$, $y = \frac{14}{x+1}$. Sketch A.

Set up integrals (but do not evaluate) for the volume of the solid obtained by rotating A about the following:

(a) $x = 21$;



For $2e^{-6} \le y \le 2$

$e^{x-6} = y/2$

$x - 6 = \ln(y/2)$

$x = 6 + \ln(y/2)$

$r(y) = 21 - (6 + \ln(y/2))$

$= 15 - \ln(y/2)$

For $2 < y \le 14$

$x + 1 = 14/y$

$x = 14/y - 1$

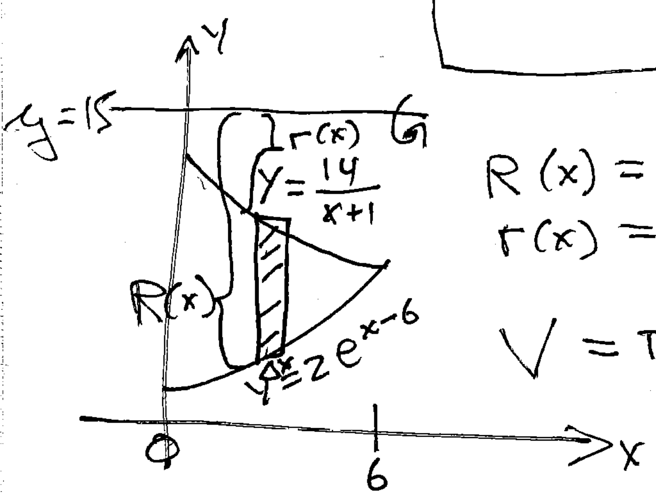
$r(y) = 21 - (\frac{14}{y} - 1)$

$= 22 - 14/y$

$$V = \pi \int_{2e^{-6}}^{2} \left[21^2 - (15 - \ln(\tfrac{y}{2}))^2\right] dy$$

$$+ \pi \int_{2}^{14} \left[21^2 - (22 - \tfrac{14}{y})^2\right] dy$$

(b) $y = 15$.



$R(x) = 15 - 2e^{x-6}$

$r(x) = 15 - \frac{14}{x+1}$

$$V = \pi \int_0^6 \left[(15 - 2e^{x-6})^2 - (15 - \tfrac{14}{x+1})^2\right] dx$$

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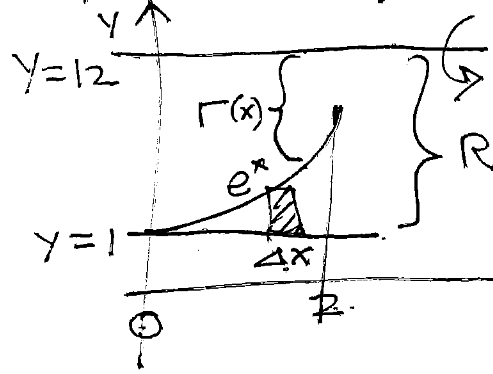


$R(x) = 12 - 1 = 11.$

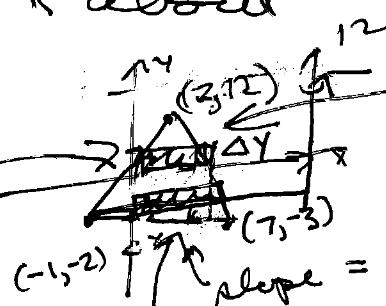
$r(x) = 12 - e^x$

$$V = \pi \int_0^2 \left[11^2 - (12 - e^x)^2\right] dx$$

The region R is a triangle with vertices $(-1,-2)$, $(2,12)$, $(7,-3)$.

Find the volume of the solid obtained by rotating R about

(a) $x = 12$



$m = \frac{14}{3}$

$y + 2 = \frac{14}{3}(x+1)$

$y = \frac{14}{3}x + \frac{8}{3}$

$3y = 14x + 8$

$x = \frac{3y - 8}{14}$

$m = \frac{15}{-5} = -3$

$y - 12 = -3(x-2)$

$= -3x + 6$

$y = -3x + 18$

$3x = 18 - y$

$x = 6 - y/3$

slope $= \frac{-2-(-3)}{-1-7} = -\frac{1}{8}$

$y + 2 = -\frac{1}{8}(x+1)$

$y = -\frac{1}{8}x - \frac{17}{8}$

$8y = -x - 17$

$x = -8y - 17$

From $y = -3$ to -2

$R(y) = 12 - (-8y - 17)$

$= 29 + 8y$

$r(y) = 12 - (6 - y/3)$

$= 6 + y/3$

From $y = -2, 12$

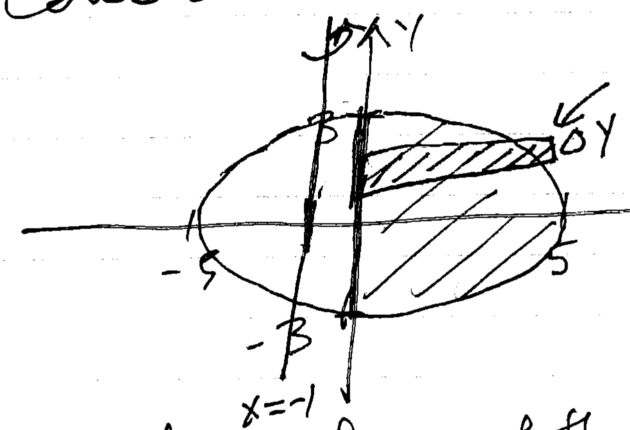
$R(y) = 12 - \left(\frac{3y-8}{14}\right)$

(2

$$V = \pi \int_{-3}^{-2} \left[(29+8y)^2 - \left(6+\frac{y}{3}\right)^2 \right] dy$$

$$+ \pi \int_{-2}^{12} \left[\left(12 - \left(\frac{3y-8}{14}\right)\right)^2 - \left(6+\frac{y}{3}\right)^2 \right] dy$$

Consider the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$



Take the $\frac{1}{2}$ ellipse for which $x \geq 0$

Find the volume of the solid obtained by rotating this region about $x=-1$

$$x^2 = 25\left(1 - \frac{y^2}{9}\right)$$

$$x = \pm 5\sqrt{1 - y^2/9}$$

$$R(y) = 5\sqrt{1 - y^2/9} + 1$$

$$r(y) = 1$$

$$V = \pi \int_{-3}^{3} \left[\left(5\sqrt{1-y^2/9} + 1\right)^2 - 1 \right] dy$$