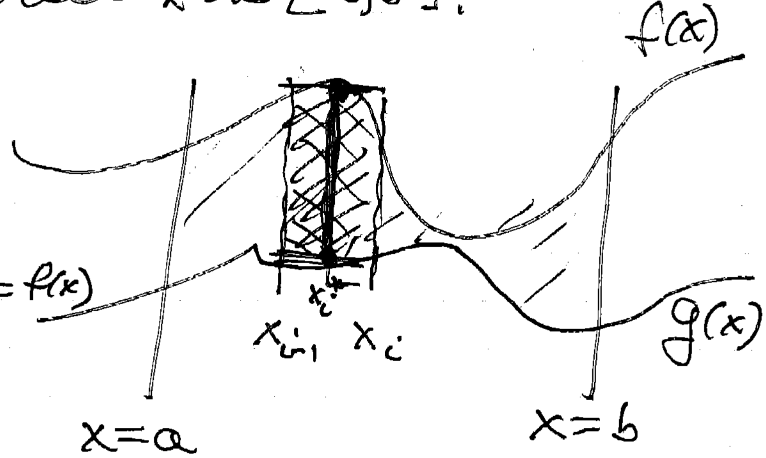
6.1 Area

Suppose $f(x) \geq g(x)$ for all x in $[a,b]$.

then $f(x) - g(x) \geq 0$.

To find the area of the region bounded by $x=a$, $x=b$, $y=g(x)$, $y=f(x)$



Partition $[a,b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Take x_i^* in $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, $\Delta x_i = x_i - x_{i-1}$ for $i=1,\ldots,n$

$$A \approx \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x_i$$

If f and g are continuous, $f(x) \geq g(x)$ on $[a,b]$

$$A = \lim_{\|P\| \to 0} \left(\sum_{i=1}^{n} (f-g)(x_i^*) \Delta x_i \right) = \int_a^b (f(x) - g(x))\,dx$$

upper - lower

6.1Area of a Region between Curves

Area is always greater than or equal to 0.

Definition of area: Assume $f\,and\,g$ are continuous functions with

$$f(x) \geq g(x)\ for\ all\ x\ in\ [a,b].$$

Then

$$f(x) - g(x) \geq 0\ for\ all\ x\ in\ [a,b].$$

Partition the interval [a, b] into n subintervals of width $\Delta x = \frac{b-a}{n}$ with endpoints

$$x_i = a + i\Delta x\ for\ i = 0,1,2,\dots,n.$$

The area of a rectangle is

$$\left(f(x_i) - g(x_i)\right)\Delta x.$$

The area of the region between the curves from a to b is approximately

$$Area \approx \sum_{i=1}^{n}\left(f(x_i) - g(x_i)\right)\Delta x.$$

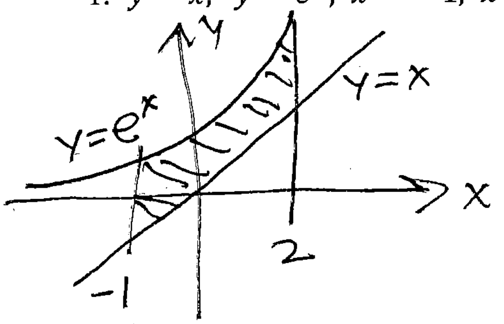
As $n \to \infty$, $f\ and\ g$ continuous implies the sum tends to an integral, and so to the area, so

$$Area = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(f(x_i) - g(x_i)\right)\Delta x\right) = \int_a^b \left(f(x) - g(x)\right)dx$$

Upper minus Lower

First sketch the region and then find the area of the region bounded by the following:

1. $y = x,\ \ y = e^x,\ \ x = -1,\ \ x = 2.$

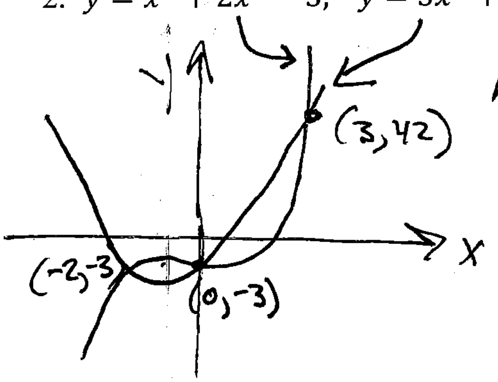


$$A = \int_{-1}^{2}(e^x - x)\,dx$$

$$= \left[e^x - \frac{x^2}{2}\right]_{-1}^{2} = e^2 - 2 - \left(e^{-1} - \frac{1}{2}\right)$$

$$= e^2 - e^{-1} - \frac{3}{2}$$

2. $y = x^3 + 2x^2 - 3,\ \ y = 3x^2 + 6x - 3.$



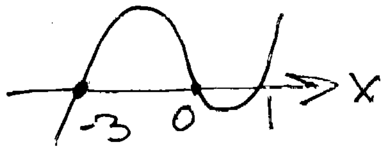
$$A = \int_{-2}^{0}\left[(x^3+2x^2-3) - (3x^2+6x-3)\right]dx$$

$$+ \int_{0}^{3}\left[3x^2+6x-3 - (x^3+2x^2-3)\right]dx$$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

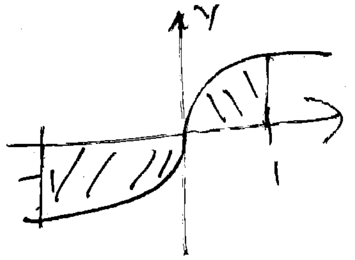
Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

3. The x-axis and $y = x^3 + 2x^2 - 3x$.



$$A = \int_{-3}^{0}(x^3+2x^2-3x)\,dx - \int_{0}^{1}(x^3+2x^2-3x)\,dx$$

4. Symmetry. $x = -1$, $x = 1$, $y = x^{1/3}$ and the x-axis.

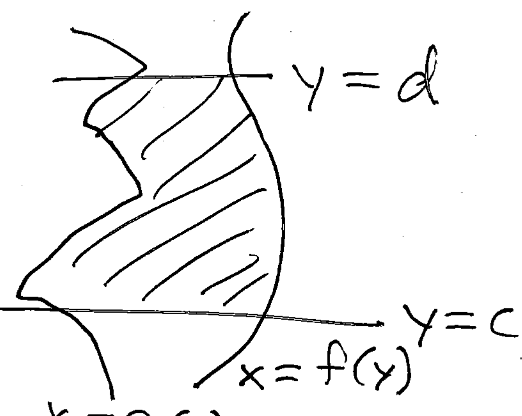


$A = 2\int_0^1 x^{1/3}\,dx$ by symmetry

Horizontal rectangles are used when x is a function of y. If $f(y) \geq g(y)$ *for all* y *in* $[c, d]$ then

$$Area = \int_c^d (f(y) - g(y))\,dy$$

Right minus Left

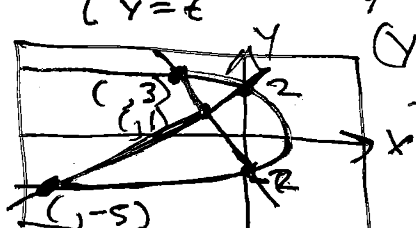


5. $3y - x = 6$, $x + y = -2$, $x + y^2 = 4$.

$x = 3y-6 \begin{cases} x = 3t-6 \\ y = t \end{cases}$

$x = -y-2 \begin{cases} x = -t-2 \\ y = t \end{cases}$

$x = 4-y^2 \begin{cases} x = 4-t^2 \\ y = t \end{cases}$



$[-25,10]$ by $[-10,10]$

Intersection points

$3y-6 = 4-y^2$

$y^2+3y-10 = 0$

$(y+5)(y-2) = 0$

$y = -5, 2$

$-y-2 = 4-y^2$

$y^2-y-6 = 0$

$(y-3)(y+2) = 0$

$y = 3, -2$

$3y-6 = -y-2$

$4y = 4$

$y = 1$

$$A = \int_{-5}^{3}(4-y^2)\,dy - \int_{-5}^{1}(3y-6)\,dy - \int_{1}^{3}(-y-2)\,dy$$

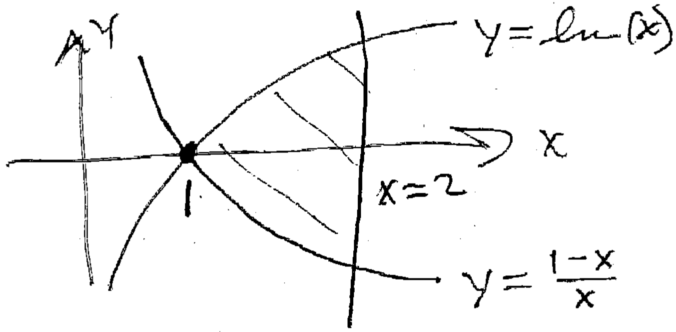
 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

6. Give integral expressions for the area of the region bounded by

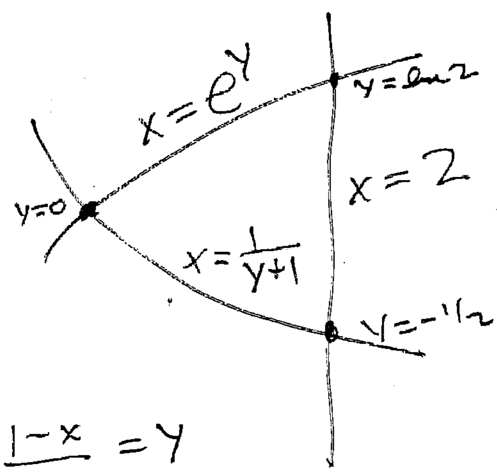
$$y = \ln(x), \qquad y = \frac{1-x}{x}, and\ x = 2$$

(a) using x as the variable of integration;



$$A = \int_1^2 \left(\ln(x) - \frac{1-x}{x}\right) dx$$

(b) using y as the variable of integration.



$$A = \int_{-1/2}^{0} \left(2 - \frac{1}{y+1}\right) dy + \int_0^{\ln 2} (2 - e^y)\, dy$$

$$\frac{1-x}{x} = y$$
$$1 - x = xy$$
$$1 = xy + x$$
$$1 = x(y+1)$$
$$x = \frac{1}{y+1}$$

Find integral expressions for the area of the region bounded by the curves below:

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

7. $x^{1/2} + y^{1/2} = \sqrt{2},\ x^2 + y^2 = 4.$

$0 \le x \le 2$

$0 \le y \le 2$

$y = (\sqrt{2} - \sqrt{x})^2$

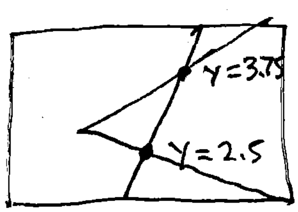
$y = \sqrt{4 - x^2}$

$$A = \int_0^2 \left(\sqrt{4-x^2} - (\sqrt{2}-\sqrt{x})^2\right) dx$$

8. $x = |y-3| + 2,\ y = 5x - 10.$, $x = \frac{1}{5}(y+10)$

$\begin{cases} x_1 = abs(t-3) + 2 \\ y_1 = t \end{cases}$

$\begin{cases} x_2 = .2(t+10) \\ y_2 = t \end{cases}$



$[1, 4]$ by $[0, 6, 6]$

$y - 1 = \frac{1}{5}(y+10)$

$5y - 5 = y + 10$

$4y = 15$

$y = \frac{15}{4} = 3.75$

$-(y-3) + 2 = \frac{1}{5}(y+10)$

$-y + 5 = \frac{1}{5}(y+10)$

$-5y + 25 = y + 10$

$15 = 6y$

$y = \frac{15}{6} = \frac{5}{2} = 2.5$

$$A = \int_{2.5}^{3.75} \left(\frac{1}{5}(y+10) - (|y-3| + 2)\right) dy$$

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.