5.6 Substitution

Chain rule $\frac{d}{dx}\left(f(u(x))\right) = f'(u(x))\,u'(x)$

$$f(u(x)) = \int d\left(f(u(x))\right) = \int f'(u(x))\,u'(x)\,dx$$

$x \longleftrightarrow u$

Example

Soln: $$\int (5x^6+2x-1)^{21}\underbrace{(30x^5+2)\,dx} = \int u^{21}\,du$$

Let $u = 5x^6+2x-1$

$\frac{du}{dx} = 30x^5+2$

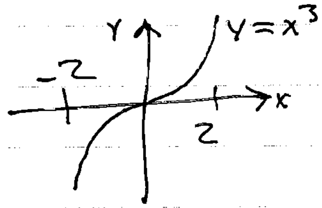
$du = (30x^5+2)\,dx$

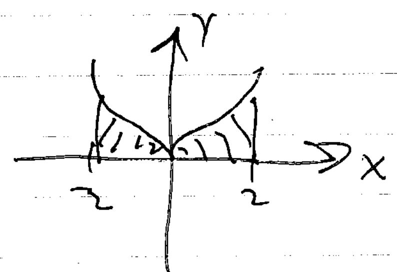
$$= \frac{u^{22}}{22} + C$$

$$= \frac{1}{22}(5x^6+2x-1)^{22} + C$$

Absolute value & symmetry

$$\int_{-2}^{2} |x^3|\,dx = 2\int_0^2 x^3\,dx = 2\frac{x^4}{4}\Big|_0^2 = \frac{1}{2}2^4 = 8$$





$$\frac{1+\sqrt{t}}{t} = (1+\sqrt{t})\,t^{-1} = t^{-1} + t^{-1/2}$$

$$\frac{t}{1+\sqrt{t}} \neq \frac{t}{1} + \frac{t}{\sqrt{t}}$$

$$= t(1+\sqrt{t})^{-1} \neq t\left(1^{-1} + (\sqrt{t})^{-1}\right)$$

$\int \sqrt{36-x^2}\,dx$ $\qquad 1-\sin^2\theta=\cos^2\theta$

$x = 6\sin\theta$

$dx = 6\cos\theta\,d\theta$

$\sqrt{36-x^2} = \sqrt{36-36\sin^2\theta}$

$= 6\sqrt{1-\sin^2\theta}$

$= 6\sqrt{\cos^2\theta}$

$= 6\cos\theta$

$\int\sqrt{36-x^2}\,dx = \int 6\cos(\theta)\,6\cos(\theta)\,d\theta$

$= 36\int\cos^2\theta\,d\theta$ $\qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

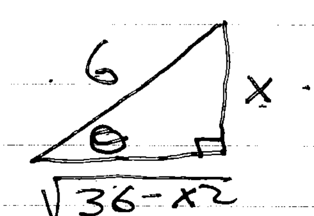
$= 36\int\frac{1+\cos(2\theta)}{2}\,d\theta$

$= 18\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$= 18(\theta + \sin(\theta)\cos(\theta)) + C$

$= 18\left(\sin^{-1}\left(\frac{x}{6}\right) + \frac{x}{6}\,\frac{\sqrt{36-x^2}}{6}\right) + C$ $\qquad \sin(\theta) = \frac{x}{6}$



$\cos(\theta) = \frac{\sqrt{36-x^2}}{6}$

5.6 Integration by Substitution

Indefinite Integrals: $\int f(u(x))u'(x)dx = \int f(u)du$

1. $\int (x^4+5)^{16} x^3 dx = \int u^{16} \cdot \frac{1}{4} du = \frac{1}{4} \frac{u^{17}}{17} + C$

Let $u = x^4+5$

$du = 4x^3 dx$

$\frac{1}{4} du = x^3 dx$

$= \frac{1}{68} (x^4+5)^{17} + C$

2. $\int sec^2(1-8x)\,dx = \int \sec^2 u \left(-\frac{1}{8} du\right) = -\frac{1}{8} \tan(u) + C$

$u = 1-8x$

$du = -8dx$

$-\frac{1}{8} du = dx$

$= -\frac{1}{8} \tan(1-8x) + C$

3. $\int \cos(ax+b)\,dx = \begin{cases} \int \cos(u) \cdot \frac{1}{a} du \\ x \cos(b) \text{ if } a=0 \; +C \end{cases} = \begin{cases} \frac{\sin(u)}{a} + C \\ \end{cases} = \begin{cases} \frac{\sin(ax+b)}{a} + C & \text{if } a \neq 0 \\ x\cos(b) + C & \text{if } a = 0 \end{cases}$

$u = ax+b$

$du = a\,dx$

$\frac{1}{a} du = dx$

4. $\int \frac{\cos(t)}{\sin^5(t)} dt = \int \frac{du}{u^5} = \int u^{-5} du = \frac{u^{-4}}{-4} + C$

Let $u = \sin(t)$

$du = \cos(t)dt$

$= \frac{-1}{4\sin^4(t)} + C$

5. $\int \frac{\sqrt{\theta}}{1-\sqrt{\theta}} d\theta = I$

Let $u = 1-\sqrt{\theta}$, $\sqrt{\theta} = 1-u$

$du = -\frac{1}{2\sqrt{\theta}} d\theta$ | $\theta = (1-u)^2$

$-2\,\theta du = \sqrt{\theta}\, d\theta$

$-2(1-u)^2 du = \sqrt{\theta} d\theta$

$\frac{-2(1-u)^2 du}{\sqrt{\theta}} = d\theta$

$I = \int \frac{-2(1-u)^2 du}{u}$

$= -2\int (1-2u+u^2) u^{-1} du$

$= -2\int (u^{-1} - 2 + u) du$

$= -2\left[\ln|u| - 2u + \frac{u^2}{2}\right] + C$

$= -2\ln|1-\sqrt{\theta}| + 4(1-\sqrt{\theta}) - (1-\sqrt{\theta})^2 + C$

CAPILANO UNIVERSITY

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

6. $\int cot^3(\theta)csc^2(\theta)d\theta = I$

$\frac{d}{d\theta}\cot(\theta) = -\csc^2(\theta)$

Let $u = \cot(\theta)$

$du = -\csc^2(\theta)\,d\theta$

$-du = \csc^2(\theta)\,d\theta$

$I = \int u^3(-du) = -\frac{u^4}{4} + C = -\frac{1}{4}\cot^4(\theta) + C$

7. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{du}{u} = -\ln|u| + C$

Let $u = \cos(x)$

$du = -\sin(x)\,dx$

$= -1\ln|\cos(x)| + C$

$= \ln|(\cos(x))^{-1}| + C$

$= \ln|\sec(x)| + C$

$r\ln(x) = \ln(x^r)$

Definite Integrals: $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$

$\boxed{\int \tan(x)dx = \ln|\sec(x)| + C}$

1. $\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int_1^3 e^u du = 2e^u\Big|_1^3 = 2(e^3 - e)$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

x	u
1	1
9	3

2. $\int_1^4 \frac{ln(x^2)}{x}dx = 2\int_1^4 \frac{\ln(x)}{x}dx = 2\int_0^{\ln(4)} u\,du = u^2\Big|_0^{\ln(4)}$

$= (\ln(4))^2$

$u = \ln(x)$

$du = \frac{1}{x}dx$

x	u
1	0
4	ln(4)

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3. $\int_1^6 \frac{2x+1}{\sqrt{x+3}}dx = \int_4^9 \frac{2(u-3)+1}{\sqrt{u}}\,du = I$

Let $u = x+3$
$du = dx$
$x = u-3$

x	u
1	4
6	9

$$I = \int_4^9 (2u-5)u^{-1/2}\,du$$

$$= \int_4^9 (2u^{1/2} - 5u^{-1/2})\,du$$

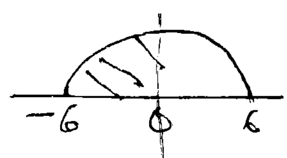
$$= \left[2\cdot\frac{2}{3}u^{3/2} - 5\cdot 2u^{1/2}\right]_4^9 = \frac{4}{\cancel{3}}\cdot\overset{9}{\cancel{27}} - 10(3) - \left(\frac{4}{3}\cdot 8 - 10\cdot 2\right) = \frac{46}{3}$$

4. $\int_{-6}^{0}(3x+2)\sqrt{36-x^2}\,dx$

$$I = 3\int_{-6}^{0} x\sqrt{36-x^2}\,dx + 2\int_{-6}^{0}\sqrt{36-x^2}\,dx$$

Let $u = 36-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

x	u
-6	0
0	36



$A = \frac{1}{4}\pi r^2$
$= \frac{1}{4}\pi 6^2 = 9\pi$

$$I = 3\left(-\frac{1}{2}\right)\int_0^{36}\sqrt{u}\,du + 2\cdot 9\pi$$

$$= -\frac{3}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^{36}$$

$$= -(216) + 18\pi$$

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