5.2 The Definite Integral

Definition: Given the interval $[a,b]$ define

a <u>partition</u>

$$P: a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

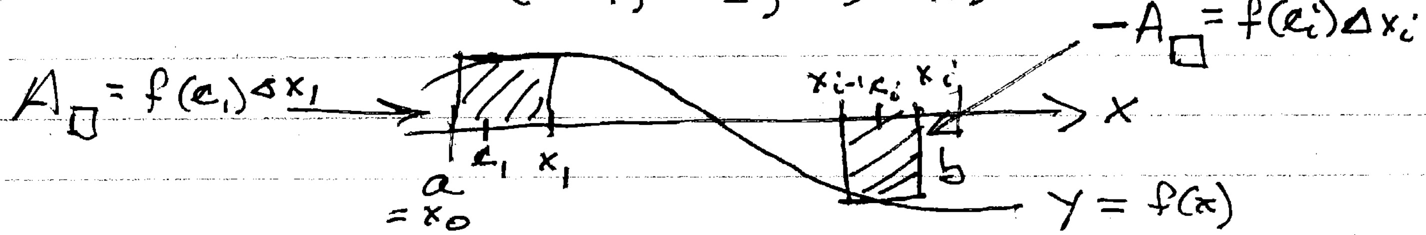
a <u>length</u>

$$\Delta x_i = x_i - x_{i-1} \text{ for } i = 1, 2, \ldots, n$$

<u>sample points</u>

$$c_i \text{ in } [x_{i-1}, x_i] \text{ for } i = 1, 2, \ldots, n$$

$$C = \{c_1, c_2, \ldots, c_n\}$$



For a function f, the <u>Riemann sum</u>

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

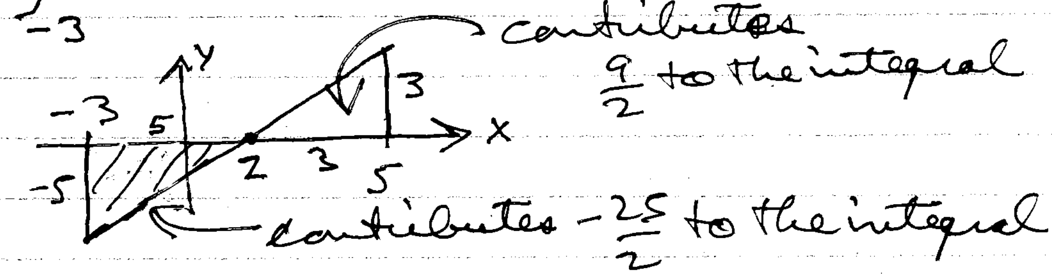
<u>norm</u> of P is $\|P\| = \max\{\Delta x_i \mid i = 1, \ldots, n\}$

define $$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \left(\sum_{i=1}^{n} f(c_i)\Delta x_i\right)$$

if the limit exists.

<u>Example</u>
By geometry find

$$\int_{-3}^{5} (x-2)\,dx = -\frac{25}{2} + \frac{9}{2} = -\frac{16}{2} = -8$$



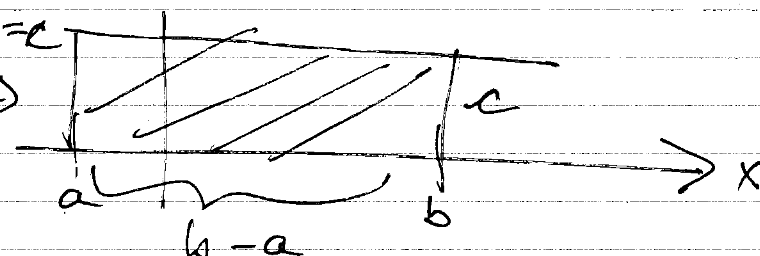
Example $\text{Area} = \int_{-3}^{5} |x-2|\,dx = \frac{25}{2} + \frac{9}{2} = \frac{34}{2} = 17$

since $|x-2| \geq 0$

Example $\left| \int_{-3}^{5} (x-2)\,dx \right| = |-8| = 8$

Properties of the definite integral

$$\int_a^b c\,dx = c(b-a)$$

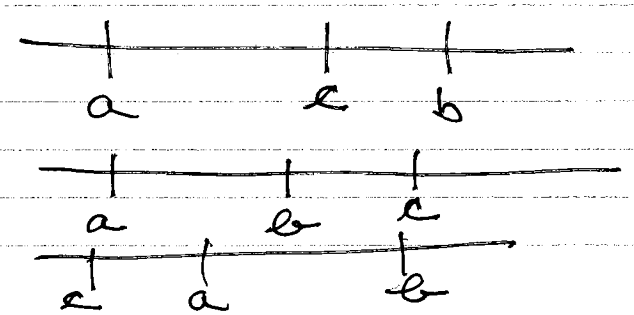


$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



5.2 **Definite Integral and Area**

1. Express the limit as a definite integral:

$\Delta x = \frac{b-a}{n} = \frac{7}{n}$, $b-a=7$ $\lim_{n\to\infty}\left[\sum_{i=1}^{n} sin\left(2+\frac{7i}{n}\right)\frac{7}{n}\right] = \int_2^9 \sin(x)\,dx$

$x_i = 2 + \frac{7i}{n} = \underline{2 + i\Delta x}$

$= \underline{a + i\Delta x}$

$a = 2$

$b - a = 7$

$b = 9$

$f(x) = \sin(x)$

2. Express each of the integrals as the limit of a Riemann sum:

$\int_3^{17} \sqrt[3]{x^5+8}\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \sqrt[3]{\left(3+\frac{14i}{n}\right)^5+8}\ \frac{14}{n}\right)$

$\Delta x = \frac{b-a}{n} = \frac{17-3}{n} = \frac{14}{n}$

$x_i = a + i\Delta x = 3 + \frac{14i}{n}$

$f(x_i) = \sqrt[3]{\left(3+\frac{14i}{n}\right)^5+8}$

$\int_{-4}^{4} \sqrt{16-x^2}\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \sqrt{16-\left(\frac{8i}{n}-4\right)^2}\ \frac{8}{n}\right) = 8\pi$

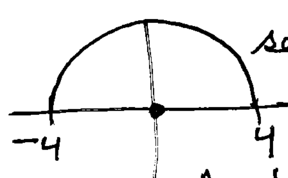
$\Delta x = 8/n$

$x_i = -4 + 8i/n$ or $8i/n - 4$

$\underline{y = \sqrt{16-x^2}}$

$y^2 = 16 - x^2$

$x^2 + y^2 = 16$

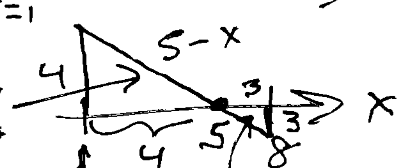


$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 4^2 = 8\pi$

$\int_1^8 (5-x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(5-\left(1+7i/n\right)\right)7/n\right) = 8 - \frac{9}{2} = \frac{7}{2}$

$\Delta x = 7/n$

$x_i = 1 + 7i/n$ or $\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(4 - 7i/n\right)7/n\right)$ or 3.5



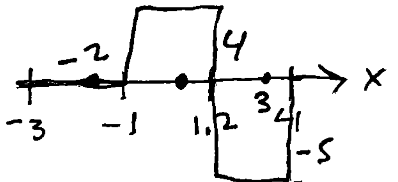
$\frac{1}{2}\cdot 4\cdot 4 = 8$

$\frac{1}{2}\cdot 3\cdot 3 = \frac{9}{2}$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

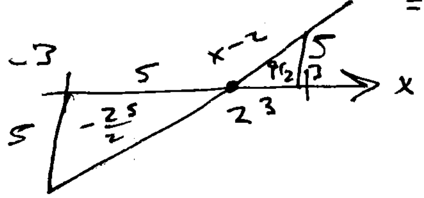
3. Given $f(x) = 4 - x^2$, $P = \{-3, -1, 1.2, 4\}$ and $C = \{-2, 0, 3\}$ calculate $R(f, P, C)$.



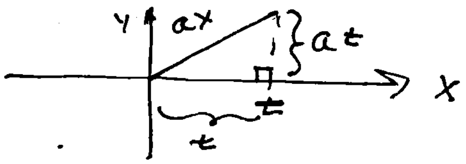
$$R(f,P,C) = 0(2) + (1.2-(-1))4 + (4-1.2)(-5)$$
$$= 2.2(4) - 2.8(5)$$
$$= 8.8 - 14 = -5.2$$

4. Use areas to calculate $\int_{-3}^{5}(x-2)dx$ and $\int_{-3}^{5}|x-2|\,dx$.

$$= \frac{-25}{2} + \frac{9}{2} = \frac{-16}{2} = -8 \qquad = \frac{25}{2} + \frac{9}{2} = \frac{34}{2} = 17$$



5. Use geometry to find $\int_0^t ax\,dx$. $= \frac{1}{2} t\, a t = \frac{1}{2} a t^2$



6. Suppose $\int_{-8}^{3} f(x)dx = 7$, $\int_0^3 f(x)dx = 2$, *and* $\int_0^3 g(x)dx = 41$. Find the following:

(a) $\int_{-8}^{0} f(x)dx = \int_{-8}^{3} f(x)dx - \int_{0}^{3} f(x)dx = 7 - 2 = 5$

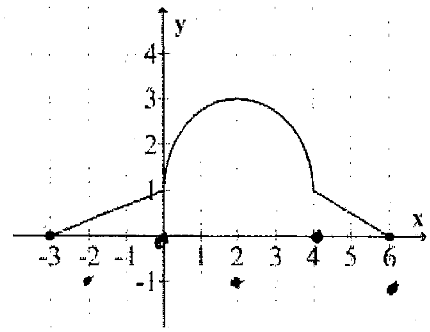
(b) $\int_0^3 (g(x) - f(x))dx = 41 - 2 = 39$

(c) $\int_0^3 [12f(x) - 3g(x)]dx = 12(2) - 3(41)$
$= 24 - 123 = -99$

(d) $\int_3^{-8} f(x)dx = -\int_{-8}^{3} f(x)dx = -7$

7. Consider the sketched function f:
The arc is a semicircle. Compute the following:

(a) $\int_{-3}^{0} f(x)dx =$ $3/2$



(b) $\int_{0}^{4} f(x)dx =$ $4+2\pi$

(c) $\int_{2}^{6} f(x)dx =$ $2+\pi+1 = 3+\pi$

(d) If the area under f from -2 to 6 is estimated using approximating rectangles, find

$\Delta x = \frac{6-(-2)}{4} = 2$

$L_4 = \left(\frac{1}{3} + 1 + 3 + 1\right)2 = \frac{32}{3}$ $R_4 = (1+3+1+0)2 = 10$

8. Properties: Given $\int_{1}^{8} f(x)dx = -4$ and $\int_{-12}^{1} f(x)dx = 6$ and $\int_{1}^{4} f(x)dx = 7$ find

a) $\int_{1}^{8} 4f(x)dx =$ -16

(b) $\int_{4}^{8} f(x)dx =$ $\int_{1}^{8} f(x)dx - \int_{1}^{4} f(x)dx = -4 - 7 = -11$

c) $\int_{1}^{-12} f(x)dx =$ $-\int_{-12}^{1} f(x)dx = -6$

(d) $\int_{-12}^{8} 3f(x)dx =$ $3\left(\int_{-12}^{1} f(x)dx + \int_{1}^{8} f(x)dx\right)$

$= 3 \cdot 2 = 6$