Mathematics 126 Fourth DE, Mixture and Orthogonal trajectories

1. A tank contains 30 kg of dissolved salt in 3000 L of water. Brine containing 0.04 kg salt per L of water is entering the tank at the rate of 20 L/min. The solution is kept thoroughly mixed and drains at the same rate. How much salt is in the tank after an hour? How much salt is in the tank after a very long time?

Let y be the amount of salt in the tank after t minutes,

$\frac{dy}{dt}$ = amount in − amount out

$= 0.04\frac{kg}{L} \cdot 20\frac{L}{min} - 20L \cdot \frac{y\,kg}{min\ 3000L}$

$= 0.8\ kg/min - \frac{y}{150}\ kg/min$

$150\frac{dy}{dt} = 120 - y$

$\int \frac{150\,dy}{120-y} = \int dt$

$-150\ln|120-y| = t + C_1$

$\ln|120-y| = -\frac{t}{150} + C_2$

$120 - y = \pm e^{-t/150} e^{C_2}$

$y = 120 - Ce^{-t/150}$

when $t=0$, $y=30$:

$30 = 120 - C$, $C = 90$

$y = 120 - 90e^{-t/150}$

After a long time, $\lim_{t\to\infty}(y) = 120$ kg

c) When will the tank contain 100 kg of salt?

$100 = 120 - 90e^{-t/150}$

$\frac{20}{90} = e^{-t/150}$

$-\frac{t}{150} = \ln(2/9)$

$t = -150\ln(2/9) \approx 226$ min after the process starts.

2. Find the orthogonal trajectories of the family of curves $3x^2 = ky^2 + 1$ where k is an arbitrary constant.

$3 \cancel{6}x = \cancel{2}kyy'$

$k = \frac{3x^2-1}{y^2}$

$3x = \frac{3x^2-1}{y^2} \times y'$

$= \frac{3x^2-1}{y} y'$

$y' = \frac{3xy}{3x^2-1}$

OT $y' = \frac{1-3x^2}{3xy}$

$\int y\,dy = \frac{1}{3}\int\left(\frac{1}{x} - 3x\right)dx$

$\frac{y^2}{2} = \frac{1}{3}\left(\ln|x| - \frac{3x^2}{2}\right) + C_1$

$y^2 = \frac{2}{3}\ln|x| - x^2 + C$

3. Find the orthogonal trajectories of the family of curves $kx^5 = y^4$ where k is an arbitrary constant.

$k \cdot 5x^4 = 4y^3y'$, $k = \frac{y^4}{x^5}$

$\frac{y^4}{x^5} 5x^4 = 4y^3y'$

$\frac{5}{4}\frac{y}{x} = y'$

OT $y' = -\frac{4}{5}\frac{x}{y}$

$\int 5y\,dy = \int -4x\,dx$

$\frac{5y^2}{2} = -2x^2 + C_1$

$5y^2 = -4x^2 + C$

$4x^2 + 5y^2 = C$

Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.
Study the (a) class notes, (b) text examples, (c) do the text exercises, and (d) do the 4th hour problems.