10.7 Taylor Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

To find Taylor series, use a table with columns $n, f^{(n)}(x), f^{(n)}(c), \frac{f^{(n)}(c)}{n!}$.

1. Find the Taylor series for $f(x) = x^4$ centered about $c = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	x^4	1	1
1	$4x^3$	4	4
2	$12x^2$	12	6
3	$24x$	24	4
4	24	24	1
5	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Taylor series

$$= 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

2. Find the Taylor series for $f(x) = \sin(x)$ centered at $c = \pi/2$.

k	n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	0	$\sin(x)$	1
	1	$\cos(x)$	0
1	2	$-\sin(x)$	-1
	3	$-\cos(x)$	0
2	4	$\sin(x)$	1
	5		$\vdots$
	$\vdots$		

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$$

The **Macluarin series** $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is the Taylor series with $c = 0$.

3. Find the Maclaurin series for $f(x) = e^x$ and find its radius of convergence.

$$e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$$

Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1 \text{ for all } x$$

Ans $R = \infty$

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The secret to mathematics is STUDY, STUDY, STUDY

4. Find the Maclaurin series for *f(x) = sin(x)* and find its radius of convergence.

k	n	$f^{(n)}(x)$	$f^{(n)}(0)$
	0	$\sin(x)$	0
0	1	$\cos(x)$	1
	2	$-\sin(x)$	0
1	3	$-\cos(x)$	-1
	4	$\sin(x)$	0
2	5	$\cos(x)$	1

so $n = 2k+1$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Ratio test

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{(-1)^{k+1} x^{2(k+1)+1} / (2(k+1)+1)!}{(-1)^k x^{2k+1} / (2k+1)!}\right|$$

$$= \lim_{k\to\infty} \left|\frac{x^{2k+3}}{x^{2k+1}} \frac{(2k+1)!}{(2k+3)!}\right|$$

5. Find the Maclaurin series for *f(x) = cos(x)* and find its radius of convergence.

$$= \lim_{k\to\infty} \left| x^2 \frac{\cancel{(2k+1)!}}{(2k+3)(2k+2)\cancel{(2k+1)!}}\right|$$

$$= \lim_{k\to\infty} \left|\frac{x^2}{(2k+3)(2k+2)}\right| = 0 < 1, \quad R = \infty$$

$I = (-\infty, \infty)$

6. Find the Maclaurin series for *f(x) = sin(2x)* and find its radius of convergence.

$$\sin(2x) = \sum \frac{(-1)^k}{(2k+1)!} (2x)^{2k+1}$$

$$= \sum \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1}$$

$R = \infty$

$I = (-\infty, \infty)$

Omit Binomial Series and Convergence of Taylor Series on pp 614 to 620.

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$$\sin(x) = \sum \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\cos(x) = \frac{d}{dx}\sin(x) = \sum \frac{d}{dx}\left(\frac{(-1)^k}{(2k+1)!} x^{2k+1}\right)$$

$$= \sum \frac{(-1)^k}{(2k+1)!} (2k+1) x^{2k}$$

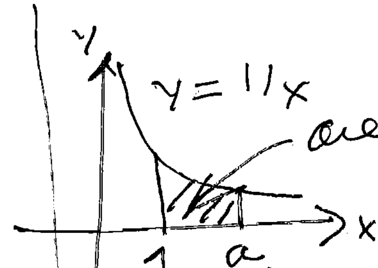
$$= \sum \frac{(-1)^k}{\cancel{(2k+1)}(2k)!} \cancel{(2k+1)} x^{2k}$$

$$\cos(x) = \sum \frac{(-1)^k}{(2k)!} x^{2k}$$

$$R = \infty, \quad I = (-\infty, \infty)$$

Evaluating infinite series

1. *Evaluate* $\sum_{k=0}^{\infty} \frac{1}{k!}$



$$e^x = \sum \frac{x^n}{n!}$$

when $x=1$, $e = \sum \frac{1}{n!} = \sum \frac{1}{k!}$

$$\int_1^a \frac{1}{x}\,dx = 1$$

$$= \ln|x| \Big|_1^a = \ln a - \ln 1$$

$$= \ln a = 1$$

$$a = e$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

2. Use the infinite series for $\tan^{-1}(x)$ to evaluate $\sum_0^\infty \frac{(-1)^k}{(2k+1)3^{(2k+1)/2}}$

$f(x) = \tan^{-1}(x)$

$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum (-x^2)^n$, $|-x^2| < 1 \Leftrightarrow |x| < 1$

(since $\frac{1}{1-x} = \sum x^n$)

$= \sum (-1)^n x^{2n}$

$f(x) = \int f'(x)\,dx = \sum (-1)^n \int x^{2n}\,dx = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{2n+1} + C$

$\tan^{-1}(0) = 0 = f(0) = \sum \frac{(-1)^n 0}{2n+1} + C = 0 + C$, $\underline{C = 0}$

so $\tan^{-1}(x) = \sum \frac{(-1)^k x^{2k+1}}{2k+1}$, $|x| < 1$

$x^{2k+1} = \frac{1}{3^{(2k+1)/2}}$, $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum \frac{(-1)^k}{(2k+1)\,3^{(2k+1)/2}}$

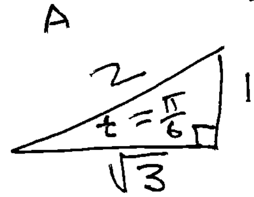
$= \left(\frac{1}{3^{1/2}}\right)^{2k+1}$

Find $\tan^{-1}(1/\sqrt{3}) = t$

T
O
A

$1/\sqrt{3} = \tan(t)$

so $x = \frac{1}{3^{1/2}} = \frac{1}{\sqrt{3}}$



so the sum $= \frac{\pi}{6}$

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