10.6 Power Series

A <u>power series</u> is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

<u>Notation</u>: $\sum a_n(x-c)^n$, assume n from 0 to ∞.

<u>Example</u>: $\sum 2^n x^n$

If $x = \frac{1}{4}$, the power series becomes

$$\sum 2^n \left(\frac{1}{4}\right)^n = \sum \frac{1}{2^n}, \text{ geometric, } r = 1/2$$

converges to $\frac{1}{1-1/2} = 2$

If $x=1$, the power series becomes

$\sum 2^n$, geometric, $r=2$, the series diverges.

<u>Theorem</u>: For the power series $\sum a_n(x-c)^n$ either
(a) there is a number R called the <u>radius of convergence</u> such that the series converge for all x in the interval $(c-R, c+R)$ and diverges on $(-\infty, c-R)$, $(c+R, \infty)$.

or (b) the series converges on $(-\infty, \infty)$, $R = \infty$

or (c) the series converges only at $x = c$, $R = 0$.

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$$\sum (-1)^n = 1-1+1-1+1-1+1-\cdots$$
$$0 = (1-1)+(1-1)+(1-1)+\cdots$$
$$= 1-(1-1)-(1-1)-(1-1)-\cdots$$
$$= 1$$

$$|(-x)| = |-1\cdot x| = |-1||x| = |x|$$

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln(x)}{\ln(x+1)} \overset{\text{L'H}\ \infty/\infty}{=} \lim_{x\to\infty} \frac{1/x}{1/x+1}$$
$$= \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty}\left(1+\underbrace{\tfrac{1}{x}}_{\to 0}\right) = 1$$

$$\sum_{0}^{\infty} x^n$$

$$x S_n = x + x^2 + x^3 + \cdots + x^{n+1}$$
$$S_n = 1 + x + x^2 + \cdots + x^n$$

Subtract $\quad x S_n - S_n = -1 + x^{n+1}$

$$S_n(x-1) = x^{n+1} - 1$$
$$S_n = \frac{x^{n+1}-1}{x-1} = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{0}^{\infty} x^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1-x^{n+1}}{1-x}\right) \qquad \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x|<1 \\ \text{diverges} & \text{if } |x|>1 \end{cases}$$

$$= \frac{1-0}{1-x} \quad \text{if } |x|<1$$

$$\therefore \sum_{0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x|<1$$

<u>Theorem</u> If $f(x) = \sum a_n (x-c)^n$, $|x-c| < R$

then $\int f(x)\,dx = \sum a_n \int (x-c)^n\,dx$

$= \sum a_n \frac{(x-c)^{n+1}}{n+1} + C$ $\quad |x-c| < R$

and $f'(x) = \sum n a_n (x-c)^{n-1}$

10.6 Power series

We use the ratio test and set $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ and solve for x to determine the open interval of convergence.

1. Find the radius of convergence and the open interval of convergence of $\sum_0^\infty 2^n x^n$.

Ratio test

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = \lim_{n\to\infty}|2x| < 1,\quad R = 1/2,\quad I = \left(-\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$|2x|<1$$

$$|x|<\tfrac{1}{2},\quad -\tfrac{1}{2}<x<\tfrac{1}{2}$$

2. Find the radius of convergence and the open interval of convergence of $\sum_1^\infty \frac{(x+2)^n}{n}$.

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}/n+1}{(x+2)^n/n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(x+2)^n}\cdot\frac{n}{n+1}\right|,\quad \frac{n}{n+1}\to 1$$

$$= |x+2|<1 \qquad R=1$$

$$-1<x+2<1 \qquad I=(-3,-1)$$

$$-3<x<-1$$

3. Find the radius of convergence and the open interval of convergence of $\sum_0^\infty n!(3x+1)^n$.

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)!\,(3x+1)^{n+1}}{n!\,(3x+1)^n}\right| = \lim_{n\to\infty}|(n+1)(3x+1)| = \infty$$

if $3x+1\neq 0$

$$R=0,\quad I=\{-1/3\}$$

5. Find the radius of convergence and the open interval of convergence of $\sum_2^\infty \frac{(-x)^n n^2}{\ln(n)}$.

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)^2/\ln(n+1)}{x^n n^2/\ln(n)}\right| = \lim_{n\to\infty}\left|x\left(\frac{n+1}{n}\right)^2\left(\frac{\ln(n)}{\ln(n+1)}\right)\right|,\quad \left(\frac{n+1}{n}\right)^2\to 1,\quad \frac{\ln(n)}{\ln(n+1)}\to 1$$

$$=|x|<1,\qquad R=1$$

$$-1<x<1 \qquad I=(-1,1)$$

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Power series representation

$$f(x) = \sum_0^\infty x^n = \frac{1}{1-x} \quad for\ |x| < 1$$

1. Express $g(x) = \dfrac{1}{1+x}$ as a power series and find its radius of convergence.

$$f(-x) = \frac{1}{1-(-x)} = \sum (-x)^n \qquad |x| < 1 \iff |-x| < 1$$

$$= g(x) = \frac{1}{1+x} = \sum_0^\infty (-1)^n x^n, \quad |x| < 1$$

$$R = 1$$

2. Express $h(x) = \dfrac{1}{1+5x}$ as a power series and find its radius of convergence.

$$h(x) = f(-5x) = \sum_0^\infty (-5x)^n = \sum_0^\infty (-5)^n x^n$$

$$|-5x| < 1$$

$$5|x| < 1$$

$$|x| < \frac{1}{5}, \quad R = 1/5$$

3. Express $k(x) = \dfrac{1}{5+x}$ as a power series and find its radius of convergence.

$$k(x) = \frac{1}{5}\,\frac{1}{1+x/5} = \frac{1}{5} f(-x/5) = \frac{1}{5}\sum (-x/5)^n$$

$$= \frac{1}{5}\sum \left(-\frac{1}{5}\right)^n x^n = \frac{1}{5}\sum \frac{(-1)^n}{5^n} x^n = \sum_0^\infty \frac{(-1)^n}{5^{n+1}} x^n$$

$$|-x/5| < 1, \quad |x| < 5, \quad R = 5$$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

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4. Find a power series representation for $g(x)=\dfrac{x^2}{1+2x}$ and find its radius of convergence.

$$g(x) = x^2 \cdot \frac{1}{1+2x} = x^2 f(-2x) = x^2 \sum (-2x)^n = x^2 \sum (-2)^n x^n$$

$$= \sum_0^{\infty} (-2)^n x^{n+2}$$

$|-2x| < 1$

$|x| < 1/2$, $R = \frac{1}{2}$

5. Express $h(x)= ln(1+x)$ as a power series and find its radius of convergence.

$$h'(x) = \frac{1}{1+x} = \sum (-x)^n, \quad |x| < 1.$$

$$= \sum (-1)^n x^n$$

$$h(x) = \int h'(x)dx = \sum (-1)^n \int x^n dx$$

$$\ln(1+x) = \sum_0^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

To find C, take $x=0$:

$$\ln(1) = \sum (-1)^n \frac{0^{n+1}}{n+1} + C$$

$$0 = 0 + C, \quad C = 0$$

$$h(x) = \ln(1+x) = \sum_0^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \quad R = 1$$

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