10.2 Series

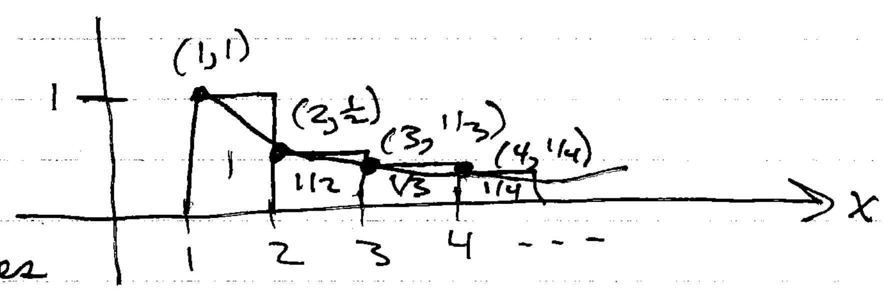
A <u>series</u> is the sum of a sequence. Given the sequence $a_1, a_2, a_3, \ldots$
the <u>sequence of partial sums</u>

$$S_1 = a_1$$
$$S_2 = a_1 + a_2 = S_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$$
$$\vdots$$
$$S_n = \sum_{i=1}^{n} a_i = S_{n-1} + a_n$$
$$\vdots$$

The <u>series</u> $s = \sum_{1}^{\infty} a_n = \lim_{n\to\infty} S_n$

<u>Harmonic Series</u> $\sum_{1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$\sum_{1}^{\infty} \frac{1}{n}$ = sum of the areas of the rectangles



$y = \frac{1}{x}$

$$\underbrace{\sum_{1}^{\infty} \frac{1}{n}}_{\text{areas of rectangles}} > \underbrace{\int_{1}^{\infty} \frac{1}{x}\,dx}_{\text{area under } y = 1/x} = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left(\ln|x| \Big|_1^t \right) = \lim_{t\to\infty} (\underbrace{\ln t}_{\to\infty} - \underbrace{\ln 1}_{=0})$$

<u>Harmonic series</u> $= \infty$

so $\sum_{1}^{\infty} \frac{1}{n} = \infty$, diverges

Study for the final exam starting now

Work through **all** the tests and sample final exams in the booklet of previous Capilano exams. Time yourself on each test and exam. Check the solutions in the back of the booklet. Discuss in the Math Learning Centre and/or with Frank any concerns you wish.

10.2 Summing an infinite series

A *partial sum* $S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$ is the sum of the first n terms of a sequence. A *series* or *infinite series* is the limit of the partial sums and is written

$$S = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

Theorem: If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$ also converge for any constant c, and $\sum a_n \pm \sum b_n = \sum(a_n \pm b_n)$ and $\sum ca_n = c\sum a_n$.

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

sequence series

Determine if each of the following series converges. If the series converges, find the sum, or approximate the sum if you cannot find it exactly. Name the test you use [for example, harmonic series, divergence test, telescoping series].

1. $\sum_{1}^{\infty} \frac{(n-4)^2}{5n(n+2)}$

$a_n = \frac{(n-4)^2}{5n(n+2)} \to \frac{1}{5}$

$\lim_{n\to\infty} a_n = \frac{1}{5} \neq 0$ so the series diverges by the divergence test.

2. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{3^n+1}$

$\lim_{n\to\infty} \frac{3^{n+1}}{3^n+1} = 3 \neq 0$ so the series diverges by the divergence test.

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Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

3. The n[th] partial sum is $s_n = \dfrac{2n-1}{5n+3}$ (a) Find the n[th] term a_n.

$$S_n = S_{n-1} + a_n, \quad a_n = S_n - S_{n-1}$$

$$a_n = \left(\frac{2n-1}{5n+3}\right) - \left(\frac{2(n-1)-1}{5(n-1)+3}\right)$$

(b) Find the sum of the series.

$$s = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2n-1}{5n+3} = \frac{2}{5}$$

The **Harmonic series** $\sum_1^{\infty} \frac{1}{n}$ diverges.

4. $\displaystyle\sum_1^{\infty} \frac{2}{5n} = \frac{2}{5} \underbrace{\sum_1 \frac{1}{n}}_{\text{Harmonic series}} = \frac{2}{5}\infty = \infty$, series diverges

Definition: A series $\sum a_n$ is **geometric** if the ratio $\frac{a_{n+1}}{a_n} = r$ is constant for all n.

Theorem: If a geometric series has ratio r and first term a, then

(i) the series sums to $\frac{a}{1-r}$ if $|r| < 1$or

(ii) diverges if $|r| \geq 1$.

Determine if each of the following series is geometric. For the geometric series, determine if the series converges, in which case find the sum.

5. $0.1 + 0.01 + 0.001 + 0.0001 + \ldots$

$= 10^{-1} + 10^{-2} + 10^{-3} + 10^{-4} + \ldots$

$n = 1 \quad 2 \quad 3 \quad 4 \quad \ldots$

$a_n = 10^{-n}$

$10^{-(n+1)}$

(1) $\dfrac{a_{n+1}}{a_n} = \dfrac{10^{-(n+1)}}{10^{-n}} = 10^{-1} = r$, geometric.

(2) $|r| = 10^{-1} < 1$, converges

(3) $\text{sum} = \dfrac{10^{-1}}{1-10^{-1}} = \dfrac{.1}{.9} = \dfrac{1}{9}$

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

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6. $\sum_{n=1}^{\infty} \frac{\ln(n)}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)/2^{n+1}}{\ln(n)/2^n} = \frac{\ln(n+1)}{2^{n+1}} \frac{2^n}{\ln(n)} = \frac{\ln(n+1)}{2\ln(n)}$$

Not geometric

7. $\sum_{n=2}^{\infty} \left(\frac{1}{3^n} - \frac{1}{4^n}\right)$

$$= \sum_2 \frac{1}{3^n} - \sum_2 \frac{1}{4^n} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

$$\frac{a_{n+1}}{a_n} = \frac{1/3^{n+1}}{1/3^n} = \frac{1}{3} = r, \ |r|<1$$

$$\sum_2 \frac{1}{3^n} = \frac{1/9}{1-1/3} = \frac{1}{9} \cdot \frac{3}{2} = \frac{1}{6}$$

$$r = \frac{1}{4}, \ |r|<1$$

$$\sum_2 \frac{1}{4^n} = \frac{1/16}{1-1/4} = \frac{1}{16} \cdot \frac{4}{3} = \frac{1}{12}$$

Telescoping Series are series in which the partial sum S_n collapses to a few terms. Find the sum of the following series, if the series converges. If the limits are not stated assume the series sums from n =1 to ∞.

8. $\sum \frac{1}{n(n+1)}$

① PF $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$1 = A(n+1) + Bn$

$n=0$: $1 = A$

$n=-1$: $1 = -B$, $B=-1$

② $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$

$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \dots + \left(\frac{1}{n-2}-\frac{1}{n-1}\right) + \left(\frac{1}{n-1}-\frac{1}{n}\right) + \left(\frac{1}{n}-\frac{1}{n+1}\right)$

$S_n = 1 - \frac{1}{n+1}$

③ $\sum \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{n+1}}_{\to 0}\right) = 1$

9. $\sum ln\left(\frac{n+1}{n}\right)$

10. **Rewriting series:** If $\sum_0^{\infty} \left(\frac{2^{n+1}}{\cos(0.8n)}\right) = \sum_1^{\infty}(b_n)$ find b_n.

$$b_1 = \frac{2}{\cos(0.8(0))}$$

$$b_2 = \frac{2^2}{\cos(0.8)}$$

$$b_n = \frac{2^n}{\cos(0.8(n-1))}$$

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