10.1 Sequences

Omit Bounded Sequences pp 543 to 545.

*A **sequence** is an ordered list of numbers* $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$.
*A **recurrence relation** of the form* $f(a_n) = a_{n+1}$ may define the sequence.
*An **explicit formula** of the form* $f(n) = a_n$ may define the sequence.

1. For the sequence $-\frac{1}{2}, \frac{2}{4}, -\frac{6}{8}, \frac{24}{16}, \ldots$ (a) find a recurrence relation defining the sequence and

$n = 1 \quad 2 \quad 3 \quad 4$ $\qquad a_{n+1} = a_n\left(-\frac{n+1}{2}\right)$

(b) find an explicit formula for the sequence. $a_n = (-1)^n \frac{n!}{2^n}$

2. Find the first 5 terms of the sequence defined by the recurrence relation

$$a_1 = -2, \; a_{n+1} = 1 + a_n/2$$

$a_1 = -2$
$a_2 = 1 + {-2}/2 = 0$
$a_3 = 1 + 0/2 = 1$
$a_4 = 1 + 1/2 = 3/2$
$a_5 = 1 + 3/4 = 7/4$

Theorem: *If f is a function with $\lim_{x\to\infty} f(x) = L$ and a_n is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$.*

The limit of the sequence equals the limit of the associated function.

1. Determine if the sequence converges and the limit, or that the sequence diverges:

$$-\frac{1}{2}, \frac{2}{4}, -\frac{6}{8}, \frac{24}{16}, \ldots ?$$

$\frac{2}{4}, \frac{24}{16}, \ldots \rightarrow \infty$

$-\frac{1}{2}, -\frac{6}{8}, \ldots \rightarrow -\infty$

The sequence diverges.

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

2. Does the sequence with n^{th} term shown below converge? If so, find the limit.

a) $a_n = (-1)^n \dfrac{n^2+1}{(1+n)(1-n^2)}$

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2+1}{|(1+n)(1-n^2)|} = \lim_{x\to\infty} \frac{x^2+1}{|(1+x)(1-x^2)|} = \lim_{x\to\infty} \frac{x^2+1}{|1-x^3-x^2+x+1|}$$

$$= \lim_{x\to\infty} \frac{x^2+1}{x^3+x^2-x-1} = 0, \text{ Converges to } 0$$

b) $a_n = \dfrac{2n+1}{1-4n}$

$$\lim_{n\to\infty} \frac{2n+1}{1-4n} = \lim_{x\to\infty} \frac{2x+1}{1-4x} \overset{L'H\ \infty/-\infty}{=} \lim_{x\to\infty} \frac{2}{-4} = -\frac{1}{2}$$

Converges to $-\frac{1}{2}$

c) $a_n = \dfrac{n\sin\left(\frac{8}{n}\right)}{(2^n+1)}$

$$\lim_{n\to\infty} \left| \frac{n\sin\left(\frac{8}{n}\right)}{2^n+1} \right| \leq \lim_{n\to\infty} \frac{n}{2^n+1} = \lim_{x\to\infty} \frac{x}{2^x+1} \overset{L'H\ \infty/\infty}{=} \lim_{x\to\infty} \frac{1}{2^x \ln(2)} = 0$$

Converges to 0

d) $a_n = \dfrac{e^n}{\ln(n)}$

$$\lim_{n\to\infty} \frac{e^n}{\ln(n)} = \lim_{x\to\infty} \frac{e^x}{\ln(x)} \overset{L'H\ \infty/\infty}{=} \lim_{x\to\infty} \frac{e^x}{1/x} = \lim_{x\to\infty} (xe^x) = \infty$$

the sequence diverges

e) $\{(-1)^{2n}\} = \{1\}$

converges to 1.

Factorial: Define $0! = 1$, $1! = 1$ and for $n > 1$, $n! = n\,(n-1)!$

f) $a_n = \dfrac{18^n}{n!}$

For $n > 18$, $a_n = \underbrace{\overbrace{\dfrac{18 \cdot 18 \cdots 18}{1 \cdot 2 \cdots 18}}^{18}}_{\text{Constant}} \cdot \underbrace{\dfrac{18}{19}}_{<1} \cdot \underbrace{\dfrac{18}{20}}_{<1} \cdots \underbrace{\dfrac{18}{n}}_{<1 \ \to 0} \to 0$

g) $\{\cos(n\pi)\}$

n	1	2	3	4
$\cos(n\pi)$	-1	1	-1	1

Sequence diverges

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.

Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

h) $a_n = \sin(n\pi)$

$\sin(n\pi) = 0$ for all integers n so the sequence converges to 0.

i) $\left\{\frac{2n+(-1)^n}{2n-(-1)^n}\right\}$ Converges to 1.

nMin = 1
u(n) = (2n + (-1)^n) / (2n - (-1)^n)
u(nMin) = 1/3
nMin = 1
nMax = 1000

[0, 1100] by [0, 1.5]

A **nonincreasing sequence** satisfies $a_n \geq a_{n+1}$ *for all n.* A **nondecreasing sequence** satisfies $a_n \leq a_{n+1}$ *for all n*. A **monotonic sequence** is either nonincreasing or nondecreasing. A **bounded sequence** satisfies $|a_n| \leq$ M for all n and some M.

Example The sequence $a_n = \frac{(-1)^n}{12^n}$ in not monotonic, but is bounded because $|a_n| \leq 1$.

Determine if the sequence $a_n = 1 - \frac{5}{n}$ is (a) monotonic,

Since $1 - \frac{5}{n} < 1 - \frac{5}{n+1}$, the sequence is monotonic

(b) bounded.

n	1	2	3
$1-\frac{5}{n}$	-4	$-\frac{3}{2}$	$-\frac{2}{3}$

$|a_n| \leq 4$, bounded

The sequence converges to 1.

 Understand the methods so you can solve similar problems.
Understand the concepts so you can solve unfamiliar problems.
Study the (a) class notes, (b) do the 4th hour problems, (c) study the text examples, (d) do the text exercises and (e) read the next text section.

A **geometric sequence** is a sequence in which $\frac{a_{n+1}}{a_n} = r$ *for all n.* The ratio of succeeding terms is constant and does not depend on *n*.

Determine if each of the following sequences is geometric. If the sequence is geometric, find the ratio *r*.

1. $\frac{1}{(-3)^n}$, $\frac{a_{n+1}}{a_n} = \frac{1/(-3)^{n+1}}{1/(-3)^n} = \frac{(-3)^n}{(-3)^{n+1}} = \frac{1}{-3}$, geometric

   $r = -\frac{1}{3}$

   $|r| = \frac{1}{3} < 1$, converges to 0

2. $(-0.8)^n$, $\frac{a_{n+1}}{a_n} = \frac{(-0.8)^{n+1}}{(-0.8)^n} = -0.8 = r$, geometric

   $|r| = 0.8 < 1$, converges to 0

3. $12\left(\frac{5}{4}\right)^n$ $\frac{a_{n+1}}{a_n} = \frac{12\left(\frac{5}{4}\right)^{n+1}}{12\left(\frac{5}{4}\right)^n} = \frac{5}{4} = r$, geometric

   $|r| = \frac{5}{4} > 1$, diverges

4. $a_n = 1 + \frac{5}{n}$ $\frac{a_{n+1}}{a_n} = \frac{1 + \frac{5}{n+1}}{1 + \frac{5}{n}} = \frac{\frac{n+6}{n+1}}{\frac{n+5}{n}} = \frac{n+6}{n+1} \cdot \frac{n}{n+5}$

   Not geometric

Theorem: Suppose a geometric sequence has ratio r and first term a.
If $|r| < 1$ the sequence converges to 0. If $|r| \geq 1$ the sequence diverges.

Determine which of the geometric sequences above converge to 0, and which diverge.