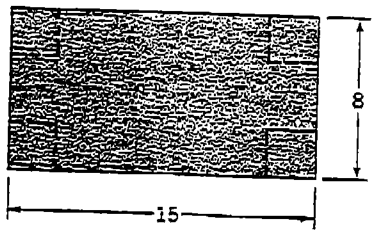
MORE APPLIED OPTIMIZATION PROBLEMS. 40

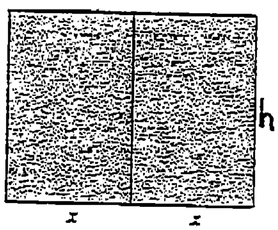
6.10

1. What are the dimensions of the base of the rectangular box of greatest volume that can be constructed from 100 square inches of cardboard if the base is to be twice as long as it is wide? Assume that the box has a top.

2. From a rectangular piece of cardboard of dimensions 8×15, four congruent squares are to be cut out, one at each corner. (See the figure below.) The remaining crosslike piece is then to be folded into an open box. What size squares should be cut out if the volume of the resulting box is to be a maximum?

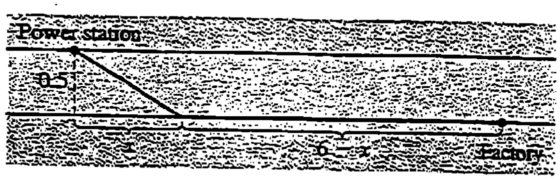


3. ***Area*** A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?



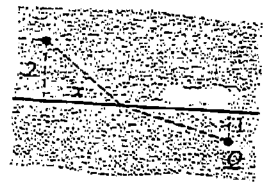
4. ***Volume*** You are designing a soft drink container that has the shape of a right circular cylinder. The container is to hold 12 fluid ounces (1 fluid ounce is approximately 1.80469 cubic inches). Find the dimensions that will use a minimum amount of construction material.

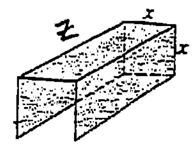
5. ***Cost*** A power station is on one side of a river that is 0.5 mile wide, and a factory is 6 miles downstream on the other side of the river (see figure). It costs \$6 per foot to run power lines overland and \$8 per foot to run them underwater. Find the most economical path for the power line from the power station to the factory.



6. ***Cost*** A small business uses a minivan to make deliveries. The cost per hour for fuel for the van is $C = v^2/600$, where v is the speed of the minivan (in miles per hour). The driver of the minivan is paid \$10 per hour. Find the speed that minimizes the cost of a 110-mile trip. (Assume there are no costs other than fuel and wages.)

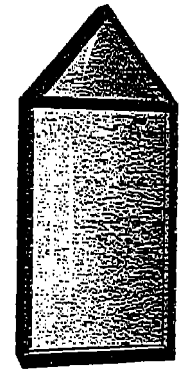
7. ***Time*** You are in a boat 2 miles from the nearest point on the coast. You are to go to point Q, 3 miles down the coast and 1 mile inland (see figure). You can row at a rate of 2 miles per hour and can walk at a rate of 4 miles per hour. Toward which point on the coast should you row in order to reach point Q in the least time?





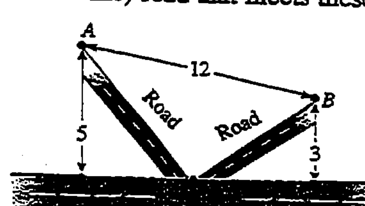
8. ***Surface Area*** A net enclosure for golf practice is open at one end (see figure). The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

9. A stained glass window in the form of an equilateral triangle is built on top of a rectangular window, as shown in Figure . The window is of clear glass and transmits twice as much light per ft² as the triangular window, which is made of stained glass. If the entire window has a perimeter of 20 ft, find the dimensions (to the nearest ft) of the rectangular window that will admit the most light.



10 A closed box with square base and vertical sides is to be built to house an ant colony. The bottom of the box and all four sides are to be made of material costing \$1/ft², and the top is to be constructed of glass costing \$5/ft². What are the dimensions of the box of greatest volume that can be constructed for \$72?

BONUS. Two towns A and B are 12 mi apart and are located 5 mi and 3 mi, respectively, from a long, straight highway. A construction company has a contract to build a road from A to the highway and then to B. How long is the *shortest* (to the nearest mile) road that meets these requirements?



ANSWERS:

5.773 x 2.887 ② $1.6\overline{6}$ ③ 25 x $33\frac{1}{3}$

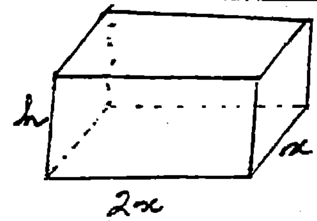
1.51 x 3.02 ⑤ x≈0.57

77.46 mph ⑦ x = 1 mi.

5m. x $3\frac{1}{3}$m. ⑨ 4ft x 4ft.

2ft x 2ft x 6ft. Bonus: L ≈ 14.283

① Let x = width of base
$2x$ = length of base
h = height.



WANT: TO MAXIMIZE THE VOLUME (V) OF THE BOX.

$$V = (2x)(x)(h) = 2x^2 h \longleftarrow \text{Express } h \text{ in terms of } x.$$

Given: TOTAL AREA = 100 SQ. IN.

$$\therefore 2(2x)(x) + 2(2x)(h) + 2(x)(h) = 100$$

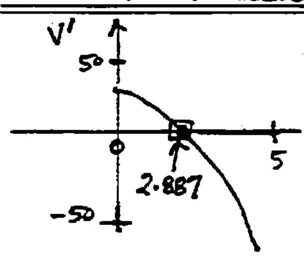
$$\Rightarrow \quad 4x^2 + 6xh = 100 \quad \Rightarrow \quad h = \frac{100 - 4x^2}{6x}.$$

$h > 0$ $\therefore x^2 < 2$… $\Rightarrow x <$

$$\therefore V = 2x^2 \cdot \left(\frac{100-4x^2}{6x}\right) \Rightarrow \boxed{V = \frac{100x - 4x^3}{3} \quad \text{for} \quad 0 < x < 5}$$

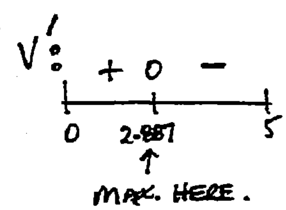
MAXIMIZE THIS!

① CRITICAL NUMBERS: $V'(x) = 0$



$\therefore x \approx 2.887.$

② CHECK:



CONCLUSION:

$\therefore$ Maximum volume when base is ≈ 2.887 in. by 5.774

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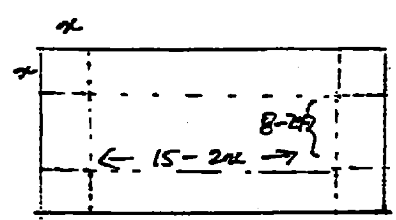
② Let x = length of each square cut out.



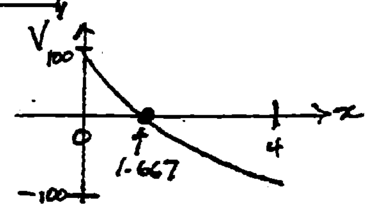
$\therefore$ Box dime…
$x \times (8-2x) \times$…

WANT: TO MAXIMIZE THE VOLUME OF THE BOX.

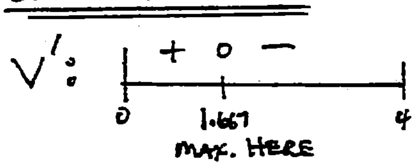
$$\boxed{V = x(8-2x)(15-2x) \quad \text{where} \quad 0 < x < 4.}$$

MAXIMIZE THIS!

① CRITICAL NUMBERS: $V'(x) = 0$
$\therefore x \approx 1.667$



② CHECK CANDIDATE:

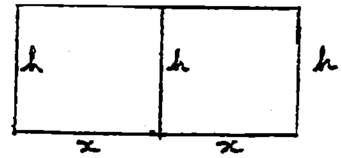


CONCLUSION:

$\therefore$ The squares should have length 1.667 uni…

(3) **WANT:** TO MAXIMIZE THE AREA.

$$A = 2x\,h \;\leftarrow\; \text{Express } h \text{ in terms of } x.$$



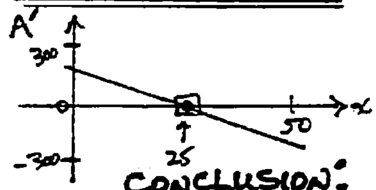
Given: Total length of fencing is 200 ft.

$\therefore\; 4x + 3h = 200 \;\Rightarrow\; h = \dfrac{200-4x}{3}$

$h > 0 \;\therefore\; 4x < 200,\; x < 50$

$$\therefore\; \boxed{A = 2x\left(\frac{200-4x}{3}\right) = \frac{400x - 8x^2}{3} \quad \text{where } 0<x<50}$$ MAXIMIZE THIS!

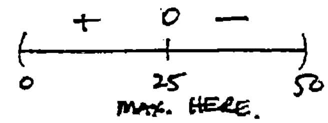
(1) CRITICAL NUMBERS: $A'(x) = 0$



$\therefore\; x = 25$

(2) CHECK CANDIDATE:

A':



CONCLUSION:

$\therefore$ Maximum area when $x = 25$ ft. and $h = 33\frac{1}{3}$ ft.

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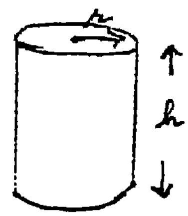
(4) Let r = radius (in.)
h = height (in.)



WANT: To MINIMIZE THE SURFACE AREA.

$$A = 2\pi r^2 + 2\pi r\,h \;\leftarrow\; \text{Express in terms of } r.$$

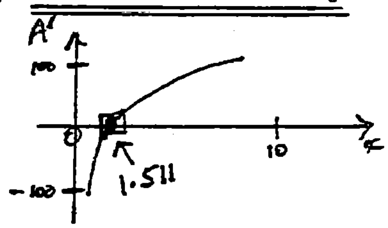
Given: Volume is $12(1.80469) = 21.65628 \text{ in}^3$.

$\therefore\; \pi r^2 \cdot h = 21.65628 \;\Rightarrow\; h = \dfrac{21.65628}{\pi r^2}$. $(r>0)$

$$\therefore\; A = 2\pi r^2 + 2\pi r\left(\frac{21.65628}{\pi r^2}\right) \quad \text{where } r>0.$$

$$\therefore\; \boxed{A = 2\pi r^2 + \frac{43.31256}{r} \quad \text{where } r>0}$$ MAXIMIZE THIS!

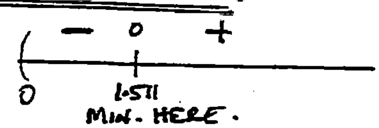
(1) CRITICAL NUMBERS: $A'(x) = 0$



$\therefore\; x \approx 1.511$

(2) CHECK CANDIDATE:

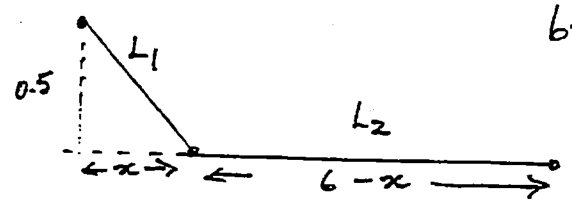
A':



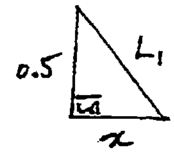
CONCLUSION:

$\therefore$ Minimum amount of material when $r \approx 1.511$ in. and $h \approx 3.019$ in.

⑤ **WANT:** TO MINIMIZE COST



$$C = 8(5280) \cdot L_1 + 6(5280) \cdot L_2$$



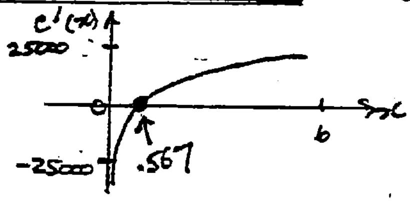
PYTHAGORAS: $x^2 + 0.5^2 = L_1^2$

$\therefore L_1 = \sqrt{x^2 + 0.5^2}$

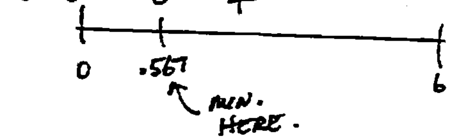
$L_2 = 6 - x$

$\therefore$ $\boxed{C(x) = 42{,}240\sqrt{x^2 + .25} + 31{,}680(6-x) \quad \text{for } 0 \le x \le 6}$ MINIMIZE THIS!

① **CRITICAL NUMBERS:** $C'(x) = 0$ $\quad \therefore x \approx .567$



② **CHECK THE CANDIDATE:**



CONCLUSION:

$\therefore$ Power line should run underwater to a point $\approx .567$ mi. downstream from the power station and then along the shoreline ≈ 5.433 mi. to the factory.

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⑥ **WANT:** TO MINIMIZE THE TOTAL COST.

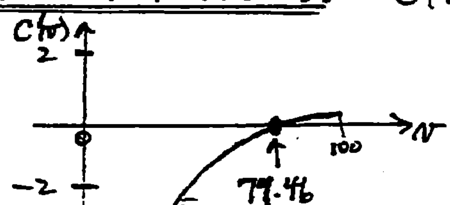
Let v = speed (mph). Then the time required is $\frac{110}{v}$.

$\therefore$ Total Cost = fuel cost + driver cost

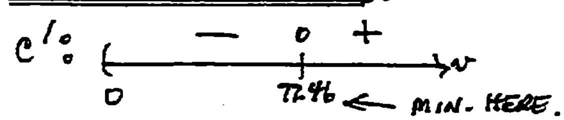
$$\therefore C(v) = \left(\frac{v^2}{600}\right)\left(\frac{110}{v}\right) + (10)\left(\frac{110}{v}\right) \Rightarrow \boxed{C(v) = \frac{11v}{60} + \frac{1100}{v} \quad \text{for } v > 0}$$

MINIMIZE THIS!

① **CRITICAL NUMBERS:** $C'(v) = 0$ $\quad \therefore v \approx 77.46$



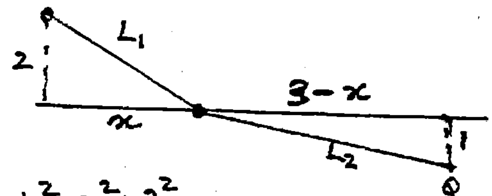
② **CHECK THE CANDIDATE:**



CONCLUSION:

$\therefore$ Driver should travel at 77.46 mph.

⑦ WANT: To minimize the total time.



$$T = \frac{L_1}{2} + \frac{L_2}{4}$$

$L_1^2 = x^2 + 2^2$

$\therefore L_1 = \sqrt{x^2+4}$

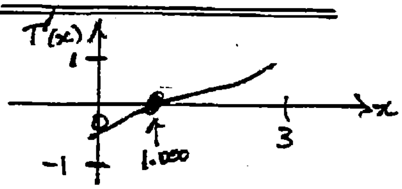
$L_2^2 = (3-x)^2 + 1$

$\therefore L_2 = \sqrt{(3-$

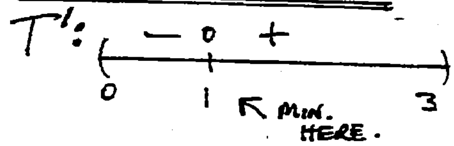
$$\therefore \boxed{T(x) = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{(3-x)^2+1}}{4} \quad \text{for } 0 \le x \le 3}$$

MINIMIZE THIS!

① CRITICAL NUMBERS: $T'(x) = 0 \quad \therefore x \approx 1.$



② CHECK THE CANDIDATE:



CONCLUSION:

$\therefore$ You should row to the point on shore 1 mi. down the coast.

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⑧ WANT: To minimize the surface area.

$A = x^2 + 3x\,Z$ ← Express in terms of x

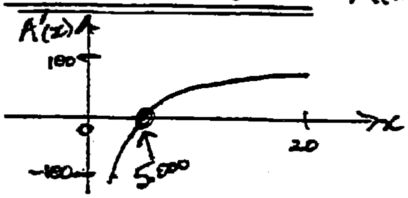
Given: Volume $= 83\frac{1}{3}\ m^3. = \frac{250}{3}\ m^3.$

$$\therefore x^2 \cdot Z = \frac{250}{3} \Rightarrow Z = \frac{250}{3x^2} \quad (x > 0)$$

$$\therefore A(x) = x^2 + 3x \cdot \left(\frac{250}{3x^2}\right) \Rightarrow \boxed{A(x) = x^2 + \frac{250}{x} \quad \text{for } x > 0.}$$

MINIMIZE THIS!

① CRITICAL NUMBERS: $A'(x) = 0 \quad \therefore x \approx 5.$



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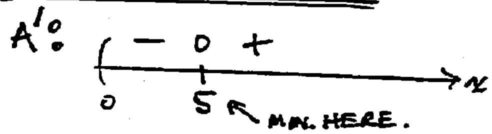


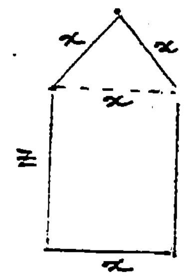
* when $x = 5$,

$$Z = \frac{250}{3(5^2)} = \frac{10}{3} =$$

CONCLUSION:

$\therefore$ The enclosure should measure $5\,m. \times 5\,m. \times 3\frac{1}{3}\,m.$

(9) For convenience, let the measure of light be 1 unit for the $\triangle$ and 2 units for the $\square$.

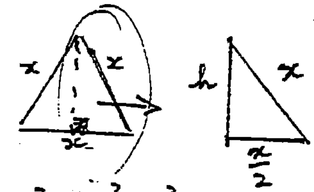


Let x = length of triangle side (ft)
z = height of rectangle (ft).

WANT: TO MAXIMIZE THE AMOUNT OF LIGHT.

$A = 1 \times$ area of $\triangle$ + $2 \times$ area of $\square$.

$\therefore A = \frac{\sqrt{3}}{4}x^2 + 2 \cdot x z$ ⟵ Express in terms of x



$h^2 + \left(\frac{x}{2}\right)^2 = x^2 \Rightarrow h^2 = x^2 - \frac{x^2}{4}$

$\Rightarrow h^2 = \frac{3x^2}{4} \Rightarrow h = \frac{\sqrt{3}x}{2}$.

$\therefore A_{\triangle} = \frac{1}{2}(x)\left(\frac{\sqrt{3}x}{2}\right) = \frac{\sqrt{3}}{4}x^2$

Given: Perimeter = 20 ft.

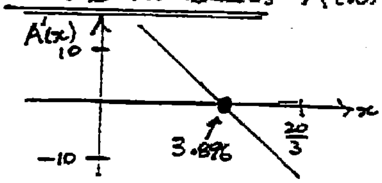
$z > 0$ $\therefore x < \frac{20}{3}$

$\therefore 3x + 2z = 20 \Rightarrow z = \frac{20-3x}{2}$

$\therefore A(x) = \frac{\sqrt{3}}{4}x^2 + 2x\left(\frac{20-3x}{2}\right)$

$\Rightarrow \boxed{A(x) = \frac{\sqrt{3}}{4}x^2 + 20x - 3x^2 \text{ for } 0 < x < \frac{20}{3}}$ MAXIMIZE THIS!

① CRITICAL NUMBERS: $A'(x) = 0$ $\therefore x \approx 3.896$



② CHECK THE CANDIDATE:

A': + 0 −
0 — 3.896 — 20/3
MAX. HERE.

* When $x \approx 3.896$
$z \approx \frac{20 - 3(3.896)}{2}$
≈ 4.156

CONCLUSION:

$\therefore$ The maximum amount of light enters when both x and z are approximately 4 ft. ⟵ (Question said to give dimensions to the nearest ft.)

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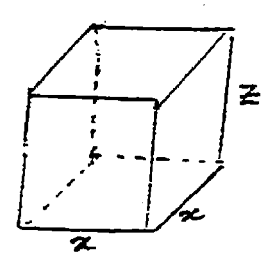
(10) Let x = length of base (ft)
z = height (ft).



WANT: TO MAXIMIZE THE VOLUME OF THE BOX

$V = x^2 \cdot z$ ⟵ Express in terms of x

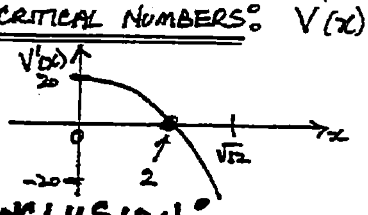
$z > 0$ $\therefore x < \sqrt{12}$

Given: Total cost = \$72

$\therefore 1(x^2) + 4(1)(xz) + 5(x^2) = 72 \Rightarrow 6x^2 + 4xz = 72 \Rightarrow z = \frac{72-6x^2}{4x}$

$\therefore V(x) = x^2 \cdot \left(\frac{72-6x^2}{4x}\right) \Rightarrow \boxed{V(x) = \frac{72x - 6x^3}{4} \text{ for } 0 < x < \sqrt{12}}$ MAXIMIZE THIS!

① CRITICAL NUMBERS: $V'(x) = 0 \Rightarrow x = 2$.



② CHECK THE CANDIDATE:

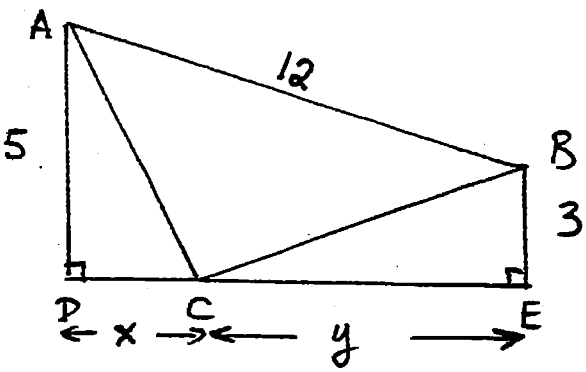
V': + 0 −
0 — 2 — $\sqrt{12}$
MAX. HERE.

* When $x = 2$:
$z = \frac{72 - 6(2^2)}{4(2)}$
$= 6$

CONCLUSION:

$\therefore$ Box should measure 2 ft × 2 ft × 6 ft.

"BONUS" QUESTION:



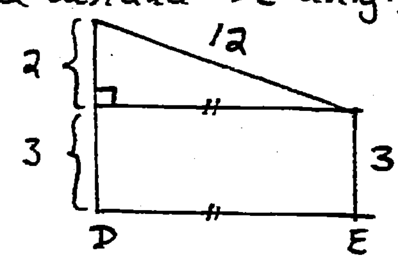
WANT:

To MINIMIZE Total Distance ACB

$$d = AC + CB$$
$$= \sqrt{5^2 + x^2} + \sqrt{3^2 + y^2}$$

(using PYTHAGORAS)

Need another equation:
Can get horizontal distance DE using PYTH.



$$DE = \sqrt{12^2 - 2^2}$$
$$= \sqrt{140}$$
$$\therefore x + y = \sqrt{140}$$
$$\text{or } y = \sqrt{140} - x$$

SUB into distance formula!

$$d = \sqrt{25 + x^2} + \sqrt{9 + (\sqrt{140} - x)^2}$$

or, in FUN. NOTATION,

$$\boxed{d(x) = \sqrt{25 + x^2} + \sqrt{9 + (\sqrt{140} - x)^2} \quad : \text{for } 0 \le x \le \sqrt{140} \approx 11.8}$$

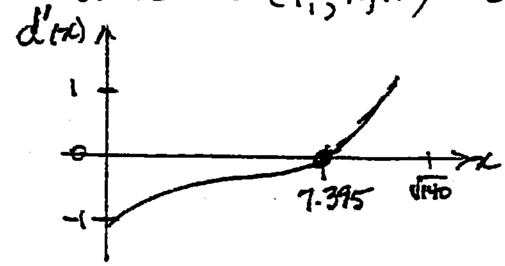
MINIMIZE THIS!

This one "LOOKS MESSY"; try GRAPHICAL SOLUTION

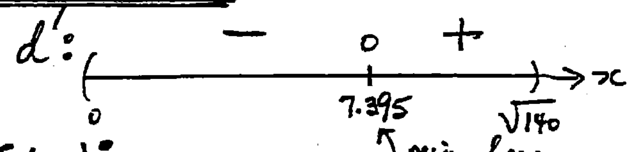
GRAPH: $Y_1 = \sqrt{(25 + x^2)} + \sqrt{(9 + (\sqrt{140} - x)^2)}$

① CRITICAL NUMBERS: $d'(x) = 0$

Want: nDeriv$(Y_1, x, x) = 0$



② CHECK THE CANDIDATE:



CONCLUSION:
Minimum length: $d(7.395) \approx 14.283$ mi.

ANALYTICAL SOLUTION:

$$d'(x) = \frac{1\,(\cancel{2}x)}{\cancel{2}\sqrt{25 + x^2}} + \frac{1\,(2(\sqrt{140} - x)(-1))}{2\sqrt{9 + (\sqrt{140} - x)^2}}$$

Set $d'(x) = 0$: $$\frac{x}{\sqrt{25 + x^2}} - \frac{(\sqrt{140} - x)}{\sqrt{9 + (\sqrt{140} - x)^2}} = 0$$

$$x\sqrt{9 + (\sqrt{140} - x)^2} = (\sqrt{140} - x)\sqrt{25 + x^2}$$
$$x^2(9 + (\sqrt{140} - x)^2) = (\sqrt{140} - x)^2(25 + x^2)$$
$$9x^2 = 25(\sqrt{140} - x)^2$$
$$3x = \pm 5(\sqrt{140} - x)$$

$$x = \frac{5}{8}\sqrt{140} \quad \text{or} \quad \cancel{\frac{5}{2}\sqrt{140}} \text{ (EXTRANE}$$

∴ "CRITICAL NUMBER" is
$$x = \frac{5}{8}\sqrt{140} \approx 7.395$$

etc . . .